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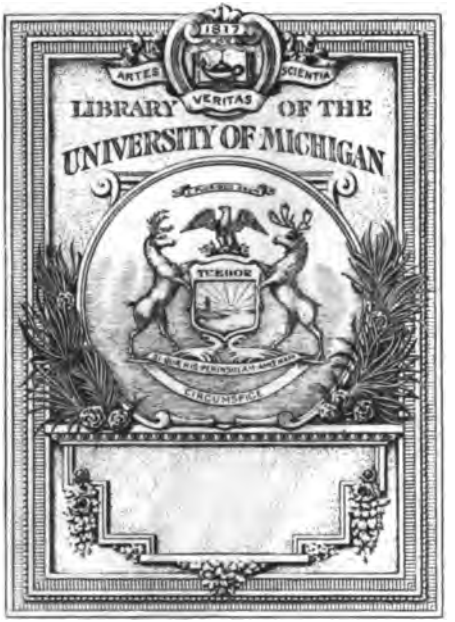

A

# TREATISE

ON

# MENSURATION,

BOTH IN

# THEORY AND PRACTICE.

THE SECOND EDITION, WITH MANY ADDITIONS.

By CHARLES HUTTON, LL. D. F. R. S. &c. &c.

PROFESSOR OF MATHEMATICS IN THE ROYAL MILITARY ACADEMY.

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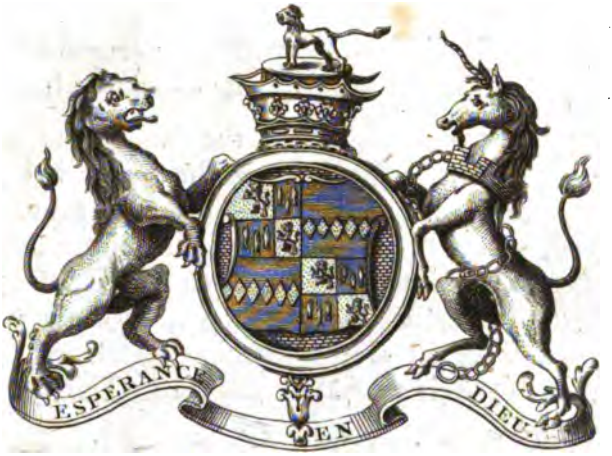


TO

THE MOST NOBLE AND PUISSANT PRINCE,

# HUGH,

DUKE AND EARL OF NORTHUMBERLAND,

EARL AND BARON PERCY, LUCY, POYNINGS, FITZPAYNE, BRYAN, LATIMER, AND WARKWORTH, AND BARONET;

LORD LIEUTENANT AND CUSTOS ROTULORUM OF THE COUNTY OF NORTHUMBERLAND, AND OF THE TOWN AND COUNTY OF THE TOWN OF NEWCASTLE UPON TYNE; VICE ADMIRAL OF THE COUNTY OF NORTHUMBERLAND; AND LIEUTENANT GENERAL OF HIS MAJESTY'S FORCES, COLONEL OF THE SECOND TROOP OF HORSE GRENADIER GUARDS, &c. &c. &c.

MY LORD,

IF the high honour this Work formerly received in the countenance of Your Grace's illuſtrious Father, had not particularly encouraged the Author's preſumption, in thus ſeeking the protection of his noble Repreſentative, he is convinced that he could not eaſily have found among the great and elevated, a character under whoſe

auſpices he ſhould more earneſtly have wiſhed to preſent it to the world in this its improved ſtate.

The great progreſs of all the uſeful arts and ſciences in this country, ſince the happy æra of the acceſſion of our auguſt Sovereign, muſt be aſcribed, next to his benign and munificent influence, to that eminent countenance which ſome of the greateſt characters have ſhewn, by their perſonal example in the cultivation of them, not leſs than by their patronage and protection.

To the hereditary fame of Your Moſt Noble Progenitors, Your Grace's perſonal bravery and military talents have added great and diſtinguiſhed luſtre. The public confidence and eſteem which neceſſarily follow ſuch accompliſhments, ſtamp a value on every thing, however inconſiderable, which is honoured with Your Grace's patronage: And the accurate judgment which Your Grace is known to poſſeſs on all ſubjects of extenſive practical utility, makes it the wiſh of ſuch as cultivate them, ardently to ſeek Your Grace's protection.

With theſe impreſſions, I preſume to lay the following performance at Your Grace's feet; and am, with the profoundeſt reſpect,

My Lord,

Your Grace's

moſt obedient, and

moſt devoted humble ſervant,

CHARLES HUTTON.

# PREFACE.

BY Menſuration I underſtand the art and ſcience which is concerned about the meaſure of extenſion, or the magnitude of figures; and it is, next to arithmetic, a ſubject of the greateſt uſe and importance, both in affairs that are abſolutely neceſſary in human life, and in every branch of the mathematics: a ſubject by which ſciences are eſtabliſhed, and commerce is conducted; by whoſe aid we manage our buſineſs, and inform ourſelves of the wonderful operations of nature; by which we meaſure the heavens and the earth, eſtimate the capacities of all veſſels, and bulks of all bodies: gauge our liquors, build edifices, meaſure our lands and the works of artificers, buy and ſell an infinite variety of things neceſſary in life, and are ſupplied with the means of making the calculations which are neceſſary for the conſtruction of almoſt all machines.

It is evident that the cloſe connection of this ſubject with the ordinary affairs of life, would very early evince its importance to mankind; and accordingly we find, that the moſt celebrated philoſophers have paid the greateſt attention to it; and the chief and moſt eſſential diſcoveries in geometry in all ages, have been made in conſequence of their attempts to improve this ſcience. Socrates thought that the prime uſe of geometry was to meaſure the ground, and indeed this buſineſs gave name to the ſubject; and moſt of the ancients ſeem to have had no other end beſides menſuration in view, in all their laboured geometrical diſquiſitions. Euclid's Elements are almoſt entirely devoted to it; and although there be contained in them many properties of geometrical figures which may be applied to other purpoſes, and indeed of which the moderns have made the greateſt uſe in moſt other ſciences; yet Euclid himſelf ſeems to have adapted them entirely to this purpoſe. For, if it be conſidered that his Elements contain a continued chain of reaſoning, and of truths, of which the former are ſucceſſively applied to the diſcovery of the latter, one propoſition depending on another, and the ſucceeding propoſitions ſtill approximating towards ſome particular object near the end of each book; and when at the laſt we find that object to be the equality, proportion, or relation between the magnitudes of figures, both plane and ſolid; it is ſcarcely

poſſible to avoid allowing this to have been Euclid's grand object. And accordingly he determined the chief properties in the menſuration of rectilineal plane and ſolid figures; and ſquared all ſuch planes, and cubed all ſuch ſolids. The only curve figures which he attempted, are the circle and ſphere; and when he could not accurately determine their meaſures, he gave an excellent method of approximating to them, by ſhewing how in a circle to inſcribe a regular polygon, which ſhould not touch another circle, concentric with the former, although their circumferences ſhould be ever ſo near together; and, in like manner, between any two concentric ſpheres to deſcribe a polyhedron, which ſhould not any-where touch the inner one: and approximations to their meaſures are all that have hitherto been given. But although he could not ſquare the circle, nor cube the ſphere, he determined the proportion of one circle to another, and of one ſphere to another, as well as the proportions of all rectilineal ſimilar figures to one another.

Archimedes took up menſuration where Euclid left it, and carried it a great length. He was the firſt who ſquared a curvilineal ſpace; unleſs Hypocrates muſt be excepted on account of his lunes. In his time the conic ſections were admitted into geometry, and he applied himſelf cloſely to the meaſuring of them, as well as other figures. Accordingly he determined the relations of ſpheres, ſpheroids, and conoids, to cylinders and cones; and the relations of parabolas to rectilineal planes whoſe quadratures had long before been determined by Euclid. He hath left us alſo his attempts upon the circle: he proved that a circle is equal to a right-angled triangle, whoſe baſe is equal to the circumference, and its altitude equal to the radius; and conſequently, that its area is equal to the rectangle of the radius, and half the circumference; and ſo reduced the quadrature of the circle to the determination of the ratio of the diameter to the circumference; but which, however, hath not yet been done. Being diſappointed of the *exact* quadrature of the circle, for want of the rectification of its circumference, which all his methods would not effect, he proceeded to aſſign an uſeful approximation to it; this he effected by the numeral calculation of the perimeters of the inſcribed and circumſcribed polygons; from which calculation it appears, that the perimeter of the circumſcribed regular polygon of 192 ſides, is to the diameter, in a leſs ratio than that of $3\frac{1}{7}$ ($3\frac{10}{71}$) to 1, and that the inſcribed polygon of 96 ſides, is to the diameter, in a greater ratio than that of $3\frac{10}{71}$ to 1; and conſequently much more that the circumference of the circle is to the diameter, in a leſs ratio than that of $3\frac{1}{7}$ to 1, but greater than that of $3\frac{10}{71}$ to 1: The firſt ratio of $3\frac{1}{7}$ to 1, reduced to whole numbers, gives that of 22 to 7, for $3\frac{1}{7} : 1 :: 22 : 7$, which therefore is nearly the ratio of the circumference to the diameter. From this ratio of the

the circumference to the diameter, he computed the approximate *area* of the circle, and found that it is to the ſquare of the diameter, as 11 is to 14.—He likewiſe determined the relation between the circle and ellipſe, with that of their ſimilar parts. It is highly probable that he likewiſe attempted the hyperbola; but it is not to be imagined that he met with any ſucceſs, ſince approximations to its area are all that can be given by the various methods that have ſince been invented.

Beſides theſe figures, he hath left us a treatiſe on the ſpiral, deſcribed by a point moving uniformly along a right line, which at the ſame time moves with an uniform angular motion; and he determined the proportion of its area to that of the circumſcribed circle; as alſo the proportion of their ſectors.

Throughout the whole works of this great man, which are chiefly on menſuration, he every where diſcovers the deepeſt deſign, and the fineſt invention; and ſeems to have been, with Euclid, exceedingly careful of admitting into his demonſtrations nothing but principles perfectly geometrical and unexceptionable: and although his moſt general method of demonſtrating the relations of curved figures to ſtraight ones, be by inſcribing polygons in them; yet to determine thoſe relations, he does not increaſe the number, and diminiſh the magnitude, of the ſides of the polygon *in infinitum*; but from this plain fundamental principle, allowed in Euclid's Elements, viz. that any quantity may be ſo often multiplied, or added to itſelf, as that the reſult ſhould exceed any propoſed finite quantity of the ſame kind, he proves that to deny his figures to have the propoſed relations, would involve an abſurdity.

He demonſtrated alſo many properties, particularly in the parabola, by means of certain numeral progreſſions, whoſe terms are ſimilar to the inſcribed figures; but ſtill without conſidering ſuch ſeries as continued *in infinitum*, and then ſumming up the terms of ſuch infinite ſeries.

He had another very curious and ſingular contrivance for determining the meaſures of figures, in which he proceeds as it were mechanically, by weighing them, or from the properties of the center of gravity.

Several other eminent men among the ancients wrote upon this ſubject, both before and after Euclid and Archimedes; but their attempts were uſually confined to particular parts of it, and made according to methods not eſſentially different from theirs. Among theſe are to be reckoned Thales, Anaxagoras, Pythagoras, Bryſon, Antiphon, Hypocrates of Chios, Plato, Apollonius, Philo, and Ptolemy; moſt of whom wrote of the quadrature of the circle; and thoſe after Archimedes, by his method, uſually extended the approximation to a greater degree of accuracy.

Many of the moderns alſo have proſecuted the ſame problem of the quadrature of the circle, after the ſame methods, to greater lengths; ſuch are Vieta, and Metius, whoſe proportion between the diameter and circumference, is that of 113 to 355, which is within about $\frac{3}{10000000}$ of the true ratio; but above all, Ludolph van Collen, or a Ceulen, who, with an amazing degree of induſtry and patience, by the ſame methods, extended the ratio to 36 places of figures, making the ratio to be that of 1 to 3·14159265358979323846264338327950288 + or 9—. And the ſame was repeated and confirmed by his editor Snellius.

The firſt material deviation from the principles uſed by the ancients, in geometrical demonſtrations, was made by Cavalerius: the ſides of their inſcribed and circumſcribed figures, they always ſuppoſed to be of a finite and aſſignable number and length; he introduced the doctrine of indiviſibles, a method which was very general and extenſive, and which, with great eaſe and expedition, ſerved to meaſure and compare geometrical figures. Very little new matter, however, was added to geometry by this method, its facility being its chief advantage. But there was great danger in uſing it, and it ſoon led the way to infinitely ſmall elements, and infiniteſimals of endleſs orders; methods which were very uſeful in reſolving difficult problems, and in inveſtigating or demonſtrating theories that are general and extenſive; but ſometimes led their incautious followers into errors and miſtakes, which occaſioned diſputes and animoſities amongſt them. There were now, however, many excellent things performed in this ſcience; not only many new properties were diſcovered concerning the old figures, but new curves were meaſured; and although ſeveral of them could not be exactly ſquared or cubed, yet general and infinite approximating ſeries were aſſigned, of which the laws of their continuation were manifeſt, and in ſome of which the terms were independent of each other. Dr. Wallis, Mr. Huygens, and Mr. James Gregory, performed wonders: Huygens in particular muſt always be admired for his ſolid, accurate, and very maſterly works.

During the preceding ſtate of things, ſeveral men, whoſe vanity ſeemed to have overcome their regard for truth, aſſerted, that they had diſcovered the quadrature of the circle, and publiſhed their attempts in the form of ſtrict geometrical demonſtrations, with ſuch aſſurance and ambiguity, as ſtaggered and miſled many who could not ſo well judge for themſelves, and perceive the fallacy of their principles and arguments. Among thoſe were Longomontanus, and our countryman Hobbes, who obſtinately refuſed all conviction of his errors.

The uſe of infinites was, however, diſliked by ſeveral people, and particularly by Sir Iſaac Newton, who, among his numerous

rous and great diſcoveries, hath given us that of the method of fluxions; a diſcovery of the greateſt importance, both in philoſophy and mathematics; being a method ſo general and extenſive, as to include all inveſtigations concerning magnitude, diſtance, motion, velocity, time, &c. with wonderful eaſe and brevity; a method eſtabliſhed by its great author upon true and inconteſtable principles; principles perfectly conſiſtent with thoſe of the ancients, and which were free from the imperfections and abſurdities attending ſome that had lately been introduced by the moderns: he rejected no quantities as infinitely ſmall, nor ſuppoſed any parts of curves to coincide with right lines; but propoſed it in ſuch a form as admits of a ſtrict geometrical demonſtration. Upon the introduction of this method, moſt ſciences aſſumed a different appearance, and the moſt abſtruſe problems became eaſy and familiar to every one; things which before ſeemed to be inſuperable, became eaſy examples, or particular caſes, of theories ſtill more general and extenſive; rectifications, quadratures, cubatures, tangencies, caſes *de maximis & minimis*, and many other ſubjects, became general problems, and were delivered in the form of general theories, which included all particular caſes: thus, in quadratures, a formula was aſſigned, which would expreſs the areas of all poſſible curves whatever, both known and unknown, and which, by proper ſubſtitutions, gave the area for any particular caſe, either in finite terms, or in infinite ſeries, of which any term, or any number of terms, could be eaſily aſſigned; and the like in other things. And although no curve, whoſe quadrature was unſucceſsfully attempted by the ancients, became by this method perfectly quadrable, yet many general methods were diſcovered for approximating to their areas, of which in all probability the ancients had not the leaſt idea or hope; and innumerable curves were ſquared which were utterly unknown to them.

The excellency of this method revived ſome hopes of ſquaring the circle; and its quadrature was attempted with eagerneſs. The quadrature of a ſpace was now reduced to the finding of the fluent of a given fluxion; but this problem, however, was found to be incapable of a general ſolution in finite terms: the fluxion of every fluent was found to be always aſſignable, but the reverſe of this problem could be effected only in particular caſes: among the exceptions, to the great mortification of geometricians, was included the caſe of the circle, with regard to all the forms of fluxions attending it.

Another method of obtaining the area was tried: of the quantity expreſſing the fluxion of any area, in general, the fluent could always be aſſigned in the form of an infinite ſeries; which ſeries, therefore, defined all areas in general, and which, on ſubſtituting for particular caſes, was often found to break off and terminate, and to afford an area in finite terms: but here

again

again the caſe of the circle failed, its area being ſtill an infinite ſeries.

All hopes of the quadrature of the circle being now at an end, the geometricians employed themſelves in diſcovering and ſelecting the beſt forms of infinite ſeries for determining its area; among which it is evident, that thoſe were to be preferred which were ſimple, and would converge quickly; but it commonly happened that theſe two properties were divided, the ſame ſeries very rarely including them both. The mathematicians in moſt parts of Europe now applied themſelves diligently to theſe new diſcoveries, and many ſeries were aſſigned on all hands, ſome admired for their ſimplicity, and others for their rate of convergency; thoſe which converged the quickeſt, and were at the ſame time ſimpleſt, which therefore were moſt uſeful in computing the area of the circle in numbers, were thoſe in which, beſides the radius, the tangent of ſome certain arc of the circle, was the quantity by whoſe powers the ſeries converged; and from ſome of theſe ſeries, the area hath been computed to a great extent of figures. Dr. Edmund Halley gave a remarkable one, from the tangent of 30 degrees, by means of which the very induſtrious Mr. Abraham Sharp computed the area of the circle to 72 places of figures; but even this was afterwards far exceeded by Mr. John Machin, who, by means hereafter deſcribed in this book, compoſed a ſeries ſo ſimple, and which converged ſo quickly, that by it, in a very little time, he extended the quadrature of the circle to 100 places of figures; from which it appears, that if the diameter be 1, the circumference will be

3,141592653589793238462643383279,502884197169399375105820974944,592307816406286208998628034825,342117067982,

and conſequently the area will be

,785398163397448309615660845819,875721049292348437764552437364,148076954101571552249657008706,335529266982.

And I have lately given, in the Philoſophical Tranſactions, various other ſeries for the ſame purpoſe, which are ſtill ſimpler in their form, and converge more readily than thoſe above mentioned.

Whilſt I have been giving the preceding account of the progreſs of this ſubject, I have at the ſame time unawares been writing its panegyric; for, from hence it appears, that moſt of the material improvements or inventions in the ſcience of geometry, have been principally made for the improvement of menſuration; which ſufficiently ſhews the dignity of this ſubject; a ſubject which, as Dr. Barrow ſays, "deſerves to be more curiouſly weighed, becauſe from hence a name is impoſed upon that mother and miſtreſs of the reſt of the mathematical ſciences, which is employed about magnitudes, and which is

went to be called geometry (a word taken from ancient uſe, becauſe it was firſt applied only to meaſuring the earth, and fixing the limits of poſſeſſions), though the name ſeemed very ridiculous to Plato, who ſubſtitutes in its place that more extenſive name of Metrics or Menſuration; and others after him gave it the title of Pantometry, becauſe it teaches the method of meaſuring all kinds of magnitudes."

But notwithſtanding the dignity and importance of this ſubject, the books which have lately been offered to the public under the title of menſuration, have treated it in a very unworthy manner, and have brought it into much contempt. It is intended in this work, therefore, to place the ſubject in a more favourable light, and in ſome meaſure to endeavour to retrieve its reputation.

The work conſiſts of five principal parts, each part being divided into ſeveral ſections, every ſection containing ſeveral problems or propoſitions, which are all demonſtrated after the ſimpleſt and ſhorteſt methods that could be deviſed; the proceſs being ſometimes by pure geometry, and ſometimes by algebra, the method of fluxions, the method of increments, &c, according as the one or the other appeared beſt ſuited to the purpoſe. And thus, by uſing various modes of demonſtration, I was enabled to make this work much more complete than it could otherwiſe have been; not only by employing that particular method which would perform the buſineſs in the ſhorteſt or cleareſt manner, but which alſo would render the ſubject ſtill more extenſive. However, I had not *always* my choice of the method of demonſtration, the ſubject ſometimes requiring one mode, and ſometimes another; for although the method of fluxions be generally the moſt conciſe, and, in other reſpects, the moſt proper for inveſtigating the meaſure of extenſions, yet there are ſome things in this buſineſs which are too ſimple for it, as well as ſome others which riſe above its reach: thus, the method of fluxions cannot determine the meaſure of a rectangle, that determination being below its root, being prior to the fluxionary method of determining areas; for in this mode, that determination is ſuppoſed, and its reſult aſſumed in the very notation; for the rectangle which is conſidered as the fluxion of a ſurface, is denoted by its length drawn into its breadth: and on this account thoſe fluxioniſts, who, in their chapter of quadratures, attempt to determine the meaſure of a rectangle, argue in a circle, or deduce as a concluſion what was neceſſarily ſuppoſed in the premiſes: the ſame may be ſaid with regard to priſms. And, on the other hand, that method alone would not extend to the determination of the equality of a triangle and hyperbola of the ſame baſe, and of an infinite length, it being there neceſſary to call in the aſſiſtance of the method of increments.

As

As extenſions are of three kinds, longitudinal, ſuperficial, and ſolid, ſo in this work the ſcience is treated of as diſtinguiſhed by nature into theſe three principal parts; that is, ſo far as the nature of geometrical figures would conveniently admit. With regard to right lines, plane ſurfaces, and ſolids, the diſtinction is general; but it does not obtain with regard to curved lines and ſurfaces. For, by preſerving this diſtinction ſo entire, as to determine the meaſure of all kinds of lines in the firſt part, and of all kinds of ſurfaces in the ſecond, I immediately perceived that the book could not be ſo conveniently adapted to the proper ſtudy of the generality of readers: on which account, in the firſt part, I have treated only of the meaſure of right lines; and in the ſecond part, only of the meaſure of planes which are bounded by right and circular lines, without any curve ſurfaces, excepting that of the ſphere. In the third part, which treats of ſolids, I have generally placed the problems which relate to the meaſures of the lines, ſurfaces, and ſolidities of each particular figure, immediately after each other, becauſe the knowledge of the one commonly led to *that* of the others. The latter parts of the book are employed chiefly about the applications of the general problems to ſeveral intereſting practical ſubjects in life.

So much for the diſtribution and order of the parts in general. I ſhall now proceed to deſcribe the contents of the parts themſelves more particularly.

The whole work conſiſts of five parts.

PART I. Contains the menſuration of right lines and right-angled angles, and is divided into three ſections.

*Sect.* 1. Contains ſeveral geometrical definitions and problems; ſome of which are new, and, it is preſumed, they are all more complete, and leſs exceptionable, than thoſe inſtead of which they have been ſubſtituted. The problems will prepare the learner for making the ſeveral figures which are afterwards treated of.

*Sect.* 2. Contains plane Trigonometry, or the meaſuring of lines and angles. All the caſes of trigonometry, both right and oblique angled, are here reduced to three only; by which means they are eaſier to be remembered, and more clearly underſtood. Beſides theſe caſes, which perform the buſineſs in the common way, by means of the ſines, tangents, and ſecants of angles, I have given a new and extenſive method, by which all the caſes of trigonometry are performed independent of ſines, tangents, and ſecants, and without any kind of tables.

*Sect.* 3. Contains the application of trigonometry to the determination of heights and diſtances; in which a great variety of caſes and methods concerning this curious ſubject are explained.

PART II. Treats of ſuperficial menſuration, or the menſuration of plane figures, and is divided into two ſections.

*Sect.*

*Sect.* 1. Treats of the areas, &c. of right-lined and circular figures; in which, beſides many things that are new and curious, are given an explanation of Profeſſor Machin's celebrated quadrature of the circle, and the demonſtrations of ſome uſeful approximations to the meaſures of circular arcs and areas, which had been given by Mr. Huygens and Sir Iſaac Newton, without demonſtrations.

*Sect.* 2. Contains a curious and uſeful collection of queſtions concerning areas, promiſcuouſly placed, and reſolved by the rules in the former ſections.

PART III. Contains the meaſuring of ſolids, and is divided into 8 ſections.

*Sect.* 1. Treats of bodies that are bounded by right or circular lines, viz. priſms, pyramids, the ſphere, and the circular ſpindle.

*Sect.* 2. Treats of the five regular ſolids or bodies.

*Sect.* 3. Treats of ſolid rings.

*Sect.* 4. Treats of the conic ſections in general; and though it be ſhort, it contains ſeveral things that are new and of great importance.

*Sect.* 5. Treats of the ellipſe and the figures generated by it, viz. ſpheroids and elliptic ſpindles.

*Sect.* 6. In like manner treats of parabolic lines, areas, ſurfaces, and ſolidities. And

*Sect.* 7. Of hyperbolic lines, areas, ſurfaces, and ſolidities.

In theſe ſections the ſeveral figures and bodies are very extenſively and particularly handled, many of the rules, &c. both here and throughout the whole book, being new and intereſting; and I have given throughout many neat approximations to the values of ſeveral things which cannot be truly expreſſed otherwiſe than by an infinite ſeries; which approximations are moſtly new, excepting two or three that were given by Sir I. Newton, and which I have demonſtrated here for the firſt time.

*Sect.* 8. Or the laſt of this part, contains a promiſcuous collection of queſtions concerning ſolids, to exerciſe the learner in the foregoing rules.

PART IV. Contains, in 3 ſections, ſeveral ſubjects relating to menſuration in general.

*Sect.* 1. Contains a treatiſe on the true quadrature and cubature of curves in general. In which are contained ſome of the moſt univerſal and important propoſitions that can be made in the ſubject.

*Sect.* 2. Contains the equidiſtant-ordinate method; or, the approximate quadrature and cubature of curves in general, by means of equidiſtant ordinates or ſections. A ſubject by which general and finite rules are diſcovered for all figures; for ſome of which they are accurately true, and for the others they are very near approximations: which are often the moſt uſeful rules that can be applied to many things in real practice.

*Sect.*

*Sect.* 3. Contains, in a very conciſe but copious treatiſe, the relations between the areas and ſolidities of figures, and the centers of gravity of their generating lines and planes.

Then the

Fifth and laſt PART, in four ſections, contains the application of the general rules to the moſt uſeful ſubjects of meaſuring that happen in ordinary life. In theſe ſubjects very material improvements are almoſt every where made, both with reſpect to the matters and the diſpoſition of them.

*Sect.* 1. Contains a very ſimple treatiſe of land ſurveying; explaining the uſe of the inſtruments, the methods of ſurveying, of planning, of computing the contents, of reducing plans, and of dividing the ground.

*Sect.* 2. Contains a very curious and complete treatiſe on gauging. As in like manner doth

*Sect.* 3. On the meaſuring of artificers works; viz, Bricklayers, Maſons, Carpenters and Joiners, Slaters, and Tilers, Plaſterers, Painters, Glaziers, Pavers, and Plumbers. Containing the deſcription of the carpenter's rule, the ſeveral meaſures uſed by each, with the methods of taking the dimenſions, and of ſquaring and ſumming them up. The whole illuſtrated by a real caſe of a building, in which are ſhewn the methods of entering the dimenſions and contents in the pocket book, of drawing out the abſtracts, and from them drawing out the forms of the bills.

*Sect.* 4. Contains a curious treatiſe on timber meaſuring; in which, among ſeveral other things, is given a new rule for meaſuring round timber, which not only gives the content very exact, but it is at the ſame time as eaſy in the operation as the common falſe one, either by the pen or the ſliding rule. It contains alſo ſome curious rules for cutting timber to the moſt advantage.

The book then concludes with a large table of the areas of circular ſegments, extended to ten times the uſual length.

It may be neceſſary to remark that, in this book, where a curve or a ſpace is ſaid to be non-quadrable, or it is ſaid that the value of a thing cannot be expreſſed except by an infinite ſeries, or any ſuch-like expreſſion is uſed; the meaning is, that it is not geometrically quadrable, or that its area or value cannot be expreſſed in a finite number of terms, by any method yet known, or by the method there uſed; but not that it is a thing naturally impoſſible in itſelf. For although a ſpace be not quadrable, by the methods yet known, it does not therefore follow that its quadrature is an impoſſible thing, or that ſome method may not hereafter be diſcovered by which it may be ſquared. All the methods uſed by the geometricians before Archimedes, were inſufficient for the quadrature of any curve ſpace whatever; but were they therefore to infer that no curve could by any means be ſquared? Archimedes diſcovered a method by which he ſquared the parabola; and by the lately-diſcovered method of fluxions, we can find

find as many quadrable curves as we pleaſe. It is true we have not yet found the area of the circle, and ſeveral other figures, in finite terms; yet for each of theſe we can aſſign infinite ſeries whoſe laws of progreſſion are viſible; which is more than the ancients could do, or perhaps ever expected could be done, if they even at all thought of ſuch things. And, perhaps, hereafter a method may be diſcovered of ſquaring any figure whatever. Which is the chief problem in geometry.

In this edition have been made many large and uſeful additions, in almoſt every ſection of the work; and it is preſumed that the whole is arranged in a more regular and perfect order than before.

*Royal Military Academy,*
*Jan.* 24, 1788.

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# A TREATISE ON MENSURATION.

## PART I.

### OF LINEAL AND ANGULAR MENSURATION, OR THE MENSURATION OF LINES AND ANGLES.

### SECTION I.

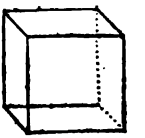
#### GEOMETRICAL DEFINITIONS AND PROBLEMS.

1. A POINT has no parts nor dimenſions, neither length, breadth, nor thickneſs.

2. A line is length, without breadth or thickneſs.

3. A ſurface, or ſuperficies, is an extenſion, or a figure, of two dimenſions, length and breadth; but without thickneſs.

4. A body or ſolid, is a figure of three dimenſions, namely, length, breadth, and thickneſs.



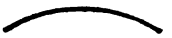
Hence ſurfaces are the extremities of ſolids; lines the extremities of ſurfaces; and points the extremities of lines.

5. Lines are either right, or curved, or mixed of theſe two.

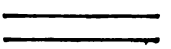
6. A right line, or ſtraight line, lies all in the ſame direction, between its extremities; and is the ſhorteſt diſtance between two points.



7. A curve continually changes its direction, between its extreme points.

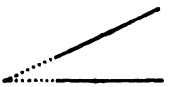


8. Lines are either parallel, oblique, perpendicular, or tangential.

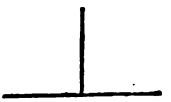


9. Parallel lines are always at the ſame diſtance; and never meet, though ever ſo far produced.

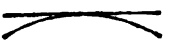
10. Oblique right lines change their diſtance, and would meet, if produced, on the ſide of the leaſt diſtance.



11. One line is perpendicular to another, when it inclines not more on the one ſide than on the other.



12. One line is tangential, or a tangent to another, when it touches it, without cutting, when both are produced.



13. An angle is the inclination, or opening, of two lines, having different directions, and meeting in a point.

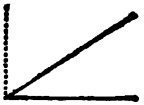
14. Angles are right or oblique, acute or obtuſe.

15. A right angle, is that which is made by one line perpendicular to another. Or when the angles on each ſide are equal to one another, they are right angles.

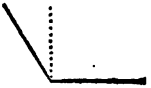
16. An

16 An oblique angle is that which is made by two oblique lines; and is either leſs or greater than a right angle.

17. An acute angle is leſs than a right angle.



18 An obtuſe angle is greater than a right angle.



19. Superficies are either plane or curved.

20. A plane, or plane ſuperficies, is that with which a right line may, every way, coincide.—But if not, it is curved.

21. Plane figures are bounded either by right lines or curves.

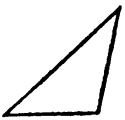
22. Plane figures bounded by right lines, have names according to the number of their ſides, or of their angles; for they have as many ſides as angles; the leaſt number being three.

23. A figure of three ſides and angles, is called a triangle. And it receives particular denominations from the relations of its ſides and angles.

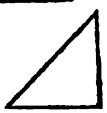
24. An equilateral triangle, is that whoſe three ſides are all equal.

25. An iſoſceles triangle, is that which has two ſides equal.

26. A ſcalene triangle, is that whoſe three ſides are all unequal.



27. A right-angled triangle, is that which has one right angle.



28. Other triangles are oblique-angled, and are either obtuſe or acute.

29. An obtuſe-angled triangle has one obtuſe angle.

30. An acute-angled triangle has all its three angles acute.

31. A figure of four ſides and angles, is called a quadrangle, or a quadrilateral.

32. A parallelogram is a quadrilateral which has both its pairs of oppoſite ſides parallel. And it takes the following particular names.

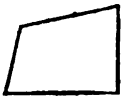
33. A rectangle is a parallelogram, having all its angles right ones.

34. A ſquare is an equilateral rectangle; having all its ſides equal, and all its angles right ones.

35. A rhomboid is an oblique-angled parallelogram.

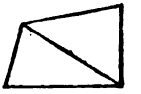
36. A rhombus is an equilateral rhomboid; having all its ſides equal, but its angles oblique.

37. A trapezium is a quadrilateral which has not both its pairs of oppoſite ſides parallel.



38. A trapezoid has only one pair of oppoſite ſides parallel.

39. A diagonal is a right line joining any two oppoſite angles of a quadrilateral.



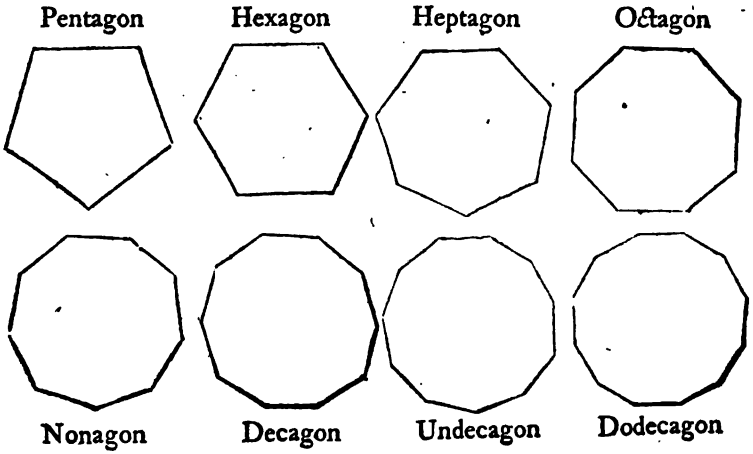
40. Plate

40. Plane figures having more than four ſides are, in general, called polygons: and they receive other particular names according to the number of their ſides or angles.

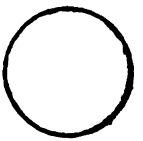
41. A pentagon is a polygon of five ſides; a hexagon has ſix ſides; a heptagon, ſeven; an octagon, eight; a nonagon, nine; a decagon, ten; an undecagon, eleven; and a dodecagon has twelve ſides.

42. A regular polygon has all its ſides and all its angles equal.—If they are not both equal, the polygon is irregular.

43. An equilateral triangle is alſo a regular figure of three ſides, and the ſquare is one of four: the former being alſo called a trigon, and the latter a tetragon.

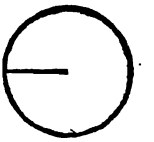



44. A circle is a plane figure bounded by a curve line, called the circumference, which is every where equi-diſtant from a certain point within, called its center.

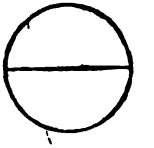


*N. B.* The circumference itſelf is often called a circle.

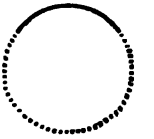
45. The radius of a circle, is a right line drawn from the center to the circumference.



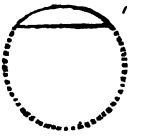
46. The diameter of a circle, is a right line drawn through the center, and terminating in the circumference on both ſides.



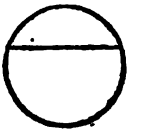
47. An arc of a circle, is any part of the circumference.



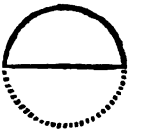
48. A chord is a right line joining the extremities of an arc.



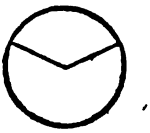
49. A ſegment is any part of a circle bounded by an arc and its chord.



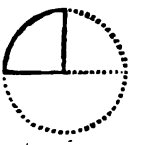
50. A ſemicircle is half the circle, or a ſegment cut off by a diameter.



51. A ſector is any part of a circle, bounded by an arc, and two radii, drawn to its extremities.

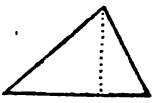


52. A quadrant, or quarter of a circle, is a ſector having a quarter of the circumference for its arc, and its two radii are perpendicular to each other.



53. The

53. The height or altitude of a figure, is a perpendicular let fall from an angle, or its vertex, to the oppoſite ſide, called the baſe.

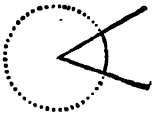


54. In a right-angled triangle, the ſide oppoſite the right angle, is called the hypotenuſe; and the other two ſides the legs, or ſometimes the baſe and perpendicular.

55. When an angle is denoted by three letters, of which one ſtands at the angular point, and the other two on the two ſides, that which ſtands at the angular point is read in the middle.

56. The circumference of every circle is ſuppoſed to be divided into 360 equal parts, called degrees; and each degree into 60 minutes, each minute into 60 ſeconds, and ſo on. Hence a ſemicircle contains 180 degrees, and a quadrant 90 degrees.

57. The meaſure of a right-lined angle, is an arc of any circle contained between the two lines which form that angle, the angular point being the center; and it is eſtimated by the number of degrees contained in that arc. Hence a right-angle is an angle of 90 degrees.

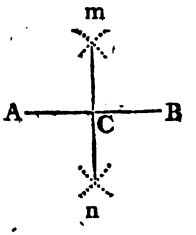


The definition of ſolids, or bodies, will be given afterwards, when we come to treat of the menſuration of ſolids.

# PROBLEMS.

## PROBLEM I.

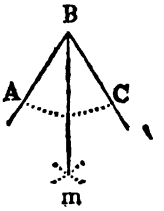
*To divide a given Line* AB *into two Equal Parts.*




From the centers A and B, with any radius greater than half AB, deſcribe arcs cutting each other in m and n. Draw the line m c n, and it will cut the given line into two equal parts in the middle point c.

## PROBLEM II.

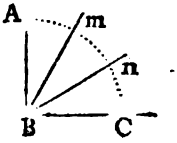
*To divide a given Angle* ABC *into two Equal Parts.*




From the center B, with any radius, deſcribe the arc AC. From A and c, with one and the ſame radius, deſcribe arcs interſecting in m. Draw the line B m, and it will biſect the angle as required.

## PROBLEM III.

*To divide a Right Angle* ABC *into three Equal Parts.*




From the center B, with any radius, deſcribe the arc AC. From the center A, with the ſame radius, croſs the arc AC in n. And with the center c, and the ſame radius, cut the arc AC in m. Then through the points m and n draw B m and B n, and they will triſect the angle as required.

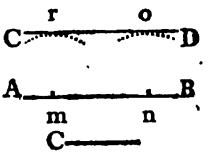
PRO-

PROBLEM IV.

*To draw a Line Parallel to a given Line* AB.

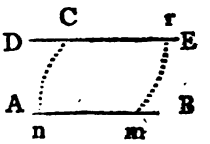
CASE I. *When the Parallel Line is to be at a given Diſtance* c.

From any two points m and n, in the line AB, with a radius equal to c, deſcribe the arcs r and o:—Draw CD to touch theſe arcs, without cutting them, and it will be the parallel required.




CASE 2. *When the Parallel Line is to paſs through a given Point* c.

From any point m, in the line AB, with the radius mc, deſcribe the arc cn.—From the center c with the ſame radius, deſcribe the arc mr.—Take the arc cn in the compaſſes, and apply it from m to r.—Through c and r draw DE, the parallel required.



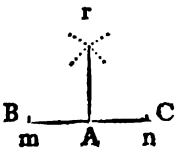

*N. B.* This problem is more eaſily effected with a parallel ruler.

PROBLEM V.

*To erect a Perpendicular from a given Point* A *in a given Line* BC.

CASE I. *When the Point is near the Middle of the Line.*

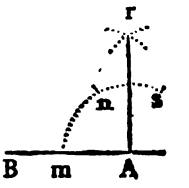
On each ſide of the point A take any two equal diſtances Am, An. From the centers m, n, with any radius greater than Am or An, deſcribe two arcs interſecting in r.—Through A and r draw the line Ar, and it will be the perpendicular as required.




CASE

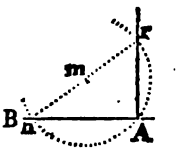
CASE 2. *When the Point is too near the End of the Line.*

With the center c, and any radius, deſcribe the arc mns.—From the point m, with the ſame radius, turn the compaſſes twice over on the arc at n and s.—Again, with the centers n and s, deſcribe arcs interſecting in r. Then draw Ar, and it will be perpendicular as required.



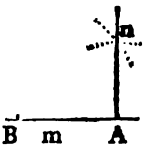

*Another Method.*

From any point m as a center, with the radius or diſtance mA, deſcribe an arc cutting the given line in n and A.—Through n and m draw a right line cutting the arc in r.—Laſtly, draw Ar, and it will be the perpendicular as required.




*Another Method.*

From any plane ſcale of equal parts ſet off Am equal to 4 parts.—With center A, and radius of 3 parts, deſcribe an arc.—And with center m, and radius of 5 parts, croſs it at n.—Draw An for the perpendicular required.




Or any other numbers in the ſame proportion as 3, 4, 5, will do the ſame.

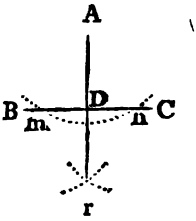
PRO-

## PROBLEM VI.

*From a given Point* A, *out of a given Line* BC, *to let fall a Perpendicular.*

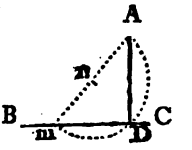
CASE 1. *When the Point is nearly oppoſite the Middle of the Line.*

With the center A, and any radius, deſcribe an arc cutting BC in m and n.—With the centers m and n, and the ſame, or any other radius, deſcribe arcs interſecting in r.—Draw ADr, for the perpendicular required.



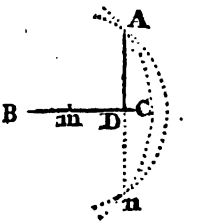

CASE 2. *When the Point is nearly oppoſite the End of the Line.*

From A draw any line Am to meet BC, in any point m.—Biſect Am at n, and with the center n, and radius An or mn, deſcribe an arc, cutting BC in D.—Draw AD the perpendicular as required.




*Another Method.*

From B or any point in BC, as a center, deſcribe an arc through the point A.—From any other center m in BC, deſcribe another arc through A, and cutting the former arc again in n.—Through A and n draw the line ADn; and AD will be the perpendicular as required.



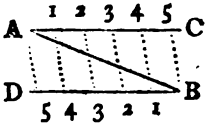

*N. B.*

*N. B.* Perpendiculars may be more readily raiſed and let fall, in practice, by means of a ſquare, or other fit inſtrument.

## PROBLEM VII.

*To divide a given Line* AB *into any propoſed Number of Equal Parts.*

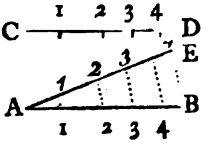
From A draw any line AC at random, and from B draw BD parallel to it.—On each of theſe lines, beginning at A and B, ſet off as many equal parts, of any length, as AB is to be divided into.—Join the oppoſite points of diviſion by the lines A 5, 1 4, 2 3, &c; and they will divide AB as required.




## PROBLEM VIII.

*To divide a given Line* AB *in the ſame Proportion as another Line* CD *is divided.*

From A draw any line AE equal to CD, and upon it transfer the diviſions of the line CD. —Join BE, and parallel to it draw the lines 1 1, 2 2, 3 3, &c; and they will divide AB as required.

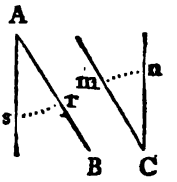



PRO-

### PROBLEM IX.

*At a given Point* A, *in a given Line* AB, *to make an Angle equal to a given Angle* C.

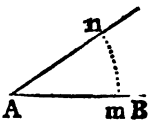
With the center C, and any radius, deſcribe an arc m n.—With the center A, and the ſame radius, deſcribe the arc r s.—Take the diſtance m n between the compaſſes, and apply it from r to s.—Then a line drawn through A and s, will make the angle A equal to the angle C as required.




### PROBLEM X.

*At a given Point* A, *in a given Line* AB, *to make an Angle of any propoſed Number of Degrees.*

With the center A, and radius equal to 60 degrees taken from a ſcale of chords, deſcribe an arc, cutting AB in m.—Then take between the compaſſes, the propoſed number of degrees from the ſame ſcale of chords, and apply them from m to n.—Through the point n draw An, and it will make the angle A of the number of degrees propoſed.



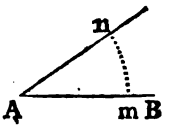

*Note.* Angles of more than 90 degrees are uſually taken off at twice.

Or the angle may be made with any divided arch or inſtrument, by laying the center to the point A, and its radius along AB; then make a mark n at the propoſed number of degrees, through which draw the line An as before.

### PROBLEM XI.

*To meaſure a given Angle* A.

Deſcribe the arc m n with the chord of 60 degrees, as in the laſt Problem. Take the arc m n between the compaſſes, and that extent, applied to the chords, will ſhew the degrees in the given angle.



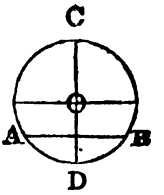

*Note.* When the diſtance m n exceeds 90 degrees, it muſt be taken off at twice, as before.

Or the angle may be meaſured by applying the radius of a graduated arc, of any inſtrument, to AB, as in the laſt problem; and then noting the degrees cut off by the other leg AN of the angle.

### PROBLEM XII.

*To find the Center of a Circle.*

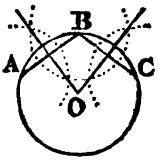
Draw any chord AB; and biſect it perpendicularly with CD, which will be a diameter.—Biſect CD in the point O, which will be the center.




### PROBLEM XIII.

*To deſcribe the Circumference of a Circle, through three given Points,* A, B, C.

From the middle point B draw chords to the other two points.—Biſect theſe chords perpendicularly by lines meeting in O, which will be the center.—Then from the center O, at the diſtance OA, or OB, or OC, deſcribe the circle.




*Note.*

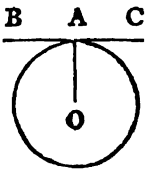
*Note.* In the same manner may the center of an arc of a circle be found.

### PROBLEM XIV.

*Through a given Point* A, *to draw a Tangent to a given Circle.*

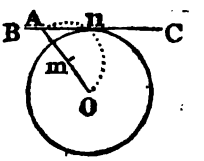
CASE 1. *When* A *is in the Circumference of the Circle.*

From the given point A, draw AO to the center of the circle.—Then through A draw BC perpendicular to AO, and it will be the tangent as required.




CASE 2. *When* A *is out of the Circumference.*

From the given point A, draw AO to the center, which bisect in the point m.—With the center m, and radius mA or mO, describe an arc cutting the given circle in n.—Through the points A and n, draw the tangent BC.

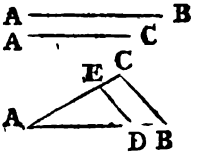



### PROBLEM XV.

*To find a Third Proportional to two given Lines* AB, AC.

Place the two given lines, AB, AC, making any angle at A, and join BC.—In AB take AD equal to AC, and draw DE parallel to BC. So shall AE be the third proportional to AB and AC.

That is, AB : AC :: AC : AE.

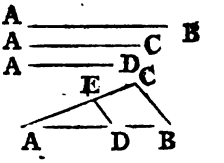


### PROBLEM XVI.

*To find a Fourth Proportional to three given Lines,* AB, AC, AD.

Place two of them, AB, AC, making any angle at A, and join BC. Place AD on AB, and draw DE parallel to BC. So ſhall AE be the fourth proportional required.

That is AB : AC :: AD : AE.

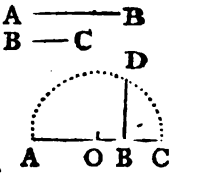



### PROBLEM XVII.

*To find a Mean Proportional between two given Lines,* AB, BC.

Join AB and BC in one ſtraight line AC, and biſect it in the point O.—With the center O, and radius OA or OC, deſcribe a ſemicircle.—Erect the perpendicular BD, and it will be the mean proportional required.

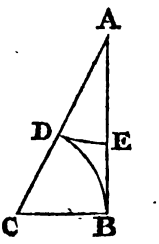
That is AB : BD :: BD : BC.




### PROBLEM XVIII.

*To divide a Line* AB *in Extreme-and-Mean Ratio.*

Raiſe BC perpendicular to AB, and equal to half AB. Join AC. With center C, and radius CB, croſs AC in D. Laſtly, with center A, and radius AD, croſs AB in E, which will divide the line AB in extreme and mean ratio, namely, ſo that the whole line is to the greater part, as the greater part is to the leſs part. That is, AB : AE :: AE : EB.

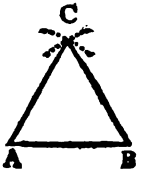



PRO-

### PROBLEM XIX.

*To make an Equilateral Triangle on a given Line* AB.

From the centers A and B, with the radius AB, deſcribe arcs, interſecting in C.—Draw AC and BC, and it is done.

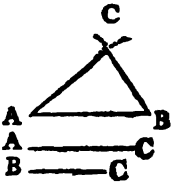



*Note.* An iſoſceles triangle may be made in the ſame manner, taking for the radius the given length of one of the equal ſides.

### PROBLEM XX.

*To make a Triangle with three given Lines* AB, AC, BC.

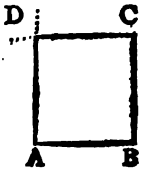
With the center A and radius AC, deſcribe an arc.—With the center B, and radius BC, deſcribe another arc, cutting the former in C.—Draw AC and BC, and ABC is the triangle required.




### PROBLEM XXI.

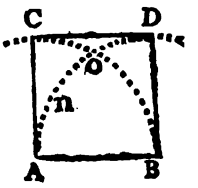
*To make a Square upon a given Line* AB.

Draw BC perpendicular and equal to AB. From A and C with the radius AB, deſcribe arcs interſecting in D.—Draw AD and CD, and it is done.




*Another Way.*

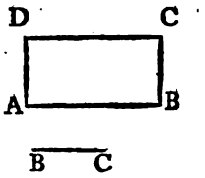
On the centers A and B, with the radius AB, deſcribe arcs croſſing at O.—Biſect AO in n.—With center O, and radius On, croſs the two arcs in C and D.—Then draw AC, BD, CD.



## PROBLEM XXII.

*To deſcribe a Rectangle, or a Parallelogram, of a given Length and Breadth.*

Place BC perpendicular to AB.—With center A, and radius AC, deſcribe an arc.—With center C, and radius AB, deſcribe another arc, cutting the former in D.—Draw AD and CD, and it is done.

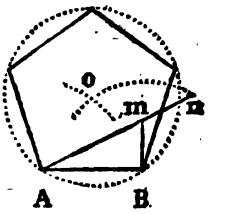



*Note.* In the ſame manner is deſcribed any oblique parallelogram, only drawing BC, making the given oblique angle with AB, inſtead of perpendicular to it.

## PROBLEM XXIII.

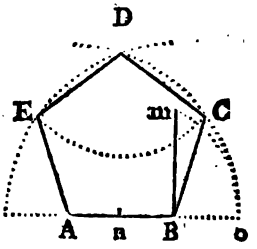
*To make a regular Pentagon on a given Line* AB.

Make Bm perpendicular and equal to half AB.—Draw Am, and produce it till mn be equal to Bm.—With centers A and B, and radius Bn, deſcribe arcs interſecting in o, which will be the center of the circumſcribing circle.—Then with the center o, and the ſame radius, deſcribe the circle; and about the circumference of it apply AB the proper number of times.



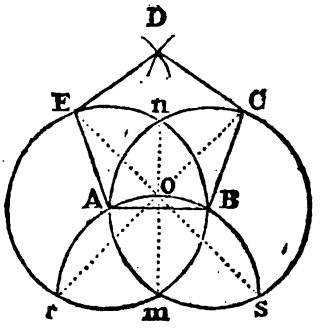

*Another*

*Another Method.*




Make Bm perpendicular and equal to AB.—Biſect AB in n; then with the center n, and radius nm, croſs AB produced in o.—With the centers A and B, and radius Ao, deſcribe arcs interſecting in D, the oppoſite angle of the pentagon. Laſtly, with center D, and radius AB, croſs thoſe arcs again in c and E, the other two angles of the figure. Then draw the lines from angle to angle, to complete the figure.

*A third Method, nearly true.*

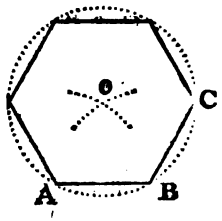



On the centers A and B, with the radius AB, deſcribe two circles, interſecting in m and n.—With the ſame radius, and the center m, deſcribe rAoBs, and draw mn cutting it in o.—Draw roc and soB, which will give two angles of the pentagon.—Laſtly, with radius AB, and centers c and E, deſcribe arcs interſecting in D, the other angle of the pentagon nearly.

### PROBLEM XXIV.

*To make a Hexagon on a given Line* AB.

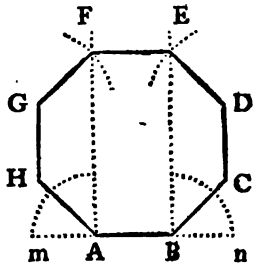
With the radius AB, and the centers A and B, deſcribe arcs interſecting in o.—With the ſame radius, and center o, deſcribe a circle, which will circumſcribe the hexagon.—Then apply the line AB ſix times round the circumference, marking out the angular points, which connect with right lines.




### PROBLEM XXV.

*To make an Octagon on a given Line* AB.

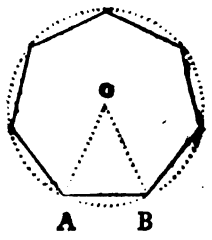
Erect AF and BE perpendicular to AB.—Produce AB both ways, and biſect the angles mAF and nBE with the lines AH and BC, each equal to AB.—Draw CD and HG parallel to AF or BE, and each equal to AB.—With radius AB, and centers G and D, croſs AF and BE in F and E.—Then join GF, FE, ED, and it is done.




### PROBLEM XXVI.

*To make any regular Polygon on a given Line* AB.

Draw AO and BO making the angles A and B each equal to half the angle of the polygon.—With the center o and radius OA, deſcribe a circle.—Then apply the line AB continually round the circumference the proper number of times, and it is done.



*Note.* The angle of any polygon, of which the angles OAB, and OBA are each one half, is found thus: divide the whole 360 degrees by the number of sides, and the quotient will be the angle at the center O; then subtract that from 180 degrees, and the remainder will be the angle of the polygon, and is double of OAB or of OBA. And thus you will find the following table, containing the degrees in the angle O at the center, and the angle of the polygon, for all the regular figures from 3 to 12 sides.

| No. of sides. | Name of the polygon. | Angle O at the center | Angle of the polyg | Angle OAB or OBA |
|---|---|---|---|---|
| 3 | Trigon | 120° | 60° | 30° |
| 4 | Tetragon | 90 | 90 | 45 |
| 5 | Pentagon | 72 | 108 | 54 |
| 6 | Hexagon | 60 | 120 | 60 |
| 7 | Heptagon | $51\frac{3}{7}$ | $128\frac{4}{7}$ | $64\frac{2}{7}$ |
| 8 | Octagon | 45 | 135 | $67\frac{1}{2}$ |
| 9 | Nonagon | 40 | 140 | 70 |
| 10 | Decagon | 36 | 144 | 72 |
| 11 | Undecagon | $32\frac{8}{11}$ | $147\frac{3}{11}$ | $73\frac{7}{11}$ |
| 12 | Dodecagon | 30 | 150 | 75 |

PROBLEM XXVII.

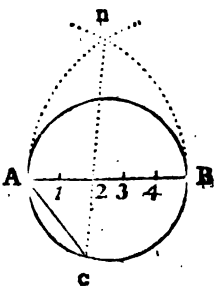
*In a given Circle to inscribe any regular Polygon; or, to divide the Circumference into any number of equal Parts.*

*(See the last figure.)*

At the center O make an angle equal to the angle at the center of the polygon, as contained in the third column of the above table of polygons.—Then the distance AB will be one side of the polygon; which being carried round the circumference the proper number of times, will complete the figure.—Or, the arc AB will be one of the equal parts of the circumference.

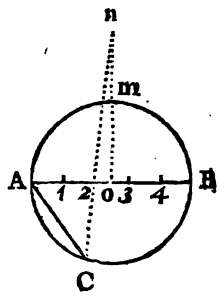
*Another Method, nearly true.*

Draw the diameter AB, which divide into as many equal parts as the figure has ſides.—With the radius AB, and centers A and B, deſcribe arcs croſſing at n; from whence draw n c through the ſecond diviſion on the diameter; ſo ſhall AC be a ſide of the polygon nearly.




*Another Method, ſtill nearer.*

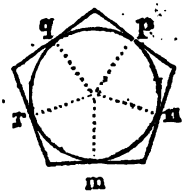
Divide the diameter AB into as many equal parts as the figure has ſides, as before.—From the center o raiſe the perpendicular o m, which produce till m n be three-fourths of the radius o m.—From n draw n c through the ſecond diviſion of the diameter, and the line AC will be the ſide of the polygon ſtill nearer than before; or the arc AC one of the equal parts into which the circumference is to be divided.




PROBLEM XXVIII.

*About a given Circle to circumſcribe any Polygon.*

Find the points m, n, p, &c, as in the laſt problem, to which draw radii m o, n o, &c, to the center of the circle.—Then through theſe points, m, n, &c, and perpendicular to theſe radii, draw the ſides of the polygon.

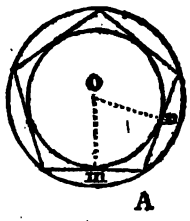



PRO-

### PROBLEM XXIX.

*To find the Center of a given Polygon, or the Center of its inſcribed or circumſcribed Circle.*

Biſect any two ſides with the perpendiculars mo, no; and their interſection will be the center.—Then with the center o, and the diſtance om, deſcribe the inſcribed circle; or with the diſtance to one of the angles, as A, deſcribe the circumſcribing circle.

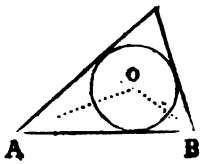



*Note.* This method will alſo circumſcribe a circle about any given oblique triangle.

### PROBLEM XXX.

*In any given Triangle to inſcribe a Circle.*

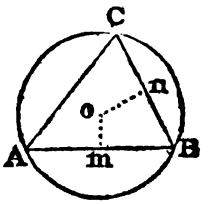
Biſect any two of the angles with the lines AO, BO, and O will be the center of the circle.—Then with the center O, and radius the neareſt diſtance to any one of the ſides, deſcribe the circle.




### PROBLEM XXXI.

*About any given Triangle to circumſcribe a Circle.*

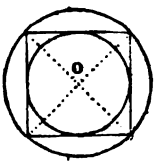
Biſect any two of the ſides AB, BC, with the perpendiculars mo, no.—With the center c, and diſtance to any one of the angles, deſcribe the circle.



## PROBLEM XXXII.

*In, or about, a given Square, to deſcribe a Circle.*

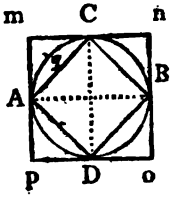
Draw the two diagonals of the ſquare, and their interſection o will be the center of both the circles.—Then with that center, and the neareſt diſtance to one ſide, deſcribe the inner circle; and with the diſtance to one angle, deſcribe the outer circle.




## PROBLEM XXXIII.

*In, or about, a given Circle, to deſcribe a Square, or an Octagon.*

Draw two diameters AB, CD, perpendicular to each other.—Then connect their extremities, and they will give the inſcribed ſquare ACBD.—Alſo through their extremities draw tangents parallel to them, and they will form the outer ſquare mnop.

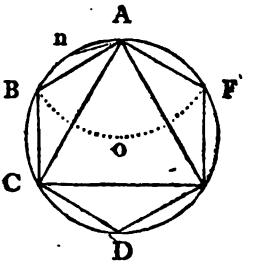



*Note.* If any quadrant, as AC, be biſected in q, it will give one-eighth of the circumference, or the ſide of the octagon.

PRO-

## PROBLEM XXXIV.

*In a given Circle, to inſcribe a Trigon, a Hexagon, or a Dodecagon.*

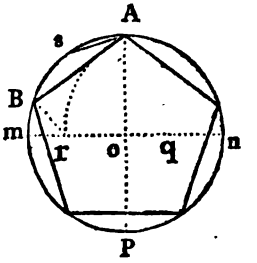



The radius is the ſide of the hexagon. Therefore from any point A in the circumference, with the diſtance of the radius, deſcribe the arc BOF. Then is AB the ſide of the hexagon; and therefore carrying it ſix times round will form the hexagon, or divide the circumference into ſix equal parts, each containing 60 degrees.—The ſecond of theſe c, will give AC the ſide of the trigon or equilateral triangle; and the arc AC one-third of the circumference, or 120 degrees.—Alſo the half of AB, or An is one-12th of the circumference, or 30 degrees, and gives the ſide of the dodecagon.

*Note.* If tangents to the circle be drawn through all the angular points of any inſcribed figure, they will form the ſides of a like circumſcribing figure.

## PROBLEM XXXV.

*In a given Circle to inſcribe a Pentagon, or a Decagon.*




Draw the two diameters AP, mn perpendicular to each other, and biſect the radius on at q.—With the center q and diſtance qA, deſcribe the arc Ar; and with the center A, and radius Ar, deſcribe the arc rB. Then is AB one-fifth of the circumference; and AB carried five times over will form the pentagon.—Alſo the arc AB biſected in s, will give As the tenth part of the circumference, or the ſide of the decagon.

*Note.*

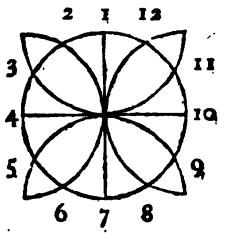
*Note.* Tangents being drawn through the angular points, will form the circumſcribing pentagon or decagon.

### PROBLEM XXXVI.

*To divide the Circumference of a given Circle into* 12 *equal Parts, each of* 30 *Degrees.*

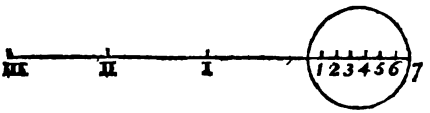
*Or to inſcribe a Dodecagon by another Method.*

Draw two diameters 1 7 and 4 10 perpendicular to each other.—Then with the radius of the circle, and the four extremities 1, 4, 7, 10, as centers, deſcribe arcs through the center of the circle; and they will cut the circumference in the points required, dividing it into 12 equal parts, at the points marked with the numbers.




### PROBLEM XXXVII.

*To draw a right Line equal to the Circumference of a given Circle.*



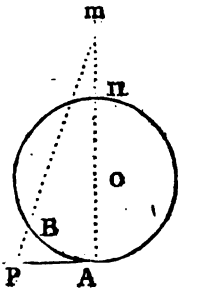

Take III 1 equal to 3 times the diameter and $\frac{1}{7}$ part more; and it will be equal to the circumference, very nearly.

PRO-

## PROBLEM XXXVIII.

*To find a Right Line equal to any given Arc* AB *of a Circle.*

Through the point A and the center draw Am, making mn equal to $\frac{3}{4}$ of the radius no.—Alſo draw the indefinite tangent AP perpendicular to it.—Then through m and B, draw mP; ſo ſhall AP be equal to the arc AB very nearly.



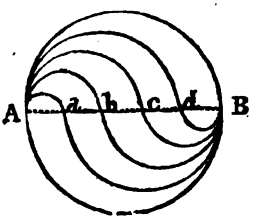

*Otherwiſe.*

Divide the chord AB into 4 equal parts.—Set one part AC on the arc from B to D.—Draw CD, and the double of it will be nearly equal to the arc ADB.

## PROBLEM XXXIX.

*To divide a given Circle into any propoſed Number of Parts by equal Lines, ſo that thoſe Parts ſhall be mutually equal, both in Area and Perimeter.*

Divide the diameter AB into the propoſed number of equal parts at the points a, b, c, &c,—Then on Aa, Ab, Ac, &c, as diameters, deſcribe ſemicircles on one ſide of the diameter AB; and on Bd, Bc, Bb, &c, deſcribe ſemicircles on the other ſide of the diameter. So ſhall the correſponding joining ſemicircles divide the given circle in

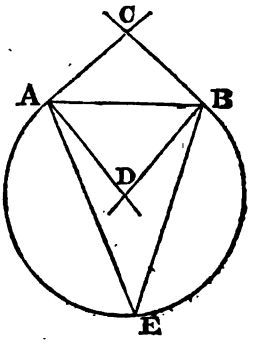


in the manner propoſed. And in like manner we may proceed when the ſpaces are to be in any given proportion.—As to the perimeters, they are always equal, whatever the proportion of the ſpaces is.

### PROBLEM XL.

*On a given Line* AB *to deſcribe the Segment of a Circle capable of containing a given Angle.*

Draw AC and BC making the angles BAC and ABC each equal the given angle.—Draw AD perpendicular to AC, and BD perpendicular to BC. With center D, and radius DA or DB, deſcribe the ſegment AEB. Then any angle, as E, made in that ſegment, will be equal to the given angle.

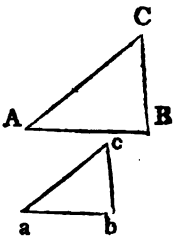



### PROBLEM XLI.

*To make a Triangle ſimilar to a given Triangle* ABC.

Let AB be one ſide of the required triangle. Make the angle a equal to the angle A, and the angle b equal to the angle B; then the triangle abc will be ſimilar to ABC as propoſed.

*Note.* If ab be equal to AB, the triangles will alſo be equal, as well as ſimilar.

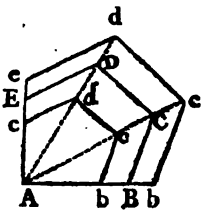



PRO-

## PROBLEM XLII.

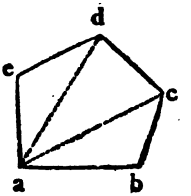
*To make a Figure similar to any other given Figure* ABCDE.

From any angle A draw diagonals to the other angles. —Take ab a side of the figure required. Then draw bc parallel to BC, and cd to CD, and de to DE, &c.



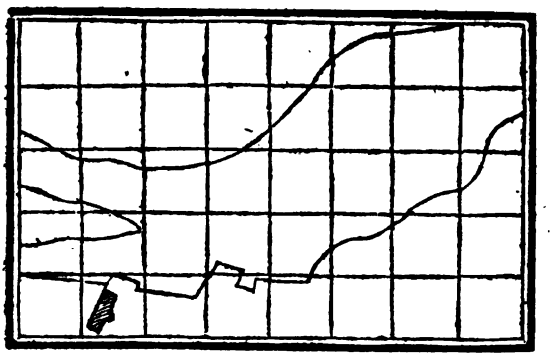

*Otherwise.*

Make the angles at a, b, e, respectively equal to the angles at A, B, E, and the lines will intersect in the corners of the figure required.

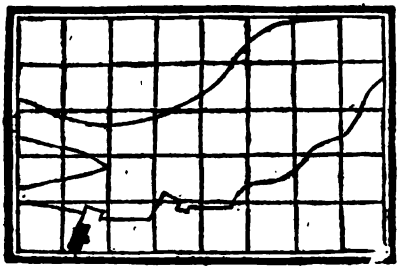



## PROBLEM XLIII.

*To reduce a Complex Figure from one Scale to another, mechanically by means of Squares.*



Divide

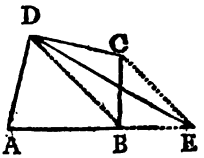


Divide the given figure, by croſs lines, into ſquares, as ſmall as may be thought neceſſary.—Then divide another paper into the ſame number of ſquares, and either greater, equal, or leſs, in the given proportion.—This done, obſerve what ſquares the ſeveral parts of the given figure are in, and draw, with a pencil, ſimilar parts in the correſponding ſquares of the new figure. And ſo proceed till the whole is copied.

### PROBLEM XLIV.

*To make a Triangle equal to a given Trapezium* ABCD.

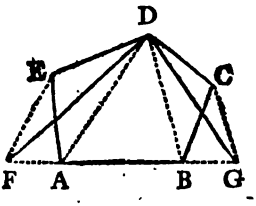
Draw the diagonal DB, and CE, parallel to it, meeting AB produced in E.—Join DE; ſo ſhall the triangle ADE be equal to the trapezium ABCD.




### PROBLEM XLV.

*To make a Triangle equal to the figure* ABCDEA.

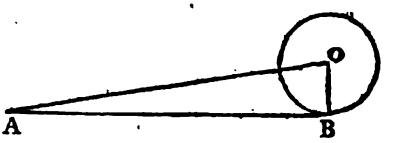
Draw the diagonals DA, DB, and the lines EF, CG, parallel to them, meeting the baſe AB, both ways produced, in F and G.—Join DF, DG; and DFG will be the triangle required equal to the given figure ABCDE.



*Note.* Nearly in the ſame manner may a triangle be made equal to any right-lined figure whatever.

### PROBLEM XLVI.

*To make a Triangle equal to a given Circle.*

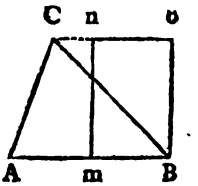



Draw any radius AO, and the tangent AB perpendicular to it.—On which take AB equal to the circumference of the circle by problem XXXVII.—Join BO, ſo ſhall ABO be the triangle required, equal to the given circle.

### PROBLEM XLVII.

*To make a Rectangle, or a Parallelogram, equal to a given Triangle* ABC.

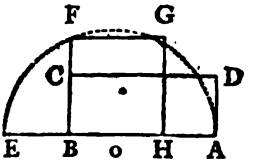
Biſect the baſe AB in m.—Through c draw cno parallel to AB.—Through m and B draw mn and BO parallel to each other, and either perpendicular to AB, or making any angle with it. And the rectangle or parallelogram mnoB will be equal to the triangle, as required.




PRO-

### PROBLEM XLVIII.

*To make a Square equal to a given Rectangle* ABCD.

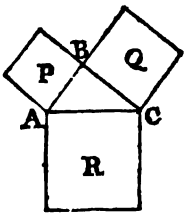



Produce one ſide, AB, till BE be equal to the other ſide BC.—Biſect AE in O; on which as a center, with radius AO deſcribe a ſemicircle, and produce BC to meet it at F.—On BF make the ſquare BFGH, and it will be equal to the rectangle ABCD, as required.

*** Thus the circle and all right-lined figures, have been reduced to equivalent ſquares.

### PROBLEM XLIX.

*To make a Square equal to two given Squares* P *and* Q.




Set two ſides AB, BC, of the given ſquares, perpendicular to each other.—Join their extremities AC; ſo ſhall the ſquare Q, conſtructed on AC, be equal to the two P and Q taken together.

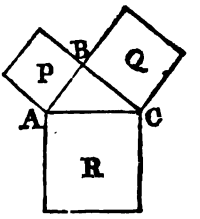
*Note.* Circles, or any other ſimilar figures, are added in the ſame manner. For, if AB and BC be the diameters of two circles, AC will be the diameter of a circle equal to both the other two. And if AB and BC be the like ſides of any two ſimilar figures, then AC will be the like ſide of another ſimilar figure equal to both the two former, and upon which the third figure may be conſtructed by problem XLII.

PRO-

## PROBLEM L.

*To make a Square equal to the Difference between two given Squares* PR.

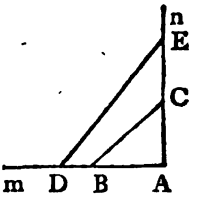
On the ſide AC of the greater ſquare, as a diameter, deſcribe a ſemicircle; in which apply AB the ſide of the leſs ſquare.—Join BC, and it will be the ſide of a ſquare equal to the difference between the two P and R, as required.




## PROBLEM LI.

*To make a Square equal to the Sum of any Number of Squares taken together.*

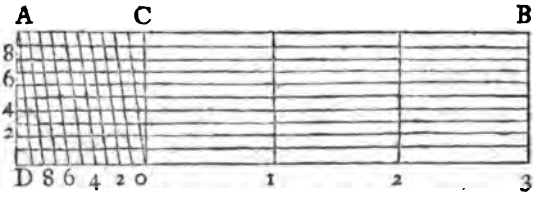
Draw two indefinite lines Am, An, perpendicular to each other at the point A. On the one of theſe ſet off AB the ſide of one of the given ſquares, and on the other AC the ſide of another of them. Join BC, and it will be the ſide of a ſquare equal to the two together.—Then take AD equal to BC, and AE equal to the ſide of the third given ſquare. So ſhall DE be the ſide of a ſquare equal to the ſum of the three given ſquares.—And ſo on continually, always ſetting more ſides of the given ſquares on the line An, and the ſides of the ſucceſſive ſums on the other line Am.




*Note.* And thus any number of any ſort of figures may be added together.

## PROBLEM LII.

### *To make plain Diagonal Scales.*




Draw any line as AB of any convenient length. Divide it into 11 equal parts*. Complete theſe into rectangles of a convenient height, by drawing parallel and perpendicular lines. Divide the altitude into 10 equal parts, if it be for a decimal ſcale for common numbers, or into 12 equal parts, if it be for feet and inches; and through theſe points of diviſion draw as many parallel lines, the whole length of the ſcale.—Then divide the length of the firſt diviſion AC into 10 equal parts, both above and below; and connect theſe points of diviſion by diagonal lines, and the ſcale is finiſhed, after being numbered as you pleaſe.

*Note.* Theſe diagonal ſcales ſerve to take off dimenſions or numbers of three figures. If the firſt large diviſions be units; the ſecond ſet of diviſions along AC, will be 10th parts; and the diviſions in the altitude, along AD, will be 100th parts. If CB be tens, AC will be units, and AD will be 10th parts. If CB be hundreds, AC will be tens, and AD units. If CB be thouſands, AC will be hundreds, and AD will be tens. And ſo on, each ſet of diviſions being tenth parts of the former ones.

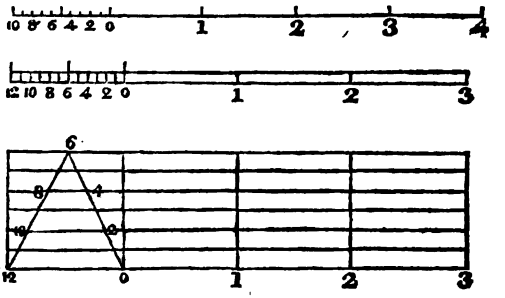
For example, ſuppoſe it were required to take off 243 from the ſcale. Fix one foot of the compaſſes at 2 of the greateſt diviſions, at the bottom of the ſcale, and extend the other to 4 of the ſecond diviſions, along

* Only 4 parts are here drawn, for want of room.

along the bottom; then, for the 3, ſlide up both points of the compaſſes by a parallel motion, till they fall upon the third longitudinal line; and in that poſition extend the ſecond point of the compaſſes to the fourth diagonal line, and you have the extent of three figures as required.

Or if you have any line to meaſure the length of. —Take it between the compaſſes, and applying it to the ſcale, ſuppoſe it fall between 3 and 4 of the large diviſions; or, more nearly, that it is 3 of the large diviſions, or 3 hundreds, and between 5 and 6 of the ſecond diviſions, or 5 tens or 50, and a little more. Slide up the points of the compaſſes by a parallel motion, keeping one foot always on the vertical diviſion of 3 hundred, till the other point fall exactly on one of the diagonal lines, which ſuppoſe to be 8, which is 8 units. Which ſhews that the length of the line, propoſed to be meaſured, is 358.

PLANE SCALES FOR TWO FIGURES.

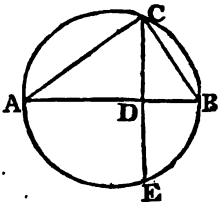



The above are three other forms of ſcales, the firſt of which is a decimal ſcale, for taking off common numbers conſiſting of two figures. The other two are duodecimal ſcales, and ſerve for feet and inches, &c.

Theſe, and other ſcales, engraven on ivory, are fitteſt for practical uſe. And the moſt convenient

form of a plane ſcale of equal diviſions, is on the very edge of the ivory, made thin at the edge, for laying along any line, and then marking on the paper oppoſite any diviſion required: which is better than taking lengths off a ſcale with compaſſes.

## REMARKS.




*Note* 1. That in a circle, the half chord DC, is a mean proportional between the ſegments AD, DB of the diameter AB perpendicular to it. That is AD : DC :: DC : DB.

2. The chord AC is a mean proportional between AD and the diameter AB. And the chord BC a mean proportional between DB and AB.

That is, AD : AC :: AC : AB,
and BD : BC :: BC : AB.

3. The angle ACB, in a ſemicircle, is always a right angle.

4. The ſquare of the hypotenuſe of a right-angled triangle, is equal to the ſquares of both the ſides.

That is, $AC^2 = AD^2 + DC^2$,
and $BC^2 = BD^2 + DC^2$,
and $AB^2 = AC^2 + BC^2$.

5. Triangles that have all the three angles of the one, reſpectively equal to all the three of the other, are called equiangular triangles, or ſimilar triangles.

6. In ſimilar triangles, the like ſides, or ſides oppoſite the equal angles, are proportional.

7. The areas, or ſpaces, of ſimilar triangles, are to each other, as the ſquares of their like ſides.

SECT.

## SECT. II.

### PLANE TRIGONOMETRY.

PLANE Trigonometry teaches the relations and calculations of the ſides and angles of plane triangles.

The angles of triangles are meaſured by the number of degrees contained in the arc cut off by the legs of the angle, and whoſe center is the angular point. A right angle is therefore an angle of 90 degrees; and the ſum of the three angles of every triangle, or two right angles, is equal to 180°. Wherefore, in a right-angled triangle, the one acute angle being ſubtracted from 90°, the remainder will be the other; and the ſum of any two angles of a triangle being taken from 180°, will leave the third angle.

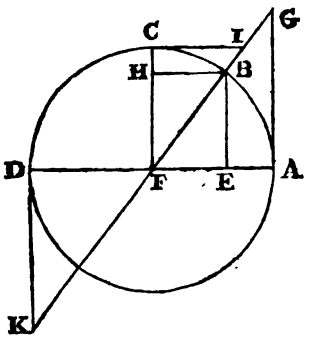
Degrees are marked at the top of the figure with a ſmall °, minutes with ′, ſeconds with ″, and ſo on. Thus 57° 30′ 12″, that is, 57 degrees 30 minutes and 12 ſeconds.

In a right-angled triangle, the ſide oppoſite the right angle, is called the hypotenuſe; and the other, the legs, or ſides, or ſometimes the baſe and perpendicular.

The complement of an arc is what it wants of a quadrant. So BC = 40° is the complement of AB = 50°.

The ſupplement of an arc is what it wants of a ſemicircle. So BCD = 130° is the complement of AB = 50°.

The ſine of an arc is the line drawn from one end of the arc perpendicularly upon the diameter drawn through the other end of the arc. So BE is the ſine of AB or of BCD.




The verſed ſine of an arc is the part of the diameter between the ſine and the beginning of the arc. So AE is the verſed ſine of AB, and DE the verſed ſine of BCD.

The tangent of an arc is the line drawn perpendicularly from one end of the diameter paſſing through one end of the arc, and terminated by the line drawn from the center through the other end of the arc. So AG or DK is the tangent of AB, or of BCD.

The ſecant of an arc is the line drawn from the center through the end of the arc, and terminated by the tangent. So FG or FK is the ſecant of AB, or of BCD.

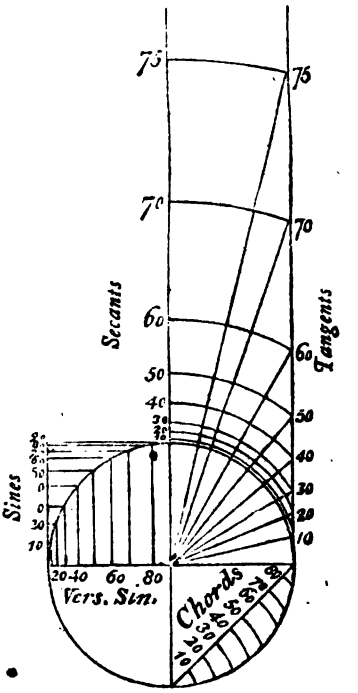
The coſine, cotangent, or coſecant of an arc, is the ſine, tangent, or ſecant of the complement of that arc. So BH is the coſine, CI the cotangent, and FI the coſecant of AB.

From the definitions it is evident that the ſine, tangent, and ſecant, are common to two arcs, which are the ſupplements of each other: ſo the ſine, tangent, or ſecant of 50°, is alſo the ſine, tangent, or ſecant of 130°.

The ſine, tangent, or ſecant of an angle, is the ſine, tangent, or ſecant of the arc, or the degrees, by which the angle is meaſured.

The ſine, tangent, and ſecant of every degree and minute in a quadrant, are calculated to the radius 1, and ranged in tables for uſe. But becauſe trigonometrical operations with theſe natural ſines, tangents, and ſecants, require tedious multiplications and diviſions, the logarithms of them are taken, and ranged in tables alſo; and the logarithmic ſines, tangents, and ſecants are commonly uſed, as they require only additions and ſubtractions, inſtead of the multiplications and diviſions.

The method of conſtructing the ſcales of chords, ſines, tangents, and ſecants, uſually placed on inſtruments, is exhibited in the following figure.




There are uſually three methods of reſolving triangles, or the caſes of trigonometry; namely, Geometrical conſtruction, Arithmetical computation, and Inſtrumental operation.

*In the firſt method*, let the triangle be conſtructed, by making the parts of the given magnitudes, namely the ſides from a ſcale of equal parts, and the angles from a ſcale of chords, or other inſtrument. Then meaſure the required parts by the ſame ſcale.

*In the ſecond method*, having ſtated the terms of the proportion according to the rule, reſolve it like all other proportions,

tions, in which a fourth term is to be found from three given terms, by multiplying the ſecond and third together, and dividing the product by the firſt, in working with the natural numbers, whether they be ſides, or ſines, tangents, or ſecants of angles. Or, in working with the logarithms, add the log. of the 2d and 3d terms together, and from the ſum ſubtract the log. of the 1ſt term; then the number anſwering to the remainder will be the 4th term required.

*To work a ſtating inſtrumentally, as ſuppoſe by the log. lines on one ſide of the two-foot ſcales.*—Extend the compaſſes from the firſt term to the ſecond, or third, which happens to be of the ſame kind with it; then that extent will reach from the other term to the fourth, taking both extents towards the ſame ſide.

*Note.* For the ſides of triangles you uſe the line of numbers (marked Num.) and for the angles, the lines of ſines or tangents (marked Sin. and Tan.) according as the proportion reſpects ſines or tangents.—If the extent upon the tangents reach beyond the line, ſet it ſo far back as it reaches over.

In a triangle there muſt be given three parts, one of which, at leaſt, muſt be a ſide; becauſe the ſame angles are common to an infinite number of triangles.

In plane trigonometry there are three caſes or varieties only, *viz.*

1. When two of the three parts given, are a ſide and its oppoſite angle.

2. When there are given two ſides and their included angle.

3. When the three ſides are given.

A Table

*A Table of Lineal Meaſures.*

| Inches | Feet | Yards | Poles | Chains | Furlongs | Mile |
|---|---|---|---|---|---|---|
| 12 | 1 | | | | | |
| 36 | 3 | 1 | | | | |
| 198 | 16½ | 5½ | 1 | | | |
| 792 | 66 | 22 | 4 | 1 | | |
| 7920 | 660 | 220 | 40 | 10 | 1 | |
| 63360 | 5280 | 1760 | 320 | 80 | 8 | 1 |

*Note alſo,* An inch is ſuppoſed equal to 3 barley-corns in length,
4 inches—a hand,
6 feet, or 2 yards—a fathom,
3 miles—a league,
60 nautical or geographical miles—a degree, or 69½ ſtatute miles nearly,
360 degrees, or 25000 miles nearly—is the circumference of the earth.

PROBLEM I.

*Given Three Parts, ſuch, that an Angle and its Oppoſite Side are Two of them, to find the reſt.*

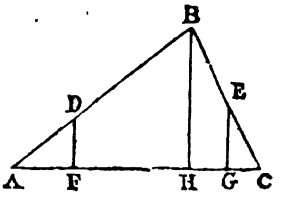
In any plane triangle, the ſides are proportional to the ſines of their oppoſite angles*. That is,

As one ſide :
Is to another ſide :
So is ſin. angle opp. the former :
To ſin. angle opp. the latter.

*Note*

* DEMONSTRATION.

Let ABC be any triangle: in AB aſſume any point D, take CE = AD, and upon AC demit the perpendiculars DF, EG, BH; then will DF and EG be the ſines of the angles A, C, to the general radius AD or CE. Now, from ſimilar triangles, we ſhall have { AB : BH :: AD : DF ; CB : BH :: AD (CE) : EG } and hence, of equality, AB : BC :: EG : DF.



*Note* 1. To find an angle, begin the proportion with a ſide oppoſite a given angle; and to find a ſide, begin with an angle oppoſite a given ſide.

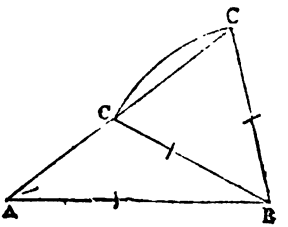
2. An angle found by this rule is always ambiguous, except it be a right angle, or except that the magnitude of the given angle prevent the ambiguity; becauſe the ſine anſwers to two angles which are the ſupplements of each other: and accordingly the conſtruction gives two triangles with the ſame given parts; and when there is no reſtriction or limitation included in the propoſition, either of them may be taken. The degrees, in the table, anſwering to the ſine, is the acute angle; and if the angle be obtuſe, take thoſe degrees from 180°, and the remainder will be the obtuſe angle.—When the given angle is obtuſe, or right, there can be no ambiguity; for then neither of the other angles can be obtuſe, and the conſtruction will produce but one triangle.

EXAMPLE I.

In the plane triangle ABC,

Given { AB 345, BC 232 } yards, ∠A 37° 20′

Required the other parts.




*Geometrically.*

1. Draw the line AB = 345 from ſome convenient ſcale of equal parts.

2. Make the angle A = 37° 20′.

3. With the center B and radius 232, taken from the ſame ſcale of equal parts, croſs AC in C.

4. Draw BC, and the triangle is conſtructed.

Then

Then the angles B and C, meaſured by the ſcale of chords, and the ſide AC, meaſured by the ſcale of equal parts, will be found to be as follows: viz.

| ∠ B | ∠ C | AC |
|---|---|---|
| 27° | 115½ | 174 |
| or 78½ | or 64½ | or 374½ |

*Arithmetically.*

As ſide BC — 232 — — 2·3654880
To ſide BA — 345 — — 2·5378191
So ſine ∠ A 37° 20′ — — 9·7827958

To ſine ∠ C 115° 36′ or 64° 24 — 9·9551269
∠ A 37 20 37 20

ſubtr. 152 56 or 101 44 ſub.
from 180 00 180 00

leaves 27 04 or 78 16 the ∠ B.

Then

As ſine ∠ A — 37° 20′ — 9·7827958
To ſine ∠ B — {27° 04′ / 78 16} — 9·6580371 / 9·9908291
So ſide BC — 232 — 2·3654880

To ſide AC — {174·07 / 374·56} — 2·2407293 / 2·5735213

*Inſtrumentally.*

In the firſt proportion, Extend from 232 to 345 upon the line of numbers; that extent will reach, upon the ſines, from 37°½ to 64°½ the angle C.

In the ſecond proportion, Extend from 37°½ to 27° or 78°¼ upon the ſines; that extent will reach, upon the numbers, from 232 to 174 or 374½, for the ſide AC.

EXAMPLE II.

In the plane triangle ABC

Given { AB 162 chains, AC 270 chains, ∠B 90° } Required the other parts. Anſ. { BC 216 chains, ∠C 36° 52′ 12″, ∠A 53 07 48

EXAMPLE III.

In the plane triangle ABC

Given { AB 365 poles, ∠B 24° 45′, ∠A 57 12 } Required the other parts. Anſ. { ∠C 98° 03′, AC 154·33, BC 309·86 } pol.

EXAMPLE IV.

In the plane triangle ABC

Given { AB 53 miles, ∠A 121° 14′, ∠C 29° 23′ } Required the other parts. Anſ. { ∠B 29° 23′, AC 53 miles, BC 92·36 miles

EXAMPLE V.

In the plane triangle ABC

Given { AB 162, ∠C 36° 52′ 12″, ∠A 53 07 48 } Required the other parts. Anſ. { ∠B 90°, AC 270 chains, BC 216 chains

EXAMPLE VI.

In the plane triangle ABC

Given { AB 365 poles, AC 154·33 poles, ∠C 98° 03′ } Required the other parts. Anſ. { ∠B 24° 45′, ∠A 57 12, BC 309·86 pol.

EX-

EXAMPLE VII.

In the plane triangle ABC

Given { AC 120 feet, BC 112 feet, ∠A 57° 27′ } Anſ. { ∠B { 64° 34′ 21″, 115 25 39 }, ∠C { 57 58 39, 7 07 21 }, AB { 112·65, 16·47 } feet }

Required the other par.

PROBLEM II.

*Given Two Sides and the Angle Included by them, to find the reſt.*

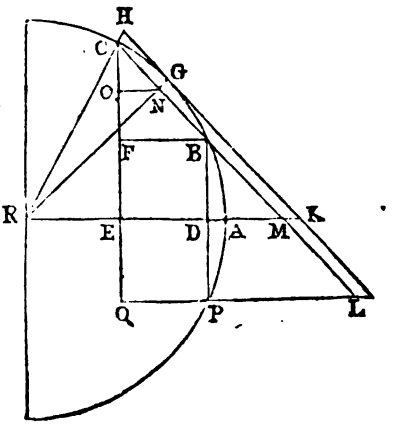
In a plane triangle, As the ſum of any two ſides is to their difference, ſo is the tangent of half the ſum of their oppoſite angles, to the tangent of half their difference. Then the half difference added to the half ſum of the angles, gives the greater; and ſubtracted, leaves the leſs angle*.

Then,

* DEMONSTRATION.

By the firſt problem, the ſides are as the ſines of their oppoſite angles, and conſequently the ſum of the ſides will be to the difference of the ſides, as the ſum of the ſines to the difference of the ſines of the ſaid oppoſite angles.

Wherefore we have only to prove that the ſum of the ſines is to the difference of the ſines of two arcs, as the tangent of half the ſum of thoſe arcs is to the tangent of half their difference: In order to which, let BD, CE, be the ſines of the arcs AB, AC; produce BD to the circumference at P, and produce CE till EQ be = DP; to the middle point G of the arc BC draw the tangent HGK, and draw CNBML parallel to it; join RH, RG, and draw ON, FB, and QL parallel to RAMK.




Now

Then, all the angles being known, find the unknown ſide by the firſt problem.

*Note.* When, in this caſe, the triangle is right-angled, the longeſt ſide will be found by extracting the ſquare root of the ſum of the ſquares of the other two ſides; and then the angles will be found by the firſt problem.

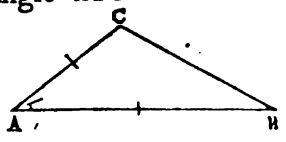
*Note alſo,* That inſtead of the tangent of the half ſum, we may uſe the cotangent of half the given angle, which is the ſame thing.

EXAMPLE I.

In the plane triangle ABC

Given { AB 345 yards
AC 174·07 yards
∠A 37° 20′

Required the other parts.




*Geometrically.*

1. Draw AB equal to 345, from a ſcale of equal parts.

2. Make

---

Now it is evident that CQ is the ſum, and CF the difference of the ſines; and that GK is the tangent of half the ſum AG, and GH the tangent of half the difference CG, of the two arcs AB, AC; alſo NM is = ½CL, for BN = NC, and BM = ML: then, in the ſimilar triangles CQL, RNM, RGK, it will be as

CQ : CF :: (½CQ or) OE : (½CF or) OC,
and NM : NC :: OE : OC,
and NM : NC :: GK : GH,
theref. CQ : CF :: GK : GH.

Which was to be demonſtrated.

And that the half ſum increaſed and diminiſhed by the half difference, gives the greater and leſs angle reſpectively, is evident from the figure.—And that two quantities of any kind may be found, by the ſame rule, from their ſum and difference, may be proved thus: Let CN repreſent the leſs, and NL the greater, of any two quantities; and let B be the middle of the right line CL. Then it is evident that BL = BC is the half ſum, and BN the half difference, as alſo that LB + BN = NL the greater quantity, and CB − BN = NC the leſs.

2. Make the angle A equal to 37° 20′.

3. Make AC equal to 174·07, by the ſcale of equal parts.

4. Join B, C, and it is done.

Then, the parts being meaſured, we have the ∠C = 115°½, the ∠B = 27°, and BC = 232 yards.

*Arithmetically.*

As ſum of ſides AB + AC 519·07 — 2·7152259
To dif. of ſides AB — AC 170·93 — 2·2328183
So tang. $\frac{\angle C + \angle B}{2}$ — 71° 20′ — 10·4712979
To tang. $\frac{\angle C - \angle B}{2}$ — 44 16 — 9·9888903

Their ſum 115 36 ∠C
Their dif. 27 04 ∠B

Then

As ſine ∠C 115° 36′ or 64° 24′ — 9·9551259
To ſine ∠A — — 37° 20′. — 9·7827958
So ſide AB 345 — — — 2·5378191
To ſide BC 232 — — — 2·3654890

*Inſtrumentally.*

In the firſt proportion, Extend from 519 to 171 on the line of numbers; that extent will reach, upon the tangents, from 71°½ (the contrary way, becauſe the tangents are ſet back again from 45°) a little beyond 45, which being ſet ſo far back from 45, falls upon 44°½ the fourth term.

In the ſecond proportion, Extend from 64°½ to 37°½ on the ſines, that extent will reach, on the numbers, from 345 to 232 the fourth term required.

EXAMPLE II.

In the plane triangle ABC

Given { AB 162, BC 216 } chains, ∠B 90°
Required the other parts.

Anſ. { ∠A 53° 07′ 48″, ∠C 36 52 12, AC 270 chains

EXAMPLE III.

In the plane triangle ABC

Given { AB 53, BC 92·36 } miles, ∠B 29° 23′
Required the other parts.

Anſ. { ∠A 121° 14′, ∠C 29 23, AC 53 miles.

EXAMPLE IV.

In the plane triangle ABC

Given { AC 120, BC 112 } poles, ∠C 57° 58′ 39″
Required the other parts.

Anſ. { ∠A 57° 27′ 00″, ∠B 64 34 21, AB 112·65 pol.

PROBLEM III.

*Given the Three Sides of a Triangle, to find the Angles.*

* In any plane triangle, having let fall a perpendicular from the greateſt angle upon the oppoſite ſide or

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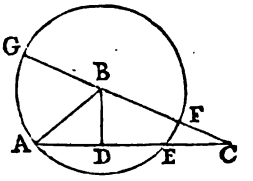
* DEMONSTRATION.




From one end B of the leaſt ſide AB, of the triangle ABC, as a center, and radius AB, deſcribe a circle cutting the other two ſides in E and F; produce CB to the circle at G, and let fall the perpendicular BD.—Then is GB = BF = AB, and (by 3. III. Eucl.) AD = DE, and conſequently EC = CD − DA the difference of the ſegments, FC = GB − BA the difference of the ſides, and

or baſe, dividing it into two ſegments, and the whole triangle into two right-angled triangles; it will be

As the baſe, or ſum of the ſegments:
Is to the ſum of the other two ſides::
So is the difference of thoſe ſides:
To the difference of the ſegments of the baſe.

Then half the difference being added to, and ſubtracted from, half their ſum, will give the greater and leſs ſegment.

Hence, in each of the right-angled triangles, will be known two ſides, and the angle oppoſite to one of them; and conſequently the other angles will be found by the firſt problem.

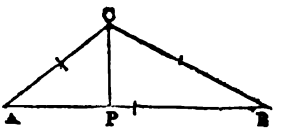
*Note.* In the above proportions, if half the difference of the ſides be taken for the third term, then the fourth term will be half the difference of the ſegments. Which will commonly be more convenient to uſe than the whole differences.

## EXAMPLE 1.

In the plane triangle ABC

Given { AB 345, AC 174·07, BC 232 } yards

Required the angles.




E *Geometrically.*

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and GC = CB + BA the ſum of the ſides. But (by Cor. to 36. III. Euel.) the rectangle CA × CE = CG × CF, or CA : CG :: CF : CE, that is AC : CB + BA :: CB − BA : CD − DA. *Q. E. D.*

And that the half ſum of two quantities increaſed and diminiſhed by their half difference, gives the greater and leſs quantities reſpectively, was proved in the laſt problem.

*Geometrically.*

1. Draw AB equal to 345 by a ſcale of equal parts.

2. With the centers A and B, and radii 174·07 and 232, taken from the ſame ſcale, deſcribe arcs interſecting in C.

3. Draw AC and BC, and it is done.

Then by meaſuring the angles, they appear to be nearly of the following dimenſions, *viz.* $\angle A = 37^\circ\frac{1}{2}$, $\angle B = 27^\circ$, and $\angle C = 115^\circ\frac{1}{2}$.

*Arithmetically.*

Having let fall the perpendicular CP, it will be As AB = 345 : BC + CA = 406·07 :: BC − CA = 57·93 : $\frac{406\cdot07 \times 57\cdot93}{345} = 68\cdot18 =$ BP − PA.

Hence, $\frac{345 + 68\cdot18}{2} = 206\cdot59 =$ BP,

and $\frac{345 - 68\cdot18}{2} = 138\cdot41 =$ AP.

Then in the triangle APC, right-angled at P,

| | | | | |
|---|---|---|---|---|
| As AC | — | 174·07 | — | 2·2407239 |
| To AP | — | 138·41 | — | 2·1411675 |
| So S. ∠P | — | 90° 00′ | — | 10·0000000 |
| To S. ∠ACP | — | 52 40 | — | 9·9004436 |
| which being taken from | | 90 00 | | |
| leaves | | 37 20 ∠A | | |

And in the triangle BPC,

| | | | | |
|---|---|---|---|---|
| As BC | — | 232 | — | 2·3654880 |
| To BP | — | 206·59 | — | 2·3151093 |
| So S. ∠P | — | 90° 00′ | — | 10·0000000 |
| To S. ∠PCB | — | 62 56 | — | 9·9496213 |
| taken from | | 90 00 | | |
| leaves | | 27 04 ∠B | | |

Alſo

Alſo 52° 40′ ∠ ACP
added to 62 56 ∠ BCP

makes 115 36 ∠ ACB

Whence the ∠ A = 37° 20′, the ∠ B = 27° 04′, and the ∠ C = 115° 36′.

*Inſtrumentally.*

In the firſt proportion.—Extend from 345 to 406, on the line of numbers; that extent will reach, upon the ſame line, from 58 to 68·2, the difference of the ſegments of the baſe.

In the ſecond proportion.—Extend from 174 to $138\frac{1}{2}$ on the numbers; that will reach, on the ſines, from 90° to $52°\frac{2}{3}$.

In the third proportion.—Extend from 232 to $206\frac{1}{2}$, and that extent will reach from 90 to 63°.

EXAMPLE II.

In the plane triangle ABC

Given { AB 162, AC 270, BC 216 } Anſ. { ∠ A 53° 07′ 48″, ∠ B 90 00 00, ∠ C 36 52 12 }
Required the angles.

EXAMPLE III.

In the plane triangle ABC

Given { AB 112·65, AC 120, BC 112 } Anſ. { ∠ A 57° 27′ 00″, ∠ B 64 34 21, ∠ C 57 58 39 }
Required the angles.

EXAMPLE IV.

In the plane triangle ABC

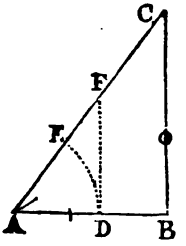
Given { AB 53, AC 53, BC 92·36 } Required the angles. } Anſ. { ∠A 121° 14′, ∠B 29 23, ∠C 29 23 }

Theſe three problems include all the caſes of plane triangles, as well right-angled as oblique; beſides which there are ſome other theorems, ſuited to ſome particular forms of triangles, which are often more expeditious in uſe than the general ones; one of which, as the caſe for which it ſerves ſo often occurs, take as follows.

PROBLEM IV.

*Given the Angles and a Leg, of a Right-angled Triangle; to find the Other Leg and the Hypotenuſe.*

* As the radius
Is to the given leg AB : :
So is tang. of angle A :
To the oppoſite leg BC : :
And ſo is ſecant of the ſame ∠A :
To the hypotenuſe AC.




EX-

* DEMONSTRATION.

With the center A and any radius AD, deſcribe an arc DE, and erect the perpendicular DF; which, it is evident, will be the tangent, and AF the ſecant of the arc DE, or angle A, to the radius AD.—And in the ſimilar triangles ADF, ABC, it will be AD : AB :: DF : BC :: AF : AC. *Q. E. D.*

GENERAL SCHOLIUM.

Beſides theſe rules, I ſhall here ſet down ſome new theorems concerning the relations of the ſides and angles of triangles, independent of any tables.

PRO-

EXAMPLE.

In the plane triangle ABC, right-angled at B,

Given $\left\{\begin{matrix} AB\ 162 \\ \angle A\ 53^\circ\ 07'\ 48'' \end{matrix}\right\}$ Required AC and BC.



*Geo-*

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PROPOSITION.

If $2a$ denote a ſide of any triangle, A the number of degrees contained in its oppoſite angle, and $r$ the radius of the circle circumſcribing the triangle: Then I ſay that A is equal to

$$57^\circ 2957795 \times ; \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9} \&c.$$

For ſince $2a$ is the chord of the arc upon which the angle, whoſe meaſure is A, inſiſts; $a$ will be the ſine of half that arc, or the ſine of the angle to the radius $r$, ſince an angle in the circumference of a circle is meaſured by half the arc upon which it ſtands: now it appears, from rule 3. prob. 6. ſect. 1. part 2. of this work, that the ſaid half arc $z$ is equal to

$a + \frac{a^3}{2.3r^2} + \frac{3a^5}{2.4.5r^4} + \frac{3.5a^7}{2.4.6.7r^6}$ &c; and, $3^\cdot 14159r$ denoting half the circumference of the ſame circle, or the arc of 180 degrees, we ſhall have $3^\cdot 14159r : 180^\circ :: z : \frac{180z}{3^\cdot 14159r} =$

$\frac{57^\cdot 2957795z}{r} = 57^\cdot 2957795 \times : \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7}$ &c

= the degrees in the angle or half arc.

*Corollary* 1. By reverting the above ſeries we obtain

$$\frac{a}{r} = \frac{A}{n} - \frac{A^3}{2.3n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7} \&c,$$

$$\text{putting } n = 57^\cdot 2957795 = \frac{180}{3^\cdot 14159\ \&c}.$$

*Corollary* 2. If $2a$ be the hypotenuſe of a right-angled triangle, $a$ will be $= r$, and then the general ſeries will become $n \times : a +$

$\frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7}$ &c $= 90$, or $\frac{90}{n} = \frac{90 \times 3^\cdot 14159\ \&c}{180} =$

$\frac{3^\cdot 14159\ \&c}{2} = 1 + \frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7} + \frac{3.5.7}{2.4.6.8.9}$ &c.

*Corollary*

*Geometrically.*

Make AB = 162, and the angle A = 53° 07′ 48″; then raiſe the perpendicular BC meeting AC in c. So ſhall AC meaſure 270, and BC 216.

*Calcu-*

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*Corollary* 3. Since the chord of 60 degrees is = the radius, or the ſine of 30 degrees = half the radius, putting $a$ for $\frac{1}{2}r$ in the general ſeries, will give $n \times : \frac{1}{2} + \frac{1}{2.3.2^3} + \frac{3}{2.4.5.2^5} + \frac{3.5}{2.4.6.7.2^7}$ &c = 30; and hence the ſum of the infinite ſeries

$$\frac{1}{2} + \frac{1}{2.3.2^3} + \frac{3}{2.4.5.2^5} + \frac{3.5}{2.4.6.7.2^7} \text{ \&c}$$

$$\text{is} = \frac{30}{n} = \frac{30 \times 3{\cdot}14159 \text{ \&c}}{180} = \frac{3{\cdot}14159 \text{ \&c}}{6} =$$

one-ſixth of the circumference of the circle whoſe diameter is 1.

*Corollary* 4. It might eaſily be ſhewn, from the principles of common geometry, that the ſine of 60 degrees is to the radius, as $\frac{1}{2}\sqrt{3}$ is to 1; ſubſtituting, then, $\frac{1}{2}r\sqrt{3}$ for $a$ in the general ſeries, we ſhall have $n\sqrt{3} \times : \frac{1}{2} + \frac{3}{2.3.2^3} + \frac{3.3^2}{2.4.5.2^5} + \frac{3.5.3^3}{2.4.6.7.2^7}$ &c = 60; and hence the ſum of the infinite ſeries

$\frac{1}{2} + \frac{3}{2.3.2^3} + \frac{3.3^2}{2.4.5.2^5} + \frac{3.5.3^3}{2.4.6.7.2^7}$ &c will be

$= \frac{60}{n\sqrt{3}} = \frac{60 \times 3{\cdot}14159 \text{ \&c}}{180\sqrt{3}} = \frac{3{\cdot}14159 \text{ \&c}}{3\sqrt{3}}$, and is, therefore, to the infinite ſeries in the third corollary, as 2 is to $\sqrt{3}$.

*Corollary* 5. If $b$, $c$ be the halves of the other two ſides of the triangle, and B, C the degrees contained in their oppoſite angles; ſince $B = n \times : \frac{b}{r} + \frac{b^3}{2.3r^3} + \frac{3b^5}{2.4.5r^5}$ &c,

and $C = n \times : \frac{c}{r} + \frac{c^3}{2.3r^3}$ &c, and the 3 angles of any triangle are equal to 180 degrees; we ſhall have $180 = A + B + C = n \times : \frac{a+b+c}{r} + \frac{a^3+b^3+c^3}{2.3r^3}$ &c, or the ſum of the infinite ſeries

$$\frac{a+b+c}{r} + \frac{1}{2.3} \cdot \frac{a^3+b^3+c^3}{r^3} + \frac{3}{2.4.5} \cdot \frac{a^5+b^5+c^5}{r^5} + \frac{3.5}{2.4.6.7} \cdot \frac{a^7+b^7+c^7}{r^7}$$

&c

*Calculation.*

As radius 1 : AB = 162 :: 1·3333284 = nat. tangent of 53° 07′ 48″ : 215·9992 = BC.

And, as 1 : 162 :: 1·6666628 = nat. secant of the same ∠A : 269·99937 = AC.



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&c will be $= \frac{180}{\pi} = \frac{180 \times 3{\cdot}14159 \text{ \&c}}{180} = 3{\cdot}14159$ &c = the circumference of a circle whose diameter is 1; $a$, $b$, $c$ being the halves of the three sides of any triangle, and $r$ the radius of its circumscribing circle.

*Corollary* 6. Since, by prob. 3, $b : a + c :: a - c : \frac{aa - cc}{b} =$ half the difference of the segments of the base ($b$) made by a perpendicular demitted from its opposite angle, and $b + \frac{aa-cc}{b} + \frac{aa+bb-cc}{b} =$ the segment adjoining to the side $2a$, we shall have $\sqrt{4a^2 - \frac{(aa+bb-cc)^2}{bb}} = \frac{\sqrt{4a^2b^2-(aa+bb-cc)^2}}{b}$ for the value of the said perpendicular to the base, and hence $\frac{\sqrt{4a^2b^2-(aa+bb-cc)^2}}{b} : 2a :: c : \frac{2abc}{\sqrt{4a^2b^2-(aa+bb-cc)^2}} = r$ the radius of the circumscribing circle.

Having now found the value of $r$, we can calculate all the cases of trigonometry without any tables, and without reducing oblique triangles to right-angled ones; for having any three parts (except the three angles) given, we can find the rest from these five equations:

1. $r = \frac{2abc}{\sqrt{4a^2b^2-(aa+bb-cc)^2}}$.

2. $A = \pi \times : \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9}$ &c.

3. $B = \pi \times : \frac{b}{r} + \frac{b^3}{2.3r^3} + \frac{3b^5}{2.4.5r^5} + \frac{3.5b^7}{2.4.6.7r^7} + \frac{3.5.7b^9}{2.4.6.8.9r^9}$ &c.

4. $C = \pi \times : \frac{c}{r} + \frac{c^3}{2.3r^3} + \frac{3c^5}{2.4.5r^5} + \frac{3.5c^7}{2.4.6.7r^7} + \frac{3.5.7c^9}{2.4.6.8.9r^9}$ &c.

5. $A + B + C = 180$.

*Inſtrumentally.*

The extent from 45° to 53° 08′, upon the tangents, will reach from 162 to 216 upon the numbers.

*Note.*

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And for the more convenience we may add the three following, which are derived from the 2d, 3d, and 4th, by reverſion of ſeries.

6. $a = r \times : \frac{A}{n} - \frac{A^3}{2.3n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7}$ &c.

7. $b = r \times : \frac{B}{n} - \frac{B^3}{2.3n^3} + \frac{B^5}{2.3.4.5n^5} - \frac{B^7}{2.3.4.5.6.7n^7}$ &c.

8. $c = r \times : \frac{C}{n} - \frac{C^3}{2.3n^3} + \frac{C^5}{2.3.4.5n^5} - \frac{C^7}{2.3.4.5.6.7n^7}$ &c.

Where $n = 57{\cdot}2957795$ &c.

EXAMPLE.

Suppoſe we take here the firſt example in prob. 1, in which are given two ſides $2b = 345$, $2c = 232$, and the angle oppoſite to $2c = 37° 20' = 37\frac{1}{3}$ degrees $= C$.

Then, $\frac{C}{n}$ being $= \frac{37\frac{1}{3} \times 3{\cdot}14159 \text{ &c}}{180} = {\cdot}651589587$, we have

$c = \frac{232}{2} = 116 = r \times : {\cdot}651589587 - {\cdot}04610744 + {\cdot}00097879 - {\cdot}000009894 + {\cdot}000000058$ &c $= r \times ({\cdot}652568435 - {\cdot}046117334) = {\cdot}6064511r$. Hence $r = \frac{116}{{\cdot}6064511} = 191{\cdot}27677$,

And $\frac{b}{r} = \frac{345 \times {\cdot}6064511}{2 \times 116} = {\cdot}9018346$.

Again $B = 57{\cdot}2957795 \times 1{\cdot}12402$ (the ſum of the ſeries in the third equation) $= 64{\cdot}4016$ degrees $= 64° 24'$.

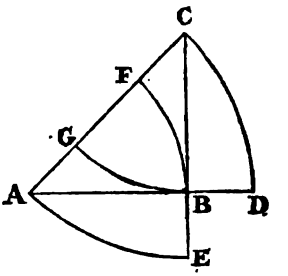
And $A = 180 - 37\frac{1}{3} - 64{\cdot}4016 = 180 - 101{\cdot}7355 = 78{\cdot}265°$ $= 78° 16'$ nearly.

Laſtly, $\frac{A}{n}$ being $= \frac{78{\cdot}265}{57{\cdot}2957795} = 1{\cdot}365982$, and $r = 191{\cdot}27677$, from the fifth equation we have $a = 191{\cdot}27677 \times : 1{\cdot}365982 - {\cdot}4247992 + {\cdot}0396379 - {\cdot}0017607 + {\cdot}0000288 - {\cdot}0000005 = 191{\cdot}27677 \times {\cdot}9790883 = 187{\cdot}27684$.

And hence $2a = 374{\cdot}55368 =$ the third ſide of the triangle.

*Corollary*

*Note.* It is common to add another method for right-angled triangles, which is this. ABC being the triangle, make a leg AB radius, that is, with center A and radius AB, deſcribe an arc BF: then it is evident that the other leg BC repreſents the tangent, and the hypotenuſe AC the ſecant of the angle A or arc BF.

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*Corollary* 7. As the ſeries by which an angle is found, often converges very ſlowly, I have inſerted the following approximation of it, viz.

$A = n \times (\frac{1}{3}\sqrt{2 - 2\sqrt{1 - \frac{aa}{rr}}} - \frac{a}{3r})$ nearly; where the letters denote the ſame quantities as in the above ſeries.

For ſince $P = \sqrt{2 - 2\sqrt{1 - \frac{aa}{rr}}}$ is $= \frac{a}{r} + \frac{a^3}{2.4r^3} + \frac{7a^5}{2.4.16r^5}$ &c,

and $\frac{A}{n}$ is $= \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5}$ &c;

we ſhall have, by taking the former of theſe from the latter, $\frac{A}{n} - P = \frac{a^3}{24r^3} + \frac{13a^5}{640r^5}$ &c. But, from the firſt ſeries, $\frac{1}{3}P - \frac{a}{3r} = \frac{a^3}{24r^3} + \frac{7a^5}{384r^5}$ &c; hence, by ſubtracting the latter from the former,

$\frac{A}{n} - P - \frac{1}{3}P + \frac{a}{3r} = \frac{A}{n} - \frac{4}{3}P + \frac{a}{3r} = \frac{a^5}{480r^5}$ &c; and

$A = n \times (\frac{4}{3}P - \frac{a}{3r}) = n \times (\frac{4}{3}\sqrt{2 - 2\sqrt{1 - \frac{aa}{rr}}} - \frac{a}{3r})$ nearly.

*Corollary* 8. And again, ſince $\frac{4}{105} \times (P - q - \frac{1}{8}q^3) = \frac{1}{480}q^5$ &c; (where $q$ is $= \frac{a}{r}$), by ſubtracting this from $\frac{A}{n} - \frac{4P - q}{3} = \frac{1}{480}q^5$ &c, and reducing, there will be obtained $A = \frac{n}{105} \times$

(144

In like manner, if the leg BC be made radius; then the leg AB will represent the tangent, and AC the secant of the arc BG, or of the angle C.

But

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$(144\,P - 39q - \frac{1}{2}q^3) = \frac{n}{105} \times (144\sqrt{2 - 2\sqrt{1-q^2}} - 39q - \frac{1}{2}q^3$ which will commonly give the angle true to within a minute of the truth. Where note that the constant quantity $\frac{n}{105} = .54567409$. And from the whole may be drawn the following problem.

### PROBLEM.

*To perform all the Cases of Trigonometry without any Tables.*

HAVING any three parts of a triangle given, except the three angles, the other three parts may be found by some of the following six general theorems.

1. $A = \frac{1}{3}n \times (4\sqrt{2 - 2\sqrt{1 - \frac{a^2}{r^2}}} - \frac{a}{r})$ nearly, Or

$$A = \frac{n}{105} \times (144\sqrt{2 - 2\sqrt{1 - \frac{a^2}{r^2}}} - 39\frac{a}{r} - \frac{a^3}{2r^3}) \text{ more nearly,}$$

2. $$A = n \times : \frac{a}{r} + \frac{a^3}{2.3r^3} + \frac{3a^5}{2.4.5r^5} + \frac{3.5a^7}{2.4.6.7r^7} + \frac{3.5.7a^9}{2.4.6.8.9r^9} \&c,$$

3. $$a = r \times : \frac{A}{n} - \frac{A^3}{2.3n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7} \&c,$$

4. $$r = \frac{a}{\frac{A}{n} - \frac{A^3}{2.3.n^3} + \frac{A^5}{2.3.4.5n^5} - \frac{A^7}{2.3.4.5.6.7n^7} \&c}$$

5. $$r = \frac{2abc}{\sqrt{4a^2b^2 - (a^2+b^2-c^2)^2}}$$

$$= \frac{2abc}{\sqrt{(a+b+c)\times(a+b-c)\times(a-b+c)\times(-a+b+c)}}$$

6. $$c = \sqrt{a^2+b^2-2ab\sqrt{1-(\frac{C}{n} - \frac{C^3}{2.3n^3} + \frac{C^5}{2.3.4.5n^5} - \frac{C^7}{2.3.4.5.6.7n^7} \&c)^2}}$$

Where

But if the hypotenuſe be made radius; then each leg will repreſent the ſine of its oppoſite angle; namely, the leg AB the ſine of the arc AE or angle C, and the leg BC the ſine of the arc CD or angle A.

And then the general rule for all theſe caſes, is this, namely, that the ſides bear to each other the ſame proportion, as the parts which they repreſent.

And this is called, making every ſide radius.

## SECT. III.

### OF HEIGHTS AND DISTANCES, &c.

BY the menſuration and protraction of lines and angles, we determine the lengths, heights, depths, or diſtances of bodies and objects. And this branch

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Where $a$, $b$, $c$, are the halves of the three ſides of the triangle, and A the number of degrees in the angle oppoſite the ſide $2a$, and C the degrees in the angle oppoſite the ſide $2c$; alſo $r$ is the radius of the circumſcribed circle;

and $n = \frac{180}{3{\cdot}141\,59} = 57{\cdot}2957795$, or $\frac{n}{105} = {\cdot}54567409$.

EXAMPLE.

Thus, if the three ſides be given, as for example $a = 13$, $b = 14$, $c = 15$. Then is $r = 16\frac{1}{4}$, and the angles by theſe theorems come out as follows, viz.

| Angles by the Theor. | | The true Angles. |
|---|---|---|
| 53° 7′ | angle A | 53° 7′ $\frac{1}{2}$ |
| 59 28 | angle B | 59 29 $\frac{1}{2}$ |
| 67 19 | angle C | 67 22 $\frac{1}{2}$ |
| 179 54 | ſum of all | 180 00 |

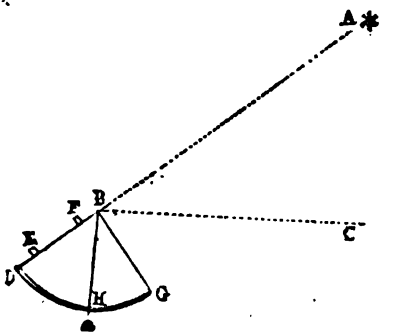
branch more immediately reſpects navigation, ſurveying, and what is commonly called altimetry and longimetry, or heights and diſtances, if indeed this muſt be diſtinguiſhed from ſurveying.

Acceſſible lines are meaſured by applying to them ſome certain meaſure, as an inch, a foot, &c. a number of times; but inacceſſible lines muſt be meaſured by taking angles, or by ſome ſuch-like method, drawn from the principles of geometry.

When inſtruments are uſed for taking the quantities of the angles in degrees, the lines are then calculated by trigonometry: in the other methods the lines are calculated from the principle of ſimilar triangles, without any regard to the quantities of the angles.

Angles of elevation, or of depreſſion, are uſually taken either with a theodolite, or with a quadrant, divided into degrees and minutes, and furniſhed with a plummet ſuſpended from the center, and two ſights fixed perpendicularly upon one of the radii.

*To take an Angle of Altitude and Depreſſion with the Quadrant.*

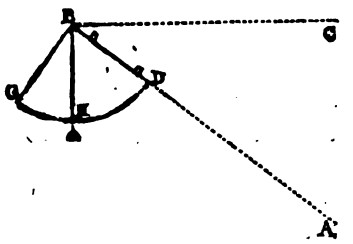



Let A be any object, as the top of a tower, hill, or other eminence; or the ſun, moon, or a ſtar; and let it be required to find the meaſure of the angle ABC which a line drawn from the object makes with the horizontal line BC.

Fix the center of the quadrant in the angular point, and move it round there as a center till with one eye at D, the other being ſhut, you perceive the

the object A through the two ſights E, F; then will the arc GH of the quadrant, cut off by the plumb line BH, be the meaſure of the angle ABC required.

The angle ABC of depreſſion of any object A is taken in the ſame manner, except that here the eye is applied to the center, and the meaſure of the angle is the arc GH.




The obſervations with the quadrant, neceſſary to determine the heights and diſtances of objects, will be ſufficiently apparent from the manner in which the following examples are propoſed; and the ſolution may eaſily be given, by any one who underſtands plane trigonometry.

The conſtruction of the figures to the following examples, are omitted; but they are to be conſtructed as in the problems of trigonometry.

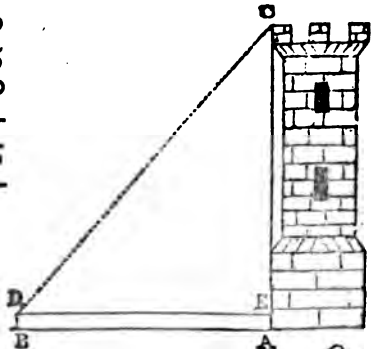
### EXAMPLE I.

Having meaſured AB equal to 100 feet from the bottom of a tower, in a direct line on a horizontal plane, I then took the angle CDE of elevation of the top, and found it to be 47° 30′, the center of the quadrant being fixed five feet above the ground: required the height of the tower.

As radius — 10·0000000
Tot. ∠ D 47° 30′ 10·0379475
So DE 100 — 2·0000000

To CE 109·13 2·0379475
Add AE 5

AC 114·13




Or,

*Or, without the Logarithms.*

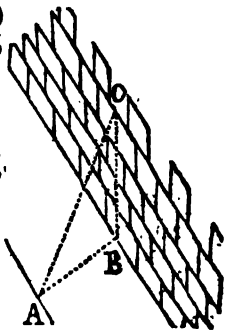
As 1 (rad.): 1·0913085 (tan. of 47° 30′) :: 100 : 100 × 1·0913085 = 109·13085 = CE.

By the calculation the height CE is found equal to 109·13, to which DB (equal to EA equal to 5 feet the height of the inſtrument) being added, gives AC equal to 114·13, the whole height.

*Note.* If you go off to ſuch a diſtance from the bottom, as that the angle of elevation ſhall be 45°, then will the height be equal to the diſtance with the height of the center of the inſtrument added.

EXAMPLE II.

From the edge of a ditch of 18 feet wide, furrounding a fort, I took the angle of elevation of the top of the wall, and found it 62° 40′: required the height of the wall, and the length of a ladder neceſſary to reach from my ſtation to the top of it.




Firſt, As 1 : 1·934702 (tan. 62° 40′) :: 18 : 1·934702 × 18 = 34·824636 = BC.

Then $\sqrt{18^2 + 34{\cdot}824636^2}$ = $\sqrt{1536{\cdot}755272}$ = 39·2014 = AC.

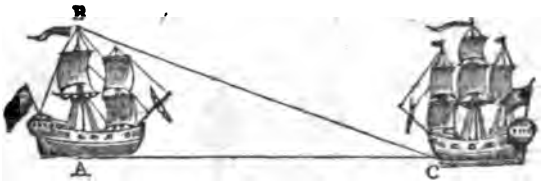
Or as 1 : 2·1778594 (ſecant of 62° 40′) :: 18 : 39·204692 = AC.

*Anſ.* The height BC = 34·82, and the length of the ladder AC = 39·2.

EXAMPLE III.

From the top of a ſhip's maſt, which was 80 feet above the water, the angle of depreſſion of another ſhip's hull, at a diſtance, upon the water, is 20°; what is their diſtance?

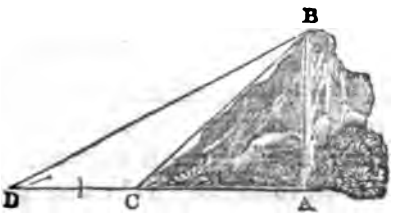
As




As 1 : 2·7474774 (tan. 70°) :: 80 : 2·7473774 × 80 = 219·790192 feet = AC the diſtance required.

EXAMPLE IV.

What is the perpendicular height of a hill whoſe angle of elevation, taken at the bottom of it, was 46°; and 100 yards farther off, on a level with the bottom of it, the angle was 31°;

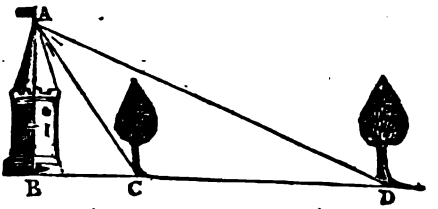



| | | |
|---|---|---|
| ∠C | 46° | } ſubtract |
| ∠D | 31 | } |
| ∠DBC | 15 | 9·4129962 |
| ∠D | 31 | 9·7118393 |
| DC | 100 | 2·0000000 |
| BC | - - | 2·2988431 |

| | | | |
|---|---|---|---|
| ∠A | 90° | — — | 10·0000000 |
| ∠C | 46 | — — | 9·8569341 |
| CB | — | — — | 2·2988431 |
| AB | 143·14 | — | 2·1557772 |

EXAMPLE V.

From the top of a tower, whoſe height was 120 feet, I took the angle of depreſſion of two trees which lay in a direct line, upon the ſame horizontal plane, with the bottom of the tower, viz. that of the nearer 57°, and that of the farther 25°½: what is the diſtance between the two trees, and the diſtance of each from the bottom of the tower?



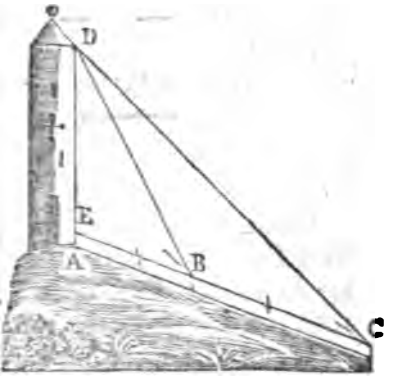

As 1 : ·6494076 (tan. ∠ BAC = 33°) :: 120 : ·6494076 × 120 = 77·928912 feet = BC the distance from the bottom of the tower to the nearest tree.

And as 1 · 2·0965436 (tan. ∠ BAD = 64° ½) :: 120 : 2·0965436 × 120 = 251·585232 feet = BD the distance of the farther tree.

Therefore, BD — BC = 251·585232 — 77·928912 = 173·65632 feet = CD the distance between the two trees.

## EXAMPLE VI.

An obelisk standing on the top of a declivity, I measured from its bottom a distance of 40 feet, and then took the angle formed by the plane and a line drawn to the top 41°; and going on in the same direction 60 feet farther, the same angle was 23° 45′, the height of the instrument being five feet: what was the height of the obelisk?




| | | |
|---|---|---|
| ∠ B 41° 00′ } subt. | | |
| ∠ C 23 45 } | | |
| ∠ BDC 17 15 | | 9·4720856 |
| ∠ C 23 45 | | 9·6050320 |
| BC 60 — | | 1·7781513 |
| BD 81·488 | | 1·9110977 |

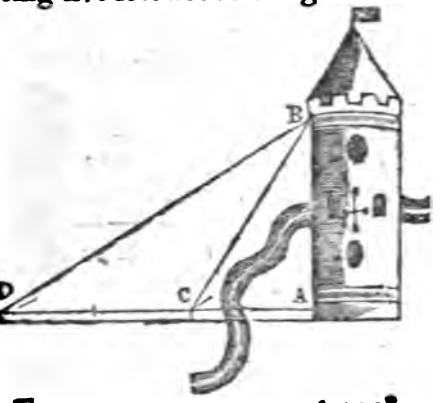
BD

| | | | | |
|---|---|---|---|---|
| BD | 81·488 | | | |
| BE | 40· | | | |
| Sum | 121·488 | — | — | 2·0845333 |
| Diff. | 41·488 | — | — | 1·6179225 |
| Tang. $\frac{E+EDB}{2}$ | 69° | 30′ | — | 10·4272623 |
| Tang. $\frac{E-EDB}{2}$ | 42 | $24\frac{1}{2}$ | — | 9·9606516 |
| Diff. = ∠EDB | 27 | $05\frac{1}{2}$ | | |
| ∠EDB | 27° | $05\frac{1}{2}$ | — | 9·6582842 |
| ∠B | 41 | 00 | — | 9·8169429 |
| BE | 40 | — | — | 1·6020600 |
| ED | 57·623 | — | — | 1·7607187 |
| add AE | 5 | | | |
| AD | 62·623 | | | |

EXAMPLE VII.

Wanting to know the height of an inacceſſible object; at the leaſt diſtance from it, upon the ſame horizontal plane, I took its angle of elevation equal to 58°, and going 100 yards directly farther from it, found the angle there to be only 32°: required its height, and my diſtance from it at the firſt ſtation, the inſtrument being five feet above the ground at each obſervation.

| | | | |
|---|---|---|---|
| ∠C | 58° | } ſubtract | |
| ∠D | 32 | | |
| ∠DBC | 26° | | 9·6418420 |
| ∠D | 32 | | 9·7242097 |
| DC | 100 | | 2· - - - - |
| BC | — | | 2·0823677 |



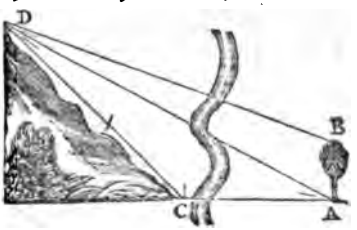
| | | | | | |
|---|---|---|---|---|---|
| ∠A | 90° | — 10·..... | ∠A | 90° | 10·------ |
| ∠C | 58 | — 9·9284205 | ∠CBA | 32° | 9·7242097 |
| BC | — | — 2·0823677 | BC | — — | 2·0823677 |
| AB | 102·51 yds | - 2·0107882 | AC | 64·05 yds | - 1·8065774 |
| | 1·66 &c yds = 5 feet | | | | |
| | 104·17 whole height | | | | |

## EXAMPLE VIII.

Wanting to know the height of, and my diſtance from, an object on the other ſide of a river, which ſeemed to be upon a level with the place where I ſtood cloſe by the ſide of the river; and not having room to go backwards, on the ſame plane, on account of the immediate riſe of the bank, I placed a mark where I ſtood, and meaſured, in a direct line from the object, up the hill, whoſe aſcent was ſo regular that I might account it for a right line, to the diſtance of 132 yards, where I perceived that I was above the level of the top of the object; I there took the angle of depreſſion of the mark by the river's ſide equal 42°, of the bottom of the object equal 27°, and of its top 19°: required the height of the object and the diſtance of the mark from its bottom.

Here 42° — 27° = 15° = ∠ADC. And 27° — 19° = 8° = ∠ADB. Alſo 90° + 19° = 109° = ∠B.

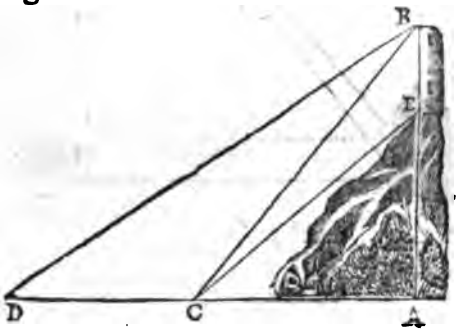
| | | | |
|---|---|---|---|
| ∠CAD | 27° | 9·6570468 | |
| ∠CDA | 15 | 9·4129962 | |
| CD | 132 | 2·1205739 | |
| CA | 75·25 | 1·8765233 | |



| | | | |
|---|---|---|---|
| ∠CAD 27° | 9·6570468 | ∠B 109° | 9·9756701 |
| ∠C 138 or 42 | 9·8255109 | ∠ADB 8 | 9·1435553 |
| CD 132 | 2·1205739 | AD | 2·2890380 |
| AD | 2·2890380 | AB 28·63 | 1·4569232 |

EXAMPLE IX.

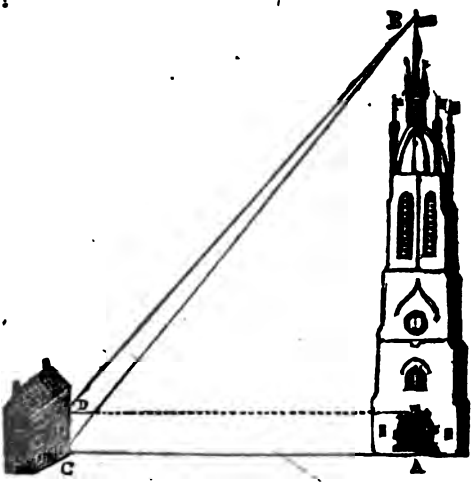
Being upon a horizontal plane, and wanting to know the height of an object on the top of an inacceſſible hill; I took the angle of elevation of the top of the hill equal 40°, and of the top of the object equal 51°; going, then, in a direct line from it to the diſtance of 100 yards farther, I found the angle of the top of the object to be 33° 45′: what is the object's height?




| | | | |
|---|---|---|---|
| ∠ACB 51° 00′ } ∠D 33 45 } subtract | | ∠BEC 130° | 9·8842540 |
| | | ∠BCE 11 | 9·2805988 |
| | | CB | 2·2726534 |
| ∠DBC 17 15 | 9·4720856 | BE 46·66574 | 1·6689982 |
| ∠D 33 45 | 9·7447390 | | |
| DC 100 | 2·0000000 | | |
| CB | 2·2726534 | | |

EXAMPLE X.

From a window near the bottom of a houſe, which ſeemed to be upon a level with the bottom of a ſteeple, I took the angle of elevation of the top of the ſteeple equal 40°, and from another window 18 feet directly above the former, the ſame angle was 37° 30′: what then is the height and diſtance of the ſteeple?




| | | | | | |
|---|---|---|---|---|---|
| From ∠C | 40° 00′ | | | | |
| Subt. ∠D | 37 30 | | | | |
| Rem. ∠CBD | 2 30 | | | | |
| ∠DBC | 2° 30′ | — | — | — | 8·6396796 |
| ∠CDB | 127½ or 52½ | — | — | — | 9·8994667 |
| DC | 18 feet | — | — | — | 1·2552725 |
| CB | — | — | — | — | 2·5150596 |

| | | | | | |
|---|---|---|---|---|---|
| ∠A | 90° | — | 10·0000000 | ∠A | 90° — 10·0000000 |
| ∠ACB | 40 | — | 9·8080675 | ∠CBA | 50 — 9·8842540 |
| CB | — | — | 2·5150596 | CB | — — 2·5150596 |
| AB | 210·44 f. | - | 2·3231271 | CA | 250·79 2·3993136 |

EX-

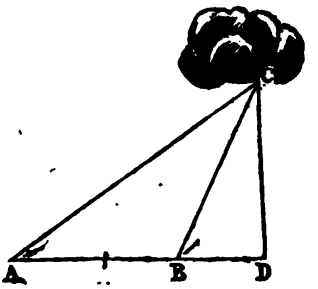
EXAMPLE XI.

What is the perpendicular height of a cloud whoſe angles of elevation are 35° and 64°, taken by two obſervers, at the ſame time, both on the ſame ſide of the cloud, and at the diſtance of 880 yards from one other, ſo placed that a vertical plane would paſs through both their ſtations and the cloud; and what is its diſtance from the two places of obſervation?

∠D 90° 90° ∠D
∠B 64 35 ∠A

∠BCD 26 55 ∠ACD
26 ∠BCD

29 ∠ACB



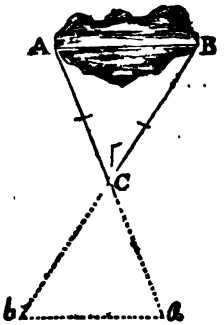

| | | | |
|---|---|---|---|
| ∠ACB | 29° | — | 9·68557 12 |
| ∠A | 35 | — | 9·7585913 |
| AB | 880 | — | 2·9444827 |
| BC | 1041·125 | - | 3·0175028 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| ∠ACB | 29° | — | 9·6855712 | ∠D | 90° | — | 10·0000000 |
| ∠ABC | 116 | — | 9·9536602 | ∠B | 64 | — | 9·9536602 |
| AB | 880 | — | 2·9444827 | BC | | — | 3·0175028 |
| AC | 1631·442 | | 3·2125717 | DC | 935·757 | - | 2·9711630 |

*Note.* In finding the diſtances of inacceſſible objects, if they be of ſuch a height as to admit of a pretty large angle of elevation, their diſtance will be found as in ſome of the foregoing examples.—If not, the theodolite, or ſome ſuch inſtrument, is uſed to take the angles of the diſtance, or the horizontal angles, of objects, as in the following examples.

EXAMPLE XII.

Suppose I wanted to know the distance of the two places A, B, to whose ends there is free access, but not to the intermediate parts, because of a hill, precipice, or water, between A and B; and that therefore I measured from A and B, to any convenient place C, the distance AC, equal 7·35 chains, and BC equal 8·4 chains, and found the angle ACB equal 55° 40′. What is the distance of the places A, B?




8·40
7·35

| | | | |
|---|---|---|---|
| Sum | 15·75 | — | 1·1972806 |
| Diff. | 1·05 | — | 0·0211893 |
| Tang. $\frac{A+B}{2}$ | 62° 10′ | - | 10·2773793 |
| Tang. $\frac{A-B}{2}$ | 7. $11\frac{11}{14}$ | - | 9·1012880 |
| ∠A | $69\ 21\frac{11}{14}$ | sum | |

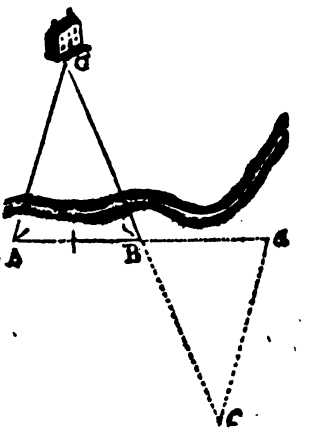
| | | | | |
|---|---|---|---|---|
| ∠A | 69° $21\frac{11}{14}'$ | — | — | 9·9711982 |
| ∠C | 55 40 | — | — | 9·9168593 |
| CB | 8·4 chains | — | — | 0·9242793 |
| AB | 7·412 chains | — | — | 0·8699404 |

*Note.* If the lines AC, BC, be produced to *a* and *b*, till C*a*, C*b*, be equal to CA, CB, or equal to CB, CA; then the distance *ba*, will be equal to the distance AB, and therefore AB will be obtained, without any calculation, by only measuring *ba*.

EXAMPLE XIII.

Being on the side of a river, and wanting to know the distance to a house which stood on the other side, I measured 200 yards in a right line by the side of the river, and found that the two angles, at each end of this line, formed by the other end and the

the houſe, were 73° 15′ and 68° 02′: What was the diſtance between each ſtation and the houſe?

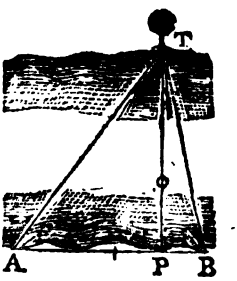



| | | | |
|---|---|---|---|
| ∠A | 73° 15′ | | |
| ∠B | 68 02 | | |
| ſum | 141 17 | | |
| from | 180 00 | | |
| ∠C | 38 43 | — | 9·7962062 |
| ∠A | 73 15 | — | 9·9811711 |
| AB | 200 | — | 2·3010300 |
| BC | 306·19 | — | 2·4859949 |
| ∠C | 38° 43′ | — | 9·7962062 |
| ∠B | 68 02 | — | 9·9672679 |
| AB | 200 | — | 2·3010300 |
| AC | 296·54 | — | 2·4720917 |

*Note.* If, in the right line AB*a*, you meaſure B*a* equal AB, and the line CB*c* be produced until the angle *a* be equal to the angle A; the diſtances B*c*, *ac*, will be equal to BC, AC.

### EXAMPLE XIV.

Wanting to know the breadth of a river, I meaſured 100 yards in a ſtraight line cloſe by one ſide of it; and at each end of this line I found the angles ſubtended by the other end and a tree cloſe by the other ſide of the river, to be 53° and 79° 12′. What is the perpendicular breadth?




| | | | |
|---|---|---|---|
| ∠A | 53° 00′ | | |
| ∠B | 79 12 | | |
| ſum | 132 12 | | |
| from | 180 | | |
| ∠ATB | 47 48 | — | 9·8697037 |
| ∠B | 79 12 | — | 9·9922385 |
| AB | 100 | — | 2·0000000 |
| AT | — — | — | 2·1225348 |

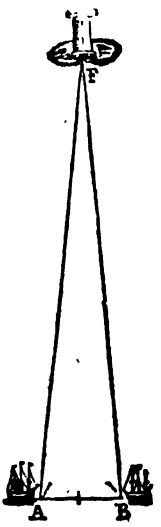
∠P

| | | | |
|---|---|---|---|
| ∠P | 90° | — | 10·0000000 |
| ∠A | 53 | — | 9·9023486 |
| AT | | — | 2·1225348 |
| TP | 105·89 | — | 2·0248834 |

EXAMPLE XV.

Two ſhips of war intending to cannonade a fort, are, by the ſhallowneſs of the water, kept ſo far from it, that they ſuſpect their guns cannot reach it; in order therefore to meaſure the diſtance, they ſeparate from each other half a mile or 880 yards; then each ſhip obſerves the angles which the other and the fort ſubtends, and finds them to be 85° 15′ and 83° 45′: What is the diſtance between each ſhip and the fort?

| | | | |
|---|---|---|---|
| ∠A | 83° 45′ | | |
| ∠B | 85 15 | | |
| ſum | 169 00 | | |
| from | 180 00 | | |
| ∠F | 11 00 | — | 9·2805988 |
| ∠A | 83 45 | — | 9·9974110 |
| AB | 880 | — | 2·9444827 |
| BF | 4584·5 | — | 3·6612949 |
| ∠F | 11° 00′ | — | 9·2805988 |
| ∠B | 85 15 | — | 9·9985058 |
| AB | 880 | — | 2·9444827 |
| AF | 4596·1 | — | 3·6623897 |

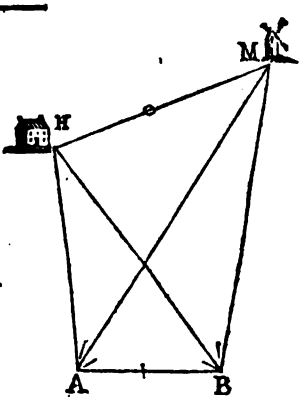



EXAMPLE XVI.

Wanting to know the diſtance between a houſe and a mill, which were ſeparated from me by a river, I took

I took another ſtation B at the diſtance of 300 yards from the firſt ſtation A: now from the firſt ſtation A, the angle ſubtended by B and the mill was 58° 20′, and by the mill and the houſe 37°; from B, the angle ſubtended by A and the houſe was 53° 30′, and by the houſe and the mill 45° 15′. What is the diſtance of the houſe and mill?

| | | | |
|---|---|---|---|
| 37° 00′ | 58° 20′ | 53° 30′ | |
| 58 20 | 53 30 | 45 15 | |
| 53 30 | 45 15 | 98 45 | ∠ABM |
| 148 50 | 157 05 | | |
| 180 00 | 180 00 | | |
| ∠AHB 31 10 | 22 55 | AMB | |




| | | |
|---|---|---|
| ∠AHB | 31 10 | - 9·7139349 |
| ∠ABH | 53 30 | - 9·9051787 |
| AB | 300 | - 2·4771213 |
| AH | 465·9776 | - 2·6683651 |

| | | | | |
|---|---|---|---|---|
| ∠AMB | 22° 55′ | | — | 9·5903869 |
| ∠ABM | 98 45 or 81 15 | | | 9·9949158 |
| AB | 300 | — | — | 2·4771213 |
| AM | 761·4655 | — | — | 2·8816502 |
| AH | 465·9776 | | | |
| Sum | 1227·4431 | — | — | 3·0890013 |
| Dif. | 295·4879 | — | — | 2·4705397 |
| Tan. | $\frac{AHM+AMH}{2}$ | 71° 30′ | — | 10·4754801 |
| Tan. | $\frac{AHM-AMH}{2}$ | 35 44 | — | 9·8570185 |
| | AMH | 35 46 | | |

∠AMH

| | | | | |
|---|---|---|---|---|
| ∠AMH | 35° 46′ | — | — | 9·7667739 |
| ∠HAM | 37 00 | — | — | 9·7794630 |
| AH | — | — | — | 2·6683651 |
| HM | 479·7933 yds | | — | 2·6810542 |

*Note.* Pretty much after the manner of theſe laſt examples many curious and uſeful problems may be reſolved: for if we can determine our diſtance from one remote object, we can do the ſame for any number of objects; or if the diſtance between two remote objects can be determined, thoſe between any number of objects may be determined likewiſe: ſo, we may determine the angles and ſides of fields, or of very large tracts of land, and that whether we be within them, or any where without them, from whence the angles can be ſeen; hence alſo ſhips at ſea may determine their diſtances from known viſible ports; and plans may be taken of countries, towns, harbours, fleets, fortifications, &c.

## EXAMPLE XVII.

Suppoſe I want to know the breadth of a river, or my diſtance from an inacceſſible object O, and that I have no inſtrument for taking angles, but only a chain or chord for meaſuring diſtances; and ſuppoſe that from each of the two ſtations A, B, which are 500 yards aſunder, I meaſure in a direct line from the object O 100 yards, viz. A*a* and B*b* each equal to 100 yards, and that the diagonal A*b* meaſures 550 yards, and the diagonal *a*B meaſures 560: What then is the diſtance of the object from each ſtation A and B?

Let

Let fall the perpendiculars AP, BQ. Then, in the triangle ABa,

$$Ba : BA + Aa :: BA - Aa : BP - Pa,$$

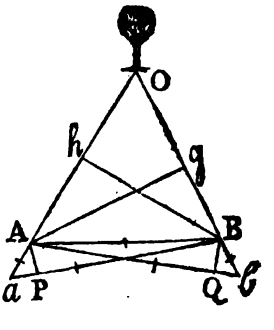
that is $560 : 600 :: 400 : 428\frac{4}{7} = BP - Pa,$

its half $214\frac{2}{7}$

half sum $280$

$BP - 494\frac{2}{7}$

$aP - 65\frac{5}{7}$




And

$$BP = \frac{560 + 428\frac{4}{7}}{2} = \frac{988\frac{4}{7}}{2} = 494\frac{2}{7},$$

$$Pa = \frac{560 - 428\frac{4}{7}}{2} = \frac{131\frac{3}{7}}{2} = 65\frac{5}{7},$$

Then

| | | | | | |
|---|---|---|---|---|---|
| Aa 100 : aP | $65\frac{5}{7}$ | :: 1 : S. | ∠aAP | 41° | 5′ |
| AB 500 : BP | $494\frac{2}{7}$ | :: 1 : S. | ∠BAP | 81 | 20 |
| | | the sum | | 122 | 25 |
| | | taken from | | 180 | 00 |
| | | leaves | ∠BAO | 57 | 35 |

Again, in the triangle ABb,

$$Ab\ 550 : AB : Bb\ 600 :: AB - Bb\ 400 : AQ - QB\ 436\frac{4}{11}$$

the half dif. $218\frac{2}{11}$

half sum $275$

$AQ - 493\frac{2}{11}$

$Qb - 56\frac{9}{11}$

Then

Then

| | | | |
|---|---|---|---|
| B*b* 100 : *b*Q 56$\frac{9}{11}$ :: 1 : s. ∠*b*BQ | 34° | 37′ |
| BA 500 : AQ 493$\frac{2}{11}$ :: 1 : s. ∠ABQ | 80 | 32 |
| the ſum | 115 | 9 |
| taken from | 180 | 0 |
| leaves ∠ABO | 64 | 51 |
| add BAO | 57 | 35 |
| the ſum | 122 | 26 |
| taken from | 180 | 00 |
| *leaves ∠O | 57 | 34 |

Whence

| | | | |
|---|---|---|---|
| ∠O 57° 34′ | 9·9263507 | ∠O 57° 34′ | 9·9263507 |
| ∠A 57 35 | 9·9264310 | ∠B 64 51 | 9·9567437 |
| AB 500 | 2·6989700 | AB 500 | 2·6989700 |
| BO 500·1 | 2·6990503 | AO 536·25 | 2·7293630 |

EX-

* Or *the angles* ABO *and* BAO *may be otherwiſe found thus*:

Draw the perpendiculars A*g*, B*b*: Then by Eucl. II. 12,

$$Bg = \frac{Ab^2 - AB^2 - bB^2}{2bB} = 212\frac{1}{2};$$

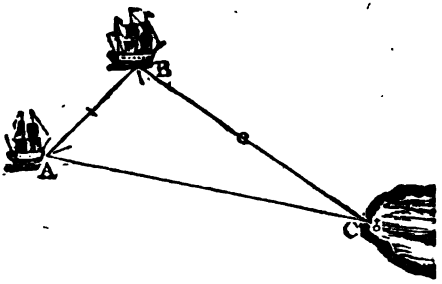
And AB : B*g* :: 1 : ·425 = coſine of 64° 51′ the ∠ABO.

In like manner $A b = \frac{Ba^2 - AB^2 - Aa^2}{2aA} = 268$;

And AB : A*b* :: 1 : ·536 the coſine of 57° 35′ the ∠BAO; both the ſame as above.

### EXAMPLE XVIII.

From a ſhip at ſea I obſerved a point of land to bear *E.* by *S.* and after ſailing *N. E.* 12 miles, I ſet it again, and found its bearing to be *S. E.* by *E.* How far was the laſt obſervation made from the point of land?




Here are given the angle A, at the place of the firſt obſervation, 5 points or 56° 15′; the angle B, 9 points or 101° 15′; and the angle C, 2 points or 22° 30′; alſo the ſide AB 12 miles: to find the ſide BC.

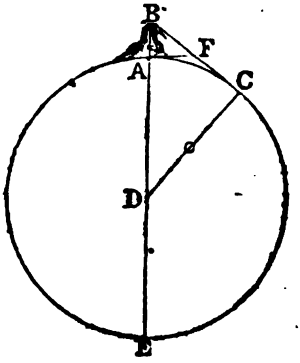
| | | |
|---|---|---|
| As s. ∠C 22° 30′ | — | 9·5828397 |
| To s. ∠A 56 15 | — | 9·9198464 |
| So is AB 12 miles | — | 1·0791812 |
| To BC 26·07281 miles | | 1·4161879 |

### EXAMPLE XIX.

If the height AB of the mountain called the pike of Teneriff be 4 miles, and the angle ABC made by a plumb line and a line BC conceived to touch the earth in the fartheſt viſible point C, be 87° 25′ 55″; required the diameter AE of the earth, and the utmoſt diſtance BC that can be ſeen from the top of the mountain, ſuppoſing the earth to be a perfect ſphere.

Draw

Draw AF perpendicular to AB, then from the principles of geometry it is known that CF is = FA; draw, alſo, the radius CD, which will be perpendicular to CB.




Hence from 90° 0′ 0″
take ∠B 87 25 55

∠D or AFB 2 34 5

Then as radius
1 : AB 4 :: 2·22960 tang. ∠B : 89·184 AF or FC
1 : AB 4 :: 2·23185 ſec. ∠B : 89·274 BF

their ſum 178·458 BC

Laſtly, As radius — — — 10·0000000
To t. ∠B — — — 11·3482280
So BC 178·458 — 2·2515360

To CD 3978·909 — 3·5997640
2

7957·818 diameter of the earth.

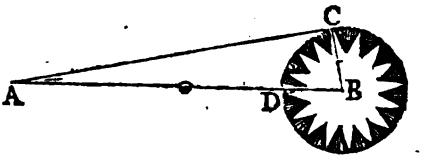
*Note.* This method of determining the magnitude of the earth, is very eaſy and ſimple; but to do it with any tolerable degree of exactneſs, the height of the mountain muſt be very accurately aſcertained, and the angle at the top muſt be taken with a quadrant divided into minutes and ſeconds, and furniſhed with a teleſcope, inſtead of the common ſights.

### EXAMPLE XX.

According to Sir Iſaac Newton, the diameter of the ſun, at a mean diſtance from the earth, ſubtends an

an angle of $32' 15''$: Then how many times his diameter in length is his mean distance from the earth equal to?

Here, in the triangle ABC, we have given CB $= \frac{1}{2}$ a diameter, the angle $C = 90°$, and the angle $A = 16' 7\frac{1}{2}''$.




| | | |
|---|---|---|
| Hence, As s. ∠A $16' 7\frac{1}{2}''$ | — | 7·6711356 |
| To s. ∠C $90°$ | — | 10·0000000 |
| So is CB 1 semi-diam | — | 0·0000000 |
| To BA 213·2379 | — | 2·3288644 |

That is, the mean distance of the sun's center is 213·2379 semi-diameters, or 106·6189 of his whole diameters. And if from AB be taken BD or $\frac{1}{2}$, we shall have remaining nearly 106 diameters for the distance of the surface.

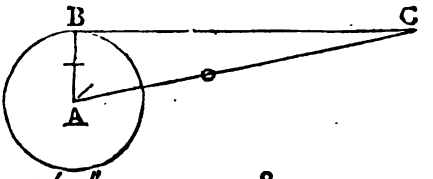
*Note.* After the same manner may be calculated the proportion of the diameter to the distance of any other celestial body, whose apparent diameter is large enough to be measured. And, in particular, the mean distance of the moon, whose mean apparent diameter is $31' 16''$, will be found to be 109·95 times her diameter.

### EXAMPLE XXI.

If when the moon appears in the horizon to a spectator on the earth, at her mean distance from it, her zenith distance, as calculated from astronomical tables, be $89° 2' 55''$; it is required to find how many of the earth's semi-diameters the said mean distance of the moon is equal to.

Here

Here AB is = one ſemi-diameter of the earth, and the ∠A = 89° 2′ 55″.




| | | |
|---|---|---|
| Hence, As s. ∠C. 57′ 5″ | — | 8·2202155 |
| To s. ∠B 90° | — | 10·0000000 |
| So is AB 1 ſemi-diam. | | 0·0000000 |
| To AC 60·226 | — | 1·7797845 |

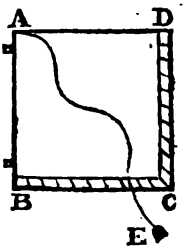
That is, the diſtance of their centers is 60·226 of the earth's ſemi-diameters, or $30\frac{1}{9}$ whole diameters.

*Note.* Hence, and from the note to the laſt example, it appears that the diameter of the earth is to that of the moon as 109·95 to $30\frac{1}{9}$, or as 11 to 3, or as $3\frac{2}{3}$ to 1 nearly.—Conſequently, as $3\frac{2}{3}$ : 1 :: 7958 (the earth's diameter in miles) : 2170 miles = the moon's diameter.—Likewiſe the ſurface of the earth is to that of the moon as ($3\frac{2}{3} \times 3\frac{2}{3}$ or) $13\frac{4}{9}$ to 1 nearly; or, the earth reflects upon the moon about $13\frac{1}{2}$ times as much light as the moon does upon the earth.—Moreover, the bulk of the earth is to that of the moon as ($3\frac{2}{3} \times 3\frac{2}{3} \times 3\frac{2}{3}$ or) 48 to 1 nearly.

## GENERAL SCHOLIUM.

There are many other methods and inſtruments uſed to find the altitudes and diſtances of objects, ſome of which are here ſubjoined.

1. One very eaſy method is by a ſquare with a plummet AE ſuſpended from one corner A, and the two ſides BC, DC, meeting in the oppoſite angle C, divided into 10, or 100, or 1000 equal parts; and two ſights on the ſide AB.

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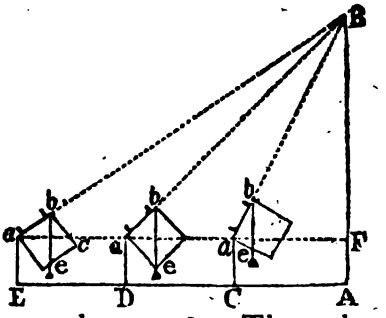


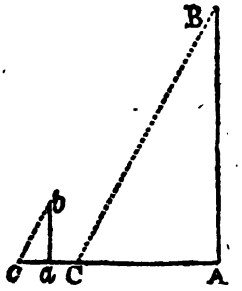

It is evident that, in taking any altitude AB with the ſquare, the plummet will always cut off from the ſquare a triangle ſimilar to that formed by the baſe line *a*F, the perpendicular FB, and B*a*.

If the angle B*a*F be equal to 45°. Then the plumb line will paſs through the oppoſite angle of the ſquare, and the diſtance DA will be equal to the altitude BF: So if DA be 60, then BF will be 60 alſo.

If the angle be greater than 45°, as at the ſtation C. Then the part of the ſide cut off *ae*, will be to the whole ſide *ab*, as *a*F, to FB: So if *ae* be 6 diviſions of which *ab* is 10, and the diſtance CA be 36 feet; then 6 : 10 :: 36 : 60 feet = the altitude BF.

If the angle be leſs than 45°, as at the ſtation E. Then the part is cut off the other decimated ſide, and *bc* : *ce* :: EA : BF: So if the parts cut off be 6, and the diſtance 100 feet, then 10 : 6 :: 100 : 60 feet = BF.

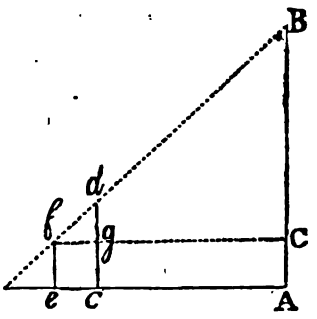
2. Another method is by ſhadows, from the property of ſimilar triangles alſo. For any object, and a pole ſet up parallel to it, are in proportion to each other as the length of their ſhadows, formed by the ſun, &c.




Let the height of the pole *ab* be 6 feet, the length of its ſhadow *ac* 4, and the ſhadow AC of the altitude 40 feet; then *ca* = 4 : *ab* = 6 :: CA = 40 : AB = 60.

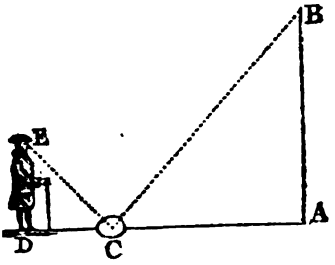
3. Another method is by two poles ſet up parallel to the object, the one longer than the other, ſo that the obſerver may ſee the top of the object over the tops of both the poles.

Let the pole *ef* be = 4 feet, *cd* = 7 feet, their diſtance aſunder *ec* = *fg* = 8 feet, and the diſtance *e*A of the ſhorter pole from the object = 160 feet.—Then, the triangles *fgd*, *f*CB being ſimilar, *fg* : *gd* :: *f*c : CB, that is 8 : 3 (= 7 − 4) :: 160 : 60 feet = BC; hence BC + CA = BC + *fe* = 60 + 4 = 64 = AB.



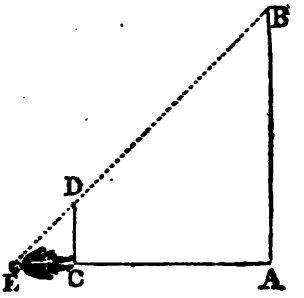

4. A fourth method is by viewing the image of the top of the object reflected from a ſmooth ſurface, as a mirror placed horizontally, or a veſſel of water.

Let c be the reflecting ſurface, at the diſtance of 84 feet from the bottom of the object AB; and let a perſon at D, 7 feet from c, with his eye 5 feet above the ground, view the image of the object at c; then becauſe the triangles CDE, CAB are ſimilar, agreeably to a principle of the opticians, we ſhall have CD : DE :: CA : AB, that is 7 : 5 :: 84 : 60 feet = AB.

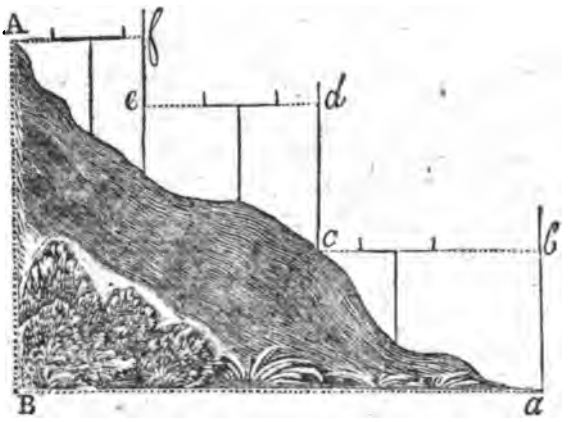



5. A

5. A fifth method is for the obſerver to fix a pole CD, equal, in length, to the height of his eye, perpendicularly at C, by trials, at ſuch a diſtance from A, that having laid himſelf upon his back, with his feet againſt the bottom of the pole, he may ſee the tops D and B of the pole and object in the ſame line: for then EA will be equal to AB, becauſe EC is equal to CD.—Or the pole may be of any length; for the diſtance from the foot to the eye of the obſerver will be in proportion to the height of the pole, as the whole diſtance is to the height of the object required.




6. When the perpendicular altitude of an irregular hill or aſcent is to be aſcertained; or when, in levelling, for conducting water, &c. it is required to find how much one aſſigned place is above another; a level and perpendicular poles, or objects, commonly are uſed.




So the level being fixed horizontally, and the places *b*, *c*, *d*, &c ſeen upon the poles through the ſights of the level, being marked; the heights *ab*, *cd*, *ef*, added all together, will give the whole height AB of A above *a*; and the diſtances *bc*, *de*, *f*A, added together, will give the horizontal diſtance *a*B of A from *a*.

Another method of levelling is ſometimes uſed, viz. The angles of elevation or depreſſion are taken, from ſtation to ſtation, with an arc of a circle, having a plummet pendent from its center, and the ſeveral diſtances, being meaſured upon the ground, are the hypotenuſes of the right-angled triangles, of which the angles at the baſes are expreſſed by the aforeſaid obſerved angles; and conſequently, the perpendiculars of theſe triangles being calculated, their ſum will be the difference of level between the extreme places, if the angles were all of elevation, or all of depreſſion; but if the angles be ſome of elevation, and ſome of depreſſion, the perpendiculars which belong to the angles of the ſame kind being added together, will give two ſums, whoſe difference will be the difference of level required.

Theſe

Theſe two methods, when accurately performed, are both very juſt, at leaſt when the ſtations are at no great diſtance from each other; yet the former is more to be depended on than the latter, in as much as the obſervations required by it, are leſs ſuſceptible of error; for beſides the inaccuracy of the meaſured hypotenuſes, on account of the unevenneſs of the ground, the angles themſelves can hardly be taken, with the generality of inſtruments for this purpoſe, to leſs than quarters of degrees.

But when the places, of which we would know whether of them is the higher, are far diſtant from each other, inſtruments of greater accuracy are to be uſed, ſuch as very exact levels with teleſcopes; but in ſuch caſes an allowance muſt be made for the rotundity of the earth; for the true water-level courſe is determined by a line of which every part is at the ſame diſtance from the center of the earth, and which is, therefore, an arc of a great circle of the earth, conſidering it not as an oblate ſpheroid, but as a ſphere, its difference from that form having, in this caſe, no ſenſible effect; and the allowance neceſſary to be made in the level, is at the rate of 8 inches nearly to a mile meaſured upon the earth; for the viſual line, when the level is properly fixed, being a tangent to the earth, or at leaſt parallel to one, that line at the diſtance of one mile from the ſtation is 7·96 inches above the water-level line, or ſo much farther from the center of the earth than the inſtrument is. And for other diſtances, above or below a mile, the allowance for the level varies in proportion as the ſquare of the diſtance.

It may be neceſſary farther to obſerve, that in taking, at ſeveral ſtations, the difference of level between two places, it is not neceſſary to go in a line from the one to the other, but every two ſucceſſive ſtations may be taken in any direction that may ſeem moſt convenient.

## PRACTICAL QUESTIONS IN TRIGONOMETRY, &c.

QUESTION 1. A may-pole, 50 feet 11 inches high, at a certain time will caſt a ſhadow 98 feet 6 inches long; what then is the breadth of a river, which, running within 20 feet 6 inches of the foot of a ſteeple, 300 feet 8 inches high, will, at the ſame time, throw the extremity of its ſhadow 30 feet 9 inches beyond the ſtream? Anſ. 530 feet 5 inches.

QUEST. 2. Required the length of a ſhoar, which being to ſtrut 11 feet from the upright of a building, will ſupport a jamb 23 feet 10 inches from the ground. Anſ. 26 ft 3 inc.

QUEST. 3. A line 27 yards long will exactly reach from the top of a fort, to the oppoſite bank of a river, known to be 23 yards broad: what is the height of the wall? Anſ. 42 ft 5 inc.

QUEST. 4. Two ſhips ſet ſail from the ſame port, one of them goes 50 leagues due eaſt, and the other 84 leagues due north: how far are they then aſunder? Anſ. $97\frac{3}{4}$ leagues.

QUEST. 5. The height of an elm, growing in the center of a circular iſland, 30 feet in diameter, plumbs 53 feet; and a line of 112 feet long ſtretched from the top of the tree ſtraight to the nearer edge of the water: required the breadth of the moat, ſuppoſing the land on either ſide of the water, to be level. Anſ. $83\frac{3}{7}$ feet.

QUEST. 6. Suppoſe a light-houſe, built on the top of a rock, the diſtance between the place of obſervation, and that part of the rock level with the eye, and directly under the building, is given 310 fathoms; the diſtance from the top of the rock, to the place of obſervation, is 423 fathoms; and from the top of the building 425: required the height of the edifice. Anſ. 17 ft 7 inc.

QUEST.

QUEST. 7. A ladder, 40 feet long, may be ſo planted, that it ſhall reach a window 33 feet from the ground, on one ſide of the ſtreet; and by turning it over, without moving the foot out of its place, it will do the ſame by a window 21 feet high, on the other ſide: required the breadth of the ſtreet.

Anſ. 56·649 feet.

QUEST. 8. There are two columns left ſtanding upright in the ruins of Perſepolis; the one is 64 feet above the plane, and the other 50: In a right line between theſe ſtands an antient ſtatue, the head of which is 97 feet from the ſummit of the higher, and 86 from that of the lower column; the baſe of which meaſures juſt 76 feet to the center of the figure's baſe: required the diſtance between the tops of the two columns. Anſ. 157 feet.

QUEST. 9. A may-pole, whoſe top was broken off by a blaſt of wind, ſtruck the ground at 15 feet diſtance from the foot of the pole: what was the height of the whole may-pole, ſuppoſing the length of the broken piece to be 39 feet? Anſ. 75 feet.

QUEST. 10. Suppoſe the breadth of a well at the top be 6 feet, and the angle formed by its ſide and a viſual diagonal line from the edge at top to the oppoſite ſide at the bottom, $18^{\circ}\ 30'$: required the depth of the well. Anſ. 17·89 feet.

QUEST. 11. At 85 feet diſtance from the bottom of a tower, the angle of its elevation was found to be $52^{\circ}\ 30'$: required the altitude of the tower.

Anſ. $110\frac{1}{2}$ feet.

QUEST. 12. At a certain place the angle of elevation of an inacceſſible tower was $26^{\circ}\ 30'$; then meaſuring 75 in a direct line towards it, the angle was then found to be $51^{\circ}\ 30'$: required the height of the tower, and its diſtance from the laſt ſtation.

Anſ. { height 62, diſtance 49.

QUEST. 13. To find the diſtance of an inacceſſible caſtle gate, I meaſured a line of 73 yards, and at each end of it took the angle of poſition of the object and the other end, and found the one to be 90°, and the other 61° 45′: required the diſtance of the caſtle from each ſtation.

Anſ. { 135·8, 154·2.

QUEST. 14. From the top of a tower by the ſea ſide of 143 feet high, I obſerved that the angle of depreſſion of a ſhip's bottom, then at anchor, was 55°; what was its diſtance from the bottom of the wall?

Anſ. 204·56 feet.

QUEST. 15. How far at ſea can the pike of Teneriff be ſeen, its height being 4 miles, and the radius of the earth 3978$\frac{7}{8}$ miles? Anſ. 178$\frac{1}{2}$ miles.

QUEST. 16. If a ſhip, in the latitude of 50° north, ſail 52 miles in the direction s wby s: what latitude is ſhe arrived in, and how much farther to the weſt?

Anſ. { lat. 49° 16·8′, weſt 28·9 miles.

QUEST. 17. Sailing w s w, I ſaw, at ſome diſtance, a point of land, which I ſet, and found its bearing w by N; and after ſailing 6 leagues farther, I ſet it again, and found its bearing N w by w. Required its diſtance. Anſ. 26·13 miles.

QUEST. 18. Obſerving three ſteeples, A, B, C, in a town at a diſtance, whoſe diſtances aſunder are known to be as follows, namely, AB 106$\frac{1}{2}$, AC 202, and BC 131 fathoms, I took their angles of poſition from the place where I ſtood D, which was neareſt the ſteeple B, and found the angle ADB 13° 30′, and the angle CDB 29° 50′. Required my diſtance from each of the three ſteeples.

Anſ. { DA 302·8 DB 214·8 DC 262·0

QUEST. 19. Suppoſing my ſtation to be fartheſt from the ſteeple B, required to find the diſtances from it, when the diſtance AB is 9 furlongs, AC 12,

and BC 6 furlongs; alſo the angle ADB 33° 45′, and the angle CDB 22° 30′.

Anſ. { DA 10·64
DB 15·64
DC 14·01

QUEST. 20. Two ſhips ſail from the ſame port; the one ſails E N E 85 miles, the other ſails E by S till the firſt ſhip bears N W by W: what is the diſtance of the ſecond ſhip from the port, and alſo from the firſt ſhip?

Anſ. { from the port 184·7
from the 1ſt ſhip 123·4

QUEST. 21. Two ports lie eaſt and weſt of each other: a ſhip ſails from each, namely, the ſhip from the weſt port ſails N E 89 leagues, and the other ſails 80 leagues, when ſhe meets the former: required the latter ſhip's courſe, and the diſtance between the two ports.

Anſ. { courſe 51° 52′
diſtance 112·3

QUEST. 22. Two ſhips ſail from a certain port; the one ſails S by E 45 leagues, and the other S S W 64 leagues. What then are their bearings and diſtance aſunder?

Anſ. { bearing 43° 14′
diſtance 36·5

QUEST. 23. A ſhip ſailing N W, two iſlands appear in ſight, of which the one bore N, and the other W N W; but after ſailing 20 leagues, the former bore N E, and the latter W by S. What is the diſtance aſunder of the two iſlands?

Anſ. 32·38 leagues.

PART

# PART II.

## OF SUPERFICIAL MENSURATION, OR THE MENSURATION OF PLANE FIGURES.

### SECTION I.

#### OF THE AREAS OF RIGHT-LINED AND CIRCULAR FIGURES.

THE meaſure of a plane figure is called its area. By the menſuration of plane figures is determined the extenſion of bodies as to length and breadth; ſuch as the quantities of lands, and the works of many artificers.

Plane figures, and the ſurfaces of bodies, are meaſured by ſquares; as ſquare inches, or ſquare feet, or ſquare yards, &c; that is, ſquares whoſe ſides are inches, or feet, or yards, &c. Our leaſt ſuperficial meaſure is the ſquare inch, other ſquares being taken from it according to the proportion in the following table.

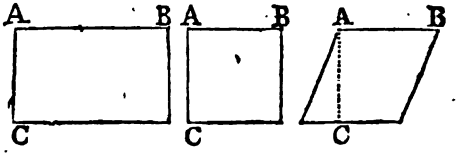
*Table of Square Meaſure.*

| Square Inches | Sq. Feet | S. Yards | S. Poles | S. Cha. | Acres | S. Mile |
|---|---|---|---|---|---|---|
| 144 | 1 | | | | | |
| 1296 | 9 | 1 | | | | |
| 39204 | 272¼ | 30¼ | 1 | | | |
| 627264 | 4356 | 484 | 16 | 1 | | |
| 6272640 | 43560 | 4840 | 160 | 10 | 1 | |
| 4014489600 | 27878400 | 3097600 | 102400 | 6400 | 640 | 1 |

PRO-

## PROBLEM I.

*To find the Area of a Parallelogram, whether it be a Square, a Rectangle, a Rhombus, or a Rhomboid.*




### RULE I.

* Multiply the length by the height or perpendicular breadth, and the product will be the area.

That is, AB × AC = the area.

*Note.* Becauſe the length of a ſquare is equal to its height, its area will be found by multiplying the ſide by itſelf.—That is AB × AB or $AB^2$ is the area of the ſquare.

EX-

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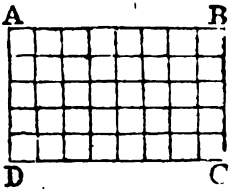
### * DEMONSTRATION.




For, let ABCD be a rectangle; and let its length AB and CD, and its breadth AD and BC, be each divided into as many equal parts, as is expreſſed by the number of times they contain the lineal meaſuring unit; and let all the oppoſite points of diviſion be connected by right lines.—Then, it is evident that, theſe lines divide the rectangle into a number of ſquares, each equal to the ſuperficial meaſuring unit; and that the number of theſe ſquares is equal to the number of lineal meaſuring units in the length, as often repeated as there are lineal meaſuring units in the breadth, or height; that is, equal to the length drawn into the breadth. But the area is equal to the number of ſquares or ſuperficial meaſuring units; and therefore the area of a rectangle is equal to the product of its length and breadth.

Again, a rectangle is equal to an oblique parallelogram of an equal length and perpendicular height, by Euclid I. 36.

Therefore the area of every parallelogram is equal to the product of its length and height. *Q. E. D.*

EXAMPLES.

1. What is the area of a parallelogram whoſe length is 12·25 chains, and its height 8·5 chains?

12·25 length
8·5 breadth

6125
9800

10) 104·125 ſquare chains

10·4125 acres
4

1·6500 roods
40

26·0000 perches

a r p
Anſ. 10 1 26

*Note.* Four roods are equal to an acre, and therefore 40 perches, or ſquare poles, make a rood.

Ex. 2. What is the area of a ſquare whoſe ſide is 35·25 chains? Anſ. 124ac. 1r. 1p.

Ex. 3. What is the area of a rectangular board, whoſe length is 12·5 feet, and breadth 9 inches?

Anſ. 9·375 ſq. feet.

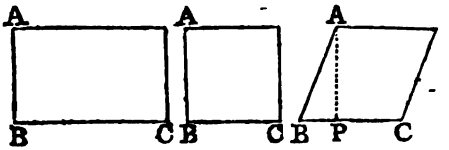
Ex. 4. How many ſquare yards of painting are in a rhombus, or a rhomboid, whoſe length is 37 feet, and perpendicular breadth 5·25 feet?

Anſ. 21·58$\frac{1}{3}$.

RULE II.

As radius (viz. ſine of 90° or tang. of 45°) :
Is to the ſine of any angle of a parallelogram ::
So is the product of the ſides including the angle:
To the area of the parallelogram.

That




* That is, AB × BC × nat. ſine of the angle B = the area.

*Note.* Becauſe the angles of a ſquare and rectangle are each 90°, whoſe ſine is 1, this rule, for them, is the ſame as the former.

EXAMPLES.

1. What is the area of a rhomboides whoſe length is 36 feet, ſlope height 25·5 feet, and one of the leſs angles 58°?

(Rad.) 1 : ·8480481 (nat. ſine of 58°) :: 918 (= 25·5 × 36) : 778·5081558 the area.

*Or, to uſe the Logarithms.*

| | | | |
|---|---|---|---|
| Radius | — | — | 10·0000000 |
| Sine of 58° | — | — | 9·9284205 |
| 918 | — | — | 2·9628427 |
| 778·5081 | — | — | 2·8912632 |

Ex. 2. What is the area of a parallelogram whoſe angle is 90°, and the including ſides 20 and 12·25 chains? Anſ. 245 acres.

Ex. 3. What is the area of a rhombus, each of whoſe ſides is 21 feet 3 inches, and each of the leſs angles 53° 20′? Anſ. 362·208757 feet.

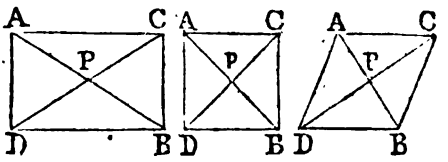
Ex. 4. How many acres are in a rhomboides whoſe leſs angle is 30°, and the including ſides 25·35 and 10·4 chains? Anſ. 13 ac. 29·12 per.

RULE

* DEMONSTRATION.

For, having drawn the perpendicular AP, the area, by the firſt rule, is AP × BC; but as rad. 1 (s. ∠P) : s. ∠B :: AB : AP = s. ∠B × AB; therefore AP × BC = BC × s. ∠B × AB, is the area; or 1 : s. ∠B :: AB × BC : s. ∠B × AB × BC = the area of the parallelogram. *Q. E. D.*

RULE III.




* As radius :
To the ſine of the angle which the diagonals of a parallelogram make with each other ::
So is the product of the diagonals :
To double the area.

That is, $\frac{AB \times CD \times \text{nat. s. } \angle P}{2}$ = the area.

*Note.* Becauſe the diagonals of a ſquare and rhombus interſect at a right angle, whoſe ſine is 1, therefore half the product of their diagonals is the area.

That is, $\frac{1}{2}AB^2$ in the ſquare, and $\frac{1}{2}AB \times CD$ in the rhombus, is the area.

EXAMPLES.

1. How many ſquare yards of pavement are in a ſquare whoſe diagonal is 27 feet 6 inches?

27·5
27·5

1375
1925
550

2)756·25

9)378·125 feet

42·013$\frac{8}{9}$ yards

Ex.

* This rule is common to all quadrilaterals, and is proved at caſe 2 of prob. 3.

Ex. 2. How many acres are in a piece of land, in the form of a rhombus, whose diagonals are 30 and 20 chains? Anſ. 30 acres.

Ex. 3. How many yards of painting are in a rectangle whose diagonals, interſecting in an angle of 30°, are each 32 feet? Anſ. $28\frac{4}{9}$.

Ex. 4. What is the area of a rhomboides whose diagonals, making an angle of 60°, are 30 and 25 feet? Anſ. 324·7595 feet.

## PROBLEM II.

### *To find the Area of a Triangle.*

### * RULE I.

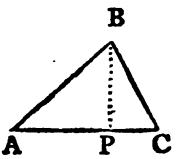
Multiply one of its ſides by the perpendicular let fall upon it from its oppoſite angle, and half the product will be the area.

That is, $\frac{AC \times BP}{2}$ = the area.

### EXAMPLES.

1. What is the area of a triangle whoſe baſe AC is 40, and the perpendicular BP is 14·52368 chains.

14·52368
20
29·047360
4
0·189440
40
7·577600




Anſ. 29 ac. or 7 p.

Ex.

* DEMONSTRATION.

This follows from rule 1 prob. 1, becauſe a triangle is half a parallelogram of the ſame baſe and height.

Ex. 2. How many ſquare feet are in a right-angled triangle whoſe baſe is 40 and perpendicular 30 feet? Anſ. 600.

Ex. 3. How many ſquare yards are in a triangle whoſe baſe is 49 feet, and perpendicular $25\frac{1}{4}$ feet. Anſ. $68\cdot736\frac{1}{3}$.

* RULE II.

As radius :
To the ſine of any angle of a triangle ::
So is the product of the ſides including the angle :
To double the area of the triangle.

That is, $\frac{\text{AB} \times \text{AC} \times \text{nat. s. of } \angle \text{A}}{2} =$ the area.

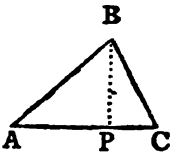
EXAMPLES.

1. What is the area of a triangle whoſe two ſides are AB 30, and AC 40, and the included angle A 28° 57′ 18″?

·4841226 ſin. ∠A
30 AB

14·5236780
20 ½AC

290·4735600 anſ.




RULE

* DEMONSTRATION.

This follows from rule 2 prob. 1, becauſe a triangle is half a parallelogram of the ſame baſe and height.

Ex. 2. How many ſquare yards are in a triangle of which one angle is 45°, and its including ſides 25 and $21\frac{1}{4}$ feet? Anſ. 20·86947.

Ex. 3. How many acres are in a right-angled triangle whoſe baſe is 40 and perpendicular 30 chains? Anſ. 60.

## * RULE III.

From half the ſum of the three ſides ſubtract each ſide ſeverally; multiply the half ſum and the three remainders



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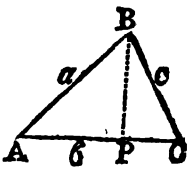
### * DEMONSTRATION.




For, $b : a+c :: a-c : \frac{aa-cc}{b} = AP-PC$;

therefore $\frac{1}{2}b + \frac{aa-cc}{2b} = \frac{bb+aa-cc}{2b} = AP$:

hence $\sqrt{aa - (\frac{bb+aa-cc}{2b})^2} =$

$\sqrt{\frac{2a^2b^2 - b^4 + 2b^2c^2 - a^4 + 2a^2c^2 - c^4}{4bb}} = BP$: then $BP \times \frac{1}{2}AC =$

$BP \times \frac{1}{2}b = \sqrt{\frac{2a^2b^2 - b^4 + 2b^2c^2 - a^4 + 2a^2c^2 - c^4}{16}} =$

$\frac{1}{4}\sqrt{\frac{-aa+bb+cc+2bc}{4} \times \frac{aa-bb-cc+2bc}{4}} =$

$\sqrt{\frac{a+b+c}{2} \times \frac{-a+b+c}{2} \times \frac{a-b+c}{2} \times \frac{a+b-c}{2}} =$

$\sqrt{\frac{a+b+c}{2} \times \frac{a+b+c}{2} - a \times \frac{a+b+c}{2} - b \times \frac{a+b+c}{2} - c} =$

the area. *Q. E. D.*

Cor.

remainders continually together, and the ſquare root of the laſt product will be the area of the triangle.

That is,

$$\sqrt{\frac{a+b+c}{2}\times\frac{a+b+c}{2}-a\times\frac{a+b+c}{2}-b\times\frac{a+b+c}{2}-c} \text{ or}$$

$$\sqrt{\frac{a+b+c}{2}\times\frac{-a+b+c}{2}\times\frac{a-b+c}{2}\times\frac{a+b-c}{2}} = \text{the area.}$$

Or, if $s$ be half the ſum of the ſides, then

$\sqrt{s\times s-a\times s-b\times s-c}$ is the area.

EX-

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*Cor.* 1. The expreſſion † is alſo $=\frac{1}{4}\sqrt{(b+c^2-a^2)\times(a^2-b\circ c^2)}=$ (putting $s=b+c$ the ſum, and $d=b\circ c$ the difference of $b$ and $c$) $\frac{1}{4}\sqrt{ss-aa\times aa-dd}$; which is another rule, very uſeful on many occaſions.

*Cor.* 2. If all the ſides be equal, the rule will become $\sqrt{\frac{3}{2}a\times\frac{1}{2}a\times\frac{1}{2}a\times\frac{1}{2}a}=\frac{1}{4}aa\sqrt{3}$, for the equilateral triangle whoſe ſide is $a$.

*Cor.* 3. If the triangle be right-angled, $a$ being the hypotenuſe, the rule will become $\frac{a+b+c}{2}\times\frac{-a+b+c}{2}$ or $\frac{1}{2}p\times\frac{1}{2}\overline{p-a}$, putting $p$ for the perimeter. For the two quantities $\frac{a+b+c}{2}\times\frac{-a+b+c}{2}$ and $\frac{a-b+c}{2}\times\frac{a+b-c}{2}$ or $\frac{-aa+bb+cc+2bc}{4}$ and $\frac{aa-bb-cc+2bc}{4}$ are then = one another, being each $=\frac{1}{2}bc$, becauſe $aa$ is $=bb+cc$.

EXAMPLES.

1. What is the area of a triangle whose three sides are 20, 30, 40 chains?

40
30
20
—
2)90
—
45 half sum
40
—
5 first dif.
—

45
30
—
15 sec. dif.
—

45
20
—
25 third dif.
15 second dif.
—
375
5 first dif.
—
1875
45 half sum
—
9375
7500
—
84375 (290·4737 sq. chains
4
—
or 29·04737 acres

49 | 443
9 | 441
—
5804 | 27500
4 | 23216
—
5808 | 4284
4066
—
218
174
—
44
—

·18948
40
—
7·5792
—

Ans. 29 ac. 7 per.

Ex. 2. How many square yards of plastering are in a triangle whose sides are 30, 40, 50 feet?

Ans. $66\frac{2}{3}$.

Ex. 3. How many acres are in a triangle whoſe ſides are 49, 50·25, 25·69 chains?

Anſ. 61ac. 1r. 39·68p.

### PROBLEM III.

### *To find the Area of a Trapezium.*

#### GENERAL RULE.

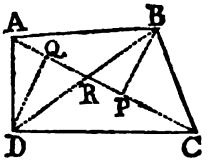
Divide it into triangles according to the manner which you judge moſt convenient; then the ſum of the areas of the triangles, calculated by the laſt problem, will be the area of the trapezium.

Obſerve, alſo, the following particular rules.

#### * RULE I.

If the trapezium be divided into two triangles by a diagonal joining two oppoſite angles, and perpendiculars be drawn to it from the other angles: then the ſum of the perpendiculars being multiplied by the diagonal, half the product will be the area.

That is, $\frac{BP + DQ}{2} \times AC =$ the area.




#### EXAMPLE.

If the diagonal AC be 20 feet, and the perpendicular BP equal 4·2 feet, and DQ equal 3·8 feet.

Then $\frac{4{\cdot}2 + 3{\cdot}8}{2} \times 20 = 8 \times 10 = 80$ ſquare feet, the area.

RULE

---

#### * DEMONSTRATION.

For the trapezium $= \triangle ABC + \triangle ADC = \frac{AC \times BP}{2} + \frac{AC \times DQ}{2} = \frac{BP + DQ}{2} \times AC$. *Q. E. D.*

### * RULE II.

If there be drawn two diagonals cutting each other: the product of the diagonals multiplied by the natural sine of the angle of intersection of the diagonals, will be double the area. And this rule, as well as the former, is common to all quadrilaterals.

That is, $\frac{AC \times BD \times \text{nat. s. } \angle R}{2} =$ the area.

Or, as radius : s. $\angle R$ :: $\frac{1}{2}AC \times BD$ : the area.

### EXAMPLE.

Suppose the two diagonals be 40 and 30 chains, and that at their intersection one of the less angles is 48°.

Then, since the nat. sine of 48° is ·7431448, the area $= \frac{40 \times 30 \times \cdot 7431448}{2} = 600 \times \cdot 7431448 = 445 \cdot 88688$ sq. chains $= 44$ ac. 2 r. 14·19008 p.

*By the Logarithms.*

| | | | |
|---|---|---|---|
| Rad. | — | — | 10·0000000 |
| Sine of 48° | — | — | 9·8710735 |
| $\frac{40 \times 30}{2} = 600$ | — | — | 2·7781513 |
| Area 445·8869 sq. chains | | — | 2·6492248 |



* DEMONSTRATION.

For the trapez. = the four $\triangle$s ARB, BRC, CRD, DRA, = $(AR \times RB + BR \times RC + CR \times RD + DR \times RA) \times \frac{1}{2}$s. $\angle R =$ $(\overline{AR + RC} \times BR + \overline{CR + RA} \times DR) \times \frac{1}{2}$s. $\angle R = \overline{AR + RC} \times \overline{BR + RD} \times \frac{1}{2}$s. $\angle R = AC \times BD \times \frac{1}{2}$s. $\angle R$. *Q. E. D.*

### * RULE III.

Square each ſide of the trapezium; add the ſquares of each pair of oppoſite ſides together; ſubtract the leſs ſum from the greater, and multiply the difference by the tangent of the angle formed by the diagonals, and $\frac{1}{4}$ of the product will be the area.

That is, $\overline{AB^2 + CD^2} \sim \overline{DA^2 + BC^2} \times \frac{1}{4}$ tan. $\angle R$ = area,
*Or* As radius : $\frac{1}{4}$ tan. $\angle R$ :: $\overline{AB^2 + CD^2} \sim \overline{DA^2 + BC^2}$ : area.

*Note.* This rule fails when the diagonals interſect at right angles: for then the tangent is infinite, and the difference of the aggregates of the ſquares is nothing.

### EXAMPLE.

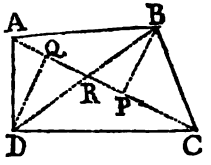
If the ſides be AB = 10, BC = 9, CD = 8, DA = 7 feet; and the leſs angle made by the diagonals equal to 80 degrees; what is the area?

The

---

### * DEMONSTRATION.

For call AR, $m$; RB, $n$; CR, $p$; RD, $q$; $s$, $c$, $t$, the ſine, coſine, and tan. $\angle R$.




Then, by 12. and 13. II. Eucl.

$BA^2 = mm + nn + 2mnc$
$AD^2 = mm + qq - 2mqc$
$BC^2 = nn + pp - 2npc$
$CD^2 = qq + pp + 2pqc$

And by add. and ſubt. $AB^2 + CD^2 - DA^2 - BC^2 = \overline{mn + pq + mq + np} \times 2c = \overline{m + p} \times \overline{n + q} \times 2c = AC \times BD \times 2c$.

But AC × BD × $\frac{1}{2}s$ = the area by the laſt rule.

Theref. $AB^2 + CD^2 - DA^2 - BC^2 = \frac{4c}{s} \times$ area $= \frac{4}{t} \times$ area

And $(\overline{AB^2 + CD^2} - \overline{DA^2 + BC^2}) \times \frac{1}{4}t$ = area. *Q. E. D.*

*Corol.* It appears from the demonſtration, that the ſum of the ſquares is greater in that pair of oppoſite ſides which ſubtend the greater, or obtuſe, angles at the interſection R; and, of conſequence, when all the angles at R are equal to each other, that is, when the diagonals interſect at right angles, the ſums of the ſquares of both pairs of oppoſite ſides will be equal to each other.

The tangent of 80° is 5·6712818, therefore $(AB^2 + CD^2 - DA^2 + BC^2) \times \frac{1}{4}$ tan. 80 is $= 164 - 130 \times \frac{1}{4} \times 5{\cdot}6712818 = 34 \times 1{\cdot}41782045 = 48{\cdot}2058953$ ſquare feet, the area.

*Or, by the Logarithms thus.*

| | | |
|---|---|---|
| Radius | — | 10·0000000 |
| Tan. of 100° | — | 10·7536812 |
| $\frac{34}{4} = 8{\cdot}5$ | — | 0·9294189 |
| Area = 48·20589 | | 1·6831001 |

* RULE IV.

If the trapezium can be inſcribed in a circle, that is, if the ſum of any two oppoſite angles be equal to two right angles, or 180°; then multiply any two adjacent ſides together, and the other two ſides together; and multiply the ſum of theſe two products by the ſine of the angle included by either of the pairs of ſides which are multiplied together; ſo ſhall half the laſt product be the area.

That is, $\frac{(AD \times DC + AB \times BC) \times \text{s. } \angle D \text{ or s. } \angle B}{2} =$ the area.

*Or* As radius : s. ∠ D or s. ∠ B :: $\frac{AD \times DC + AB \times BC}{2}$ : area.

EXAMPLE.

If the ſides be AB equal to 7·5, BC equal to 5·5, CD equal to 6, DA equal to 4, and the angle B equal



* DEMONSTRATION.

For this product will be equal to the two triangles ADC, ABC, found by rule 2, prob. 2, ſince the ſines of the oppoſite angles D, B, of a trapezium inſcribed in a circle, are equal to each other.

to 74° $40\frac{1}{4}'$, and, therefore, the angle D equal to 105° $19\frac{3}{4}'$; what is the area?

Here the fine of 74° $40\frac{1}{4}'$ is ·9644229.

Then $\frac{4\times 6+7\cdot 5\times 5\cdot 5}{2}\times \cdot 9644229 = 31\cdot 46429$ ſquare feet, the area.

| | | |
|---|---|---|
| Or, Radius — — | | 10·0000000 |
| Sine ∠B 74° $40\frac{1}{4}'$ — | | 9·9842675 |
| $\frac{4\times 6+7\cdot\times 55\cdot 5}{2} = 32\cdot 625$ | | 1·5135505 |
| Area = 31·4643 — — | | 1·4978180 |

* RULE V.

Or, when the trapezium can be inſcribed in a circle, the area may be, otherwiſe, found thus:

From

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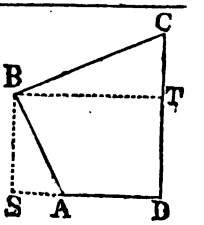
* DEMONSTRATION.




For, upon the ſides AD, DC, let fall the perpendiculars BS, BT: put $s$ = the ſine of the angle A, or of the angle C; and $a$, $b$, $c$, $d$ = the four ſides AB, BC, CD, DA reſpectively,

Then $1 : s :: \begin{cases} a : as = BS, \\ b : bs = BT, \end{cases}$ and $1 : \sqrt{1-ss} :: \begin{cases} a : a\sqrt{1-ss} = AS, \\ b : b\sqrt{1-ss} = CT, \end{cases}$

hence $SD = d + a\sqrt{1-ss}$, and $DT = c - b\sqrt{1-ss}$. But by right-angled triangles, $BS^2 + SD^2 = DT^2 + TB^2$, that is, $dd + 2ad\sqrt{1-ss} + aa = cc - 2bc\sqrt{1-ss} + bb$, and hence $\sqrt{1-ss} = \frac{bb+cc-aa-dd}{2ad+2bc}$, and $s = \frac{\sqrt{(2ad+2bc)^2-(bb+cc-aa-dd)^2}}{2ad+2bc}$; then, by the laſt rule, the area will be $\frac{1}{4}\sqrt{(2ad+2bc)^2-(bb+cc-aa-dd)^2} = \frac{1}{4}\sqrt{(b+c^2-a-d^2)\times(a+d^2-b-c^2)} = \frac{1}{4}\sqrt{(a+b+c-d)\times(a+b-c+d)\times(a-b+c+d)\times(-a+b+c+d)} = \sqrt{s-a\times s-b\times s-c\times s-d}$, if $s$ be half the ſum of the four ſides. Q. E. D.

From half the ſum of the four ſides ſubtract each ſide ſeverally; multiply the four remainders continually together, and the ſquare root of the laſt product will be the area.

That is,

$\sqrt{\frac{a+b+c-d}{2} \times \frac{a+b-c+d}{2} \times \frac{a-b+c+d}{2} \times \frac{-a+b+c+d}{2}}$ = the area. Or if $s$ be half the ſum of the ſides, then $\sqrt{(s-a) \times (s-b) \times (s-c) \times (s-d)}$ is the area.

The letters $a$, $b$, $c$, $d$, denoting the four ſides of the trapezium.

EXAMPLE.

The four ſides of a trapezium inſcribed in a circle, are 6, 5·5, 7·5, 4 feet; required the area.

$\sqrt{\frac{6+5\cdot5+7\cdot5-4}{2} \times \frac{6+5\cdot5-7\cdot5+4}{2} \times \frac{6-5\cdot5+7\cdot5+4}{2} \times \frac{-6+5\cdot5+7\cdot5+4}{2}}$

$= \frac{1}{4}\sqrt{15 \times 8 \times 12 \times 11} = \frac{1}{4}\sqrt{15840} = \frac{125\cdot85708}{4} =$ 31·46427 ſquare feet the area.

RULE

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*Corol.* 1. The expreſſion marked † is a very pretty theorem, and may be uſeful on many occaſions.

*Corol.* 2. Hence may be deduced rule 3 for the triangle; for, if we here ſuppoſe one of the ſides, as $d$, to be nothing, or to decreaſe till it vaniſh, the rule will become $\sqrt{(s-a) \times (s-b) \times (s-c) \times s}$, the ſame as in the triangle.

*Corol.* 3. If, in the trapezium, $a = d$, and $b = c$, the rule will be barely $ab$.

*Corol.* 4. When all the four ſides are equal, the rule becomes $\sqrt{a \times a \times a \times a} = aa$.

*Corol.* 5. If the ſides be in arithmetical proportion, either continued or diſcontinued; the ſquare root of the continual product of all the four ſides will be the area. Or, take a mean proportional between any two ſides, and alſo a mean proportional between the other two ſides; then the figure will be equal to the rectangle of theſe mean proportionals. For, half the ſum of the ſides is equal to the ſum of the extremes, or equal to the ſum of the means; and each extreme and mean being taken from it, will leave the other extreme and mean; and conſequently the four remainders will be equal to the four given ſides. Wherefore, &c. That is, $\sqrt{abcd}$ = the area.

### RULE VI.

In a trapezoid, multiply the ſum of the parallel ſides by the diſtance between them, and half the product will be the area.*

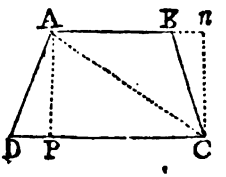
That is, if AB, DC be the parallel ſides, and AP perpendicular to DC; then $\frac{AB+CD}{2} \times AP$ will be the area.

*Note.* This caſe is very uſeful in ſurveying. For many irregular pieces of land may be, very conveniently, divided into a number of trapezoids, of that kind in which the parallel ſides are perpendicular to one of the other ſides, which, in that caſe, is the diſtance of the parallel ſides.

EXAMPLE.

If AB be 7·5, CD 12·25, and AP 15·4 chains; what is the area?

12·25
7·50
19·75
15·4
7900
9875
1975
2)304·150
Anſ. 15ac. 33·2 per.
15,2075
4
·8300
40
33·2000




SOME

* DEMONSTRATION.

For, the diagonal AC divides the trapezoid into two triangles, whoſe baſes are the two parallel ſides, and heights each equal to the diſtance AP; therefore, &c.

## SOME PROMISCUOUS EXAMPLES FOR PRACTICE.

### EXAMPLE I.

How many ſquare yards of paving are in the trapezium ABCD, whoſe diagonal AC is 65 feet, and the perpendiculars BP equal to 33.5, and DQ equal to 28 feet? By rule 1.

33·5
28·0
61·5
65
3075
3690
2)3997·5
9)1998·75 feet
Anſ. 222·08$\frac{1}{3}$ yards.

### EXAMPLE II.

What is the area of a trapezium whoſe ſouth ſide is 27·4 chains, eaſt ſide 35·75 chains, north ſide 37·55 chains, and weſt ſide 41·05 chains; alſo the diagonal from ſouth-weſt to north-eaſt 48·35 chains?

By rule 3 of prob. 2,

$\frac{27\cdot4+35\cdot75+48\cdot35}{2}=\frac{111\cdot5}{2}=55\cdot75=$ half the ſum of the ſides of the ſouth-eaſt triangle. And

$\frac{37\cdot55+41\cdot05+48\cdot35}{2}=\frac{126\cdot95}{2}=63\cdot475=$ half the ſum of

of the ſides of the north-weſt triangle. Whence $\sqrt{55{\cdot}75\times55{\cdot}75-27{\cdot}4\times55{\cdot}75-35{\cdot}75\times55{\cdot}75-48{\cdot}35}=\sqrt{55{\cdot}75\times28{\cdot}35\times20\times7{\cdot}4}=\sqrt{233915{\cdot}85}=483{\cdot}6485$ the area of the former triangle. And

$\sqrt{63{\cdot}475\times63{\cdot}475-37{\cdot}55\times63{\cdot}475-41{\cdot}05\times63{\cdot}475-48{\cdot}35}$
$=\sqrt{63{\cdot}475\times25{\cdot}925\times22{\cdot}425\times15{\cdot}125}=\sqrt{558148{\cdot}1}$
&c $=747{\cdot}0932$ the area of the latter. And their ſum is 1230·7417 ſquare chains $=$ 123 ac. 11·8672 per. the area required,

## EXAMPLE III.

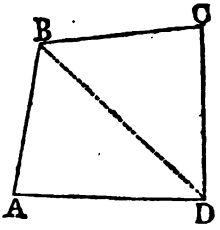
How many ſquare feet are in a trapezium whoſe ſides are AB equal to 12, BC equal to 13, CD equal to 14, DA equal to 15 feet, and the angle A equal to 83° 30′?

Firſt, by rule 2, prob. 2, $\frac{\cdot9935719\times12\times15}{2}=$ $\cdot9935719\times90=89{\cdot}421471=$ the area of the triangle ABD.

Again, by trigonometry,

As 27 $=$ DA $+$ AB : 3 $=$ DA $-$ AB :: 1·1204053 (tang. $\frac{\text{DBA}+\text{ADB}}{2}=48°\ 15'$) : $\frac{1{\cdot}1204053}{9}=$ $\cdot1244895=$ tang. $\frac{\text{DBA}-\text{ADB}}{2}=7°\ 06'$. Therefore $48°\ 15'-7°\ 06'=41°\ 09'=$ ADB. And,

| | | |
|---|---|---|
| ∠ADB | 41° 09′ | 9·8182474 |
| ∠A | 83 30 | 9·9971993 |
| AB | 12 | 1·0791812 |
| BD | 18·119 | 1·2581331 |




Then,

Then, by rule 3, prob. 2,

$$\sqrt{\tfrac{18\cdot119+14+13}{2}\times\tfrac{18\cdot119+14-13}{2}\times\tfrac{18\cdot119-14+13}{2}\times\tfrac{-18\cdot119+14+13}{2}}$$

$=\sqrt{22\cdot5595\times9\cdot5595\times8\cdot5595\times4\cdot4405}=\sqrt{8196\cdot812}$
$=90\cdot53625=$ the area of the triangle DCB.

The sum of these two is $179\cdot957721=$ the area of the trapezium required.

## EXAMPLE IV.

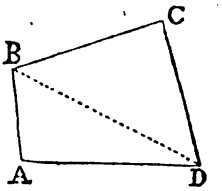
In a quadrangular field the south side is 23·4, the east side 19·75, and the north side 20·5 chains; also the south-east and north-east angles are 73° and 87° 30′: what is the area?

First, by rule 2 prob. 2, $\frac{\cdot9990482\times19\cdot75\times20\cdot5}{2}=$
$\cdot4995241\times19\cdot75\times20\cdot5=202\cdot24482$ the area of the north-east triangle BCD.

Again, $40\cdot25=$ BC + CD : $\cdot75=$
BC − CD :: 1·0446136 (tang. $\frac{\text{BDC}+\text{CBD}}{2}=46°\ 15'$) :
$\frac{1\cdot0446136\times3}{161}=\cdot01946485=$ the tang. of $\frac{\text{BDC}-\text{CBD}}{2}=$
1° 07′. Wherefore ∠ BDC = 46° 15′ + 1° 07′ = 47° 22′.

And ∠ ADB = (ADC − CDB = 73° − 47° 22′ =)
25° 38′. But,

| | | |
|---|---|---|
| ∠ BDC | 47° 22′ | 9·8667026 |
| ∠ C | 87 30 | 9·9995865 |
| CB | 20·5 | 1·3117539 |
| BD | — | 1·4446378 |




Whence,

Whence, by rule 2, prob. 2, alſo,

| | | | | |
|---|---|---|---|---|
| $\frac{1}{2}$AD | — | 11·7 | — | 1·0681859 |
| BD | — | | — | 1·4446378 |
| S. ADB | — | 25° 38′ | — | 9·6360969 |
| 140·9031 the area of the △ ABD | | | - | 2·1489206 |

Their ſum is 343·1479 ſquare chains = 34 acres 1 rood and 10·3664 perches, the area required.

EXAMPLE V.

What is the area of a trapezium, inſcribed in a circle, whoſe four ſides are 24, 26, 28, 30 yards?

Here $\frac{24+26+28+30}{2} = \frac{108}{2} = 54$ the half ſum of the ſides.

Then, by rule 5, $\sqrt{54-24\times54-26\times54-28\times54-30} = \sqrt{30\times28\times26\times24} = 24\sqrt{5\times7\times13\times2} = 24\sqrt{910} = 24\times30{\cdot}1662062 = 723{\cdot}9889488$ ſquare yards, the area required.

EXAMPLE VI.

How many ſquare feet are in a board whoſe length is $12\frac{1}{2}$ feet, the breadth of the greater end $1\frac{1}{4}$ feet, and of the leſs 11 inches?

By rule 6, $\frac{1\frac{1}{4}+\frac{11}{12}}{2}\times12\frac{1}{2} = \frac{13}{12}\times12\frac{1}{2} = 13\frac{6\frac{1}{2}}{12} = 13{\cdot}541\frac{2}{3}$ ſquare feet, the area required.

EXAMPLE VII.

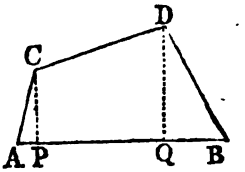
In a trapezoid, the length of the two parallel ſides are 30 and 46 chains, and the length of one of the other ſides, which is perpendicular to the parallel ſides, is 60·37 chains: what is the area?

By rule 6, $\frac{30+46}{2} \times 60{\cdot}37 = 38 \times 60{\cdot}37 = 2294{\cdot}06$ ſquare chains = 229 acres 1 rood 24·96 perches, the area required.

### EXAMPLE VIII.

In meaſuring along one ſide AB of a quadrangular field, that ſide and the two perpendiculars upon it from the oppoſite corners, meaſured as in the field book below: required the area.

| FIELD BOOK. | | | |
|---|---|---|---|
| | chains | | chains |
| AP = | 1·10 | PC = | 3·52 |
| AQ = | 7·45 | QD = | 5·95 |
| AB = | 11·10 | | |




AP × PC = 1·10 × 3·52 = 3·872 = 2APC
PQ × $\frac{PC + QD}{2}$ = 6·35 × 9·47 = 60·1345 = 2CPQD
DQ × QB = 5·95 × 3·65 = 21·7175 = 2DQB

the ſum = 85·724 = double the whole figure, whoſe half = 42·862 ſqu. chains = 4 acres 1 rood 5·792 perches is the area required.

### PROBLEM IV.

*To find the Area of any Irregular Figure.*

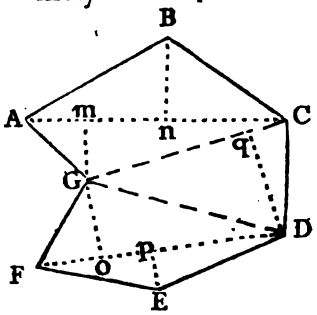
Draw diagonals dividing the figure into trapeziums and triangles. Then find the areas of all theſe ſeparately, and their ſum will be the content of the whole irregular figure.

### EXAMPLES.

1. To find the content of the irregular figure ABCDEFGA,

ABCDEFGA, in which are given the following diagonals and perpendiculars: namely

AC 5·5
FD 5·2
GC 4·4
Gm 1·3
Bn 1·8
Go 1·2
Ep 0·8
Dq 2·3




AC × $\overline{\text{Gm} + \text{Bn}}$ = 5·5 × 3·1 = 17·05 = 2ABCG
FD × $\overline{\text{Go} + \text{Ep}}$ = 5·2 × 2·0 = 10·40 = 2GFED
GC × Dq = 4·4 × 2·3 = 10·12 = 2GCD

2) 37·57
answer 18·785

## PROBLEM V.

*To find the Areas of Regular Figures or Polygons.*

### RULE I.

Multiply the perimeter of the figure by the radius of its inſcribed circle, or by the perpendicular demitted from its center to one of the ſides, and half the product will be the area.*

*Note*

* DEMONSTRATION.

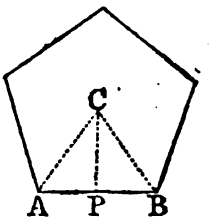
For, the polygon conſiſting of as many triangles as it hath ſides, whoſe heights and baſes are each equal to the perpendicular, and the ſide of the polygon reſpectively; if $s$ be a ſide of the polygon, $p$ the perpendicular, and $n$ the number of ſides; then $\frac{1}{2}ps$ will be one triangle, and $\frac{1}{2}nps$ all the triangles, or the whole polygon. *Q. E. D.*

*Note* 1. The perimeter of a figure is the ſum of its ſides, and in a regular figure is equal to the length of one ſide multiplied by the number of ſides.

2. The center of the figure is the ſame as the center of the inſcribed or circumſcribed circle.

EXAMPLE.

Required the area of a regular pentagon whoſe ſide is 25 yards.




Since $\frac{360^\circ}{5} = 72^\circ = \angle ACB$;

hence $\frac{72}{2} = 36^\circ = \angle ACP$;

and $90 - 36 = 54^\circ = \angle CAP$;

Then as $1 = \text{rad.}$ :

$1{\cdot}3763819 = \text{tang. } 54^\circ :: 12\frac{1}{2}$

$= AP : 17{\cdot}2047737 = PC.$

Whence $\frac{17{\cdot}2047737 \times 25 \times 5}{2} = 1075{\cdot}298356$, the area required.

* RULE II.

Multiply the ſquare of the ſide, of any regular figure, by the multiplier ſtanding oppoſite its name,



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* DEMONSTRATION.

This rule is founded upon the property, that like polygons, as ſimilar figures, are to one another as the ſquares of their like ſides; which has been proved.

Now, the multipliers in the table, are the areas of the polygons to the ſide 1; whence the rule is manifeſt.

SCHOLIUM.

The table is formed thus: As radius $= 1$ : tang. $\angle CAP = t$ :: $AP : PC = t \times AP = \frac{1}{2}t$, ſuppoſing $AB = 1$; then $\frac{1}{4}t$ $(= CP \times PA) =$ the $\triangle ACB$, and $\frac{1}{4}nt =$ the polygon; where $n$ is the number of ſides: So that, by finding the tangent of the $\angle CAP$, by the

in the following table; and the product will be the area.

| N° of sides | Names. | Multipliers, or area when the side is 1. |
|---|---|---|
| 3 | Trigon or equi. △ | $0.4330127 = \frac{1}{4}\tan.30° = \frac{1}{4}\sqrt{3}$ |
| 4 | Tetragon or □ | $1.0000000 = \frac{1}{4}\tan.45° = 1 \times 1$ |
| 5 | Pentagon | $1.7204774 = \frac{5}{4}\tan.54° = \frac{5}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}$ |
| 6 | Hexagon | $2.5980762 = \frac{6}{4}\tan.60° = \frac{3}{2}\sqrt{3}$ |
| 7 | Heptagon | $3.6339124 = \frac{7}{4}\tan.64\frac{2}{7}°$ |
| 8 | Octagon | $4.8284271 = \frac{8}{4}\tan.67\frac{1}{2}° = 2 \times \overline{1+\sqrt{2}}$ |
| 9 | Nonagon | $6.1818242 = \frac{9}{4}\tan.70°$ |
| 10 | Decagon | $7.6942088 = \frac{10}{4}\tan.72° = \frac{5}{2}\sqrt{5+2\sqrt{5}}$ |
| 11 | Undecagon | $9.3656399 = \frac{11}{4}\tan.73\frac{7}{11}°$ |
| 12 | Dodecagon | $11.1961524 = \frac{12}{4}\tan.75° = 3 \times \overline{2+\sqrt{3}}$ |

EX-

the table of tangents, and multiplying it by the number of sides, $\frac{1}{4}$ of the product will be the multiplier in the table.

All these multipliers may be derived from other methods, and, indeed, many of them from such as are very simple.

Thus, in the trigon, or equilateral triangle, $\sqrt{1^2 - \frac{1^2}{2}} = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$ = the perpendicular, and $\frac{1}{4}\sqrt{3}$ is the area.—In the trigon the angle CAP is 30°, and therefore the tangent of 30° is $= \sqrt{\frac{1}{3}} = \frac{1}{3}\sqrt{3}$.

In the tetragon, or square, the side drawn into itself $1 \times 1 = 1$, is the area.

In the pentagon, the ∠CAP is 54°, whose sine is $\frac{\sqrt{5}+1}{4}$, and its cosine $\frac{\sqrt{10-2\sqrt{5}}}{4}$; and, because the tangent is equal to the sine divided by the cosine, we shall have the tangent of 54° or $t = \frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}} = \sqrt{\frac{6+2\sqrt{5}}{10-2\sqrt{5}}} = \sqrt{1+\frac{2}{5}\sqrt{5}}$; therefore the area of the pentagon is $\frac{5}{4}\sqrt{1+\frac{2}{5}\sqrt{5}}$.

In the hexagon, all the triangles are equilateral, and therefore it will be 6 times the trigon, or 6 times $\frac{1}{4}\sqrt{3} = \frac{6}{4}\sqrt{3} = \frac{3}{2}\sqrt{3}$.—And, since, in it, the ∠CAP is = 60°, the tangent of 60° will be $= \sqrt{3}$ = triple the tangent of 30°.

In the octagon, the ∠CAP is $67\frac{1}{2}°$, whose sine is $\frac{1}{2}\sqrt{2+\sqrt{2}}$, and its cosine $\frac{1}{2}\sqrt{2-\sqrt{2}}$; whence the tangent of $67\frac{1}{2}°$ or $t = \sqrt{\frac{2+\sqrt{2}}{2-\sqrt{2}}} = 1+\sqrt{2}$; and therefore the area of the octagon is $\frac{8}{4} \times (1+\sqrt{2}) = 2 \times (1+\sqrt{2})$.

In the decagon, the angle CAP is 72°, whose sine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$, and its cosine $\frac{1}{4} \times (\sqrt{5}-1)$; whence $t$ the tangent of 72° is

$\sqrt{}$

EXAMPLE I.

What is the area of a pentagon whoſe ſide is 25 feet?



$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1} = \sqrt{\frac{10+2\sqrt{5}}{6-2\sqrt{5}}} = \sqrt{5+2\sqrt{5}}$; and therefore the area of the decagon is $\frac{10}{4}\sqrt{5+2\sqrt{5}} = \frac{5}{2}\sqrt{5+2\sqrt{5}}$.

In the dodecagon, the angle CAP is 75°, whoſe ſine is $\frac{1}{2}\sqrt{2+\sqrt{3}}$, and its coſine $\frac{1}{2}\sqrt{2-\sqrt{3}}$; whence the tangent of 75° is $\sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}} = 2+\sqrt{3}$; and therefore the area of the dodecagon is $\frac{12}{4} \times (2+\sqrt{3}) = 3 \times (2+\sqrt{3})$.

The tangents of the angle CAP in the heptagon, nonagon, and undecagon, are not ſo eaſily found, independent of a table of tangents; but we may find equations whoſe roots ſhall be the required tangent, in the following manner.

Put $y$ and $x$ for the ſine and coſine of the angle ACB at the center, ſubtended by the ſide of any polygon, and $n$ = the number of ſides: then it is known that the ſine of $n$ times that angle, or of 360°, the whole circumference of a circle, is

$$nx^{n-1}y - \frac{n.n-1.n-2}{1.2.3}x^{n-3}y^3 + \frac{n.n-1.n-2.n-3.n-4}{1.2.3.4.5}x^{n-5}y^5 \&c.$$

But the ſine of the whole circumference is nothing, and therefore this ſeries, or the ſame divided by $ny$, will be equal to nothing alſo, viz.

$$x^{n-1} - \frac{n-1.n-2}{2.3}x^{n-3}y^2 + \frac{n-1.n-2.n-3.n-4}{2.3.4.5}x^{n-5}y^4 \&c = 0.$$

And ſince $n$ is an integer number, it is evident that the ſeries will always terminate. Now exterminating $y$ by its value $\sqrt{1-xx}$, the ſeries will become

$$x^{n-1} - \frac{n-1.n-2}{2.3}x^{n-3}(1-xx) + \frac{n-1.n-2.n-3.n-4}{2.3.4.5}x^{n-5}(1-xx)^2 \&c = 0,$$

$$\text{or } 1 - \frac{n-1.n-2}{2.3}\cdot\frac{1-xx}{xx} + \frac{n-1.n-2.n-3.n-4}{2.3.4.5}\left(\frac{1-xx}{xx}\right)^2 \&c = 0.$$

Or exterminating $x$ by its value $\sqrt{1-y^2}$ will give

$$(1-y^2)^{\frac{n-1}{2}} - \frac{n-1.n-2}{2.3}(1-y^2)^{\frac{n-3}{2}}y^2 + \frac{n-1.n-2.n-3.n-4}{2.3.4.5}(1-y^2)^{\frac{n-5}{2}}y^4 \&c = 0.$$

$$\text{or } 1 - \frac{n-1.n-2}{2.3}(1-y^2)y^2 + \frac{n-1.n-2.n-3.n-4}{2.3.4.5}(1-y^2)^2y^4 \&c = 0.$$

And by writing any particular value for $n$, the root of the equation

The multiplier for the pentagon being 1·7204774, we ſhall have 1·7204774 × 25 × 25 = 1075·298375 the area required.

EX-

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equation reſulting will be the coſine, or ſine, of the angle ACB.

Thus, if $n$ be $= 7$, the equation will be

$\left\{\begin{matrix} x^6 - \frac{5}{4}x^4 + \frac{3}{8}x^2 - \frac{1}{64} = 0 \\ y^6 - \frac{7}{4}y^4 + \frac{7}{8}y^2 - \frac{7}{64} = 0 \end{matrix}\right\}$ whoſe root $\left\{\begin{matrix} x \\ y \end{matrix}\right\}$ is the $\left\{\begin{matrix} \text{coſine} \\ \text{ſine} \end{matrix}\right\}$ of double the complement of the ∠CAP ($64\frac{2}{7}°$) in the heptagon.

If $n = 9$, the ſeries will become

$\left\{\begin{matrix} x^8 - \frac{7}{4}x^6 + \frac{15}{16}x^4 - \frac{5}{32}x^2 + \frac{1}{256} = 0 \\ y^8 - \frac{9}{4}y^6 + \frac{27}{16}y^4 - \frac{15}{32}y^2 + \frac{9}{256} = 0 \end{matrix}\right\}$ whoſe root $\left\{\begin{matrix} x \\ y \end{matrix}\right\}$ is the $\left\{\begin{matrix} \text{coſine} \\ \text{ſine} \end{matrix}\right\}$ of double the complement of the angle CAP (70°) in the nonagon.

If $n = 11$, there will come out the equation

$\left\{\begin{matrix} x^{10} - \frac{9}{4}x^8 + \frac{7}{4}x^6 - \frac{35}{64}x^4 + \frac{15}{128}x^2 - \frac{1}{1024} = 0 \\ y^{10} - \frac{11}{4}y^8 + \frac{11}{4}y^6 - \frac{77}{64}y^4 + \frac{55}{128}y^2 - \frac{11}{1024} = 0 \end{matrix}\right\}$ whoſe root is $\left\{\begin{matrix} \text{coſine} \\ \text{ſine} \end{matrix}\right\}$ of double the complement of the ∠CAP ($73\frac{7}{11}°$) in the undecagon.

When the root $x$ of the equation is extracted, the tangent of the ∠CAP may be eaſily found thus. Call that tangent $t$; then $t$ being the cotangent of an angle of the double of which

$\left\{\begin{matrix} x \\ y \end{matrix}\right\}$ is the $\left\{\begin{matrix} \text{coſine} \\ \text{ſine} \end{matrix}\right\}$ therefore $t = \sqrt{\frac{1+x}{1-x}} = \frac{1+\sqrt{1-y^2}}{y} = \frac{y}{1-\sqrt{1-y^2}}$

Or, by writing $\frac{tt-1}{tt+1}$ inſtead of $x$, or $\frac{2t}{1+tt}$ inſtead of $y$, in the three equations above, we ſhall have,

Firſt, $t^{12} - 26t^{10} + 143t^8 - 245\frac{1}{3}t^6 + 143t^4 - 26t^2 + 1 = 0$, an equation whoſe root is the tangent of the ∠CAP ($64\frac{2}{7}°$) in the heptagon.

Secondly, $t^{16} - 45\frac{1}{3}t^{14} + 476t^{12} - 1768t^{10} + 2701\frac{1}{3}t^8 - 1768t^6 + 476t^4 - 45\frac{1}{3}t^2 + 1 = 0$, an equation whoſe root is the tangent of the ∠CAP (70°) in the nonagon.

And laſtly, $t^{20} - 70t^{18} + 1197t^{16} - 7752t^{14} + 22610t^{12} - 32065\frac{1}{11}t^{10} + 22610t^8 - 7752t^6 + 1197t^4 - 70t^2 + 1 = 0$, the root of which equation is the tangent of the angle CAP ($73\frac{7}{11}°$) in the undecagon.

MOREOVER,

EXAMPLE II.

What is the area of a hexagon whose side is 20?
Here $2{\cdot}598076 \times 20 \times 20 = 1039{\cdot}2304 =$ the area.



MOREOVER, If in a square be inscribed a circle, and in the circle the several regular polygons; then

The square will be to 4 as the circle to 3·14159265 &c, and as the inscribed

| | | | |
|---|---|---|---|
| Trigon | to | $\frac{3}{2}$ × sine of 120° | $= \frac{3}{4}\sqrt{3}$ |
| Tetragon | to | $\frac{4}{2}$ × sine of 90 | $= 2$ |
| Pentagon | to | $\frac{5}{2}$ × sine of 72 | $= \frac{5}{4}\sqrt{10+2\sqrt{5}}$ |
| Hexagon | to | $\frac{6}{2}$ × sine of 60 | $= \frac{3}{2}\sqrt{3}$ |
| Heptagon | to | $\frac{7}{2}$ × sine of $51\frac{3}{7}$ | $= \frac{7}{4}x$ |
| Octagon | to | $\frac{8}{2}$ × sine of 45 | $= 2\sqrt{2}$ |
| Nonagon | to | $\frac{9}{2}$ × sine of 40 | $= \frac{9}{2}y$ |
| Decagon | to | $\frac{10}{2}$ × sine of 36 | $= \frac{5}{4}\sqrt{10-2\sqrt{5}}$ |
| Undecagon | to | $\frac{11}{2}$ × sine of $32\frac{8}{11}$ | $= \frac{11}{2}z$ |
| Dodecagon | to | $\frac{12}{2}$ × sine of 30 | $= 3$ |
| &c. | | &c. | |

Where $x$, $y$, $z$ are the roots of the equations

$$x^6 - \tfrac{7}{4}x^4 + \tfrac{7}{16}x^2 - \tfrac{7}{64} = 0$$

$$y^8 - \tfrac{9}{4}y^6 + \tfrac{21}{16}y^4 - \tfrac{15}{16}y^2 + \tfrac{9}{64} = 0$$

$$z^{10} - \tfrac{11}{4}z^8 + \tfrac{11}{4}z^6 - \tfrac{77}{64}z^4 + \tfrac{55}{256}z^2 - \tfrac{11}{1024} = 0$$

as appears from page 116.

For, if the side of the square, or diameter of the circle, be 2, their areas will be 4 and 3·14159 &c. And, if radii be drawn to each angular point of the polygons, they will divide the polygons into as many equal triangles, as they have sides; and if, in one of these triangles, a perpendicular be demitted from the extremity of one radius to the other, half this perpendicular will express the area of the triangle, because the base upon which it falls is equal to 1, and consequently, half the number of sides drawn into the perpendicular, will denote the area of the polygon: but the perpendiculars are the sines of the angles above expressed; and consequently the truth of the proportions is manifest.

As to the values of the sines, they are sufficiently well known, excepting those denoted by $x$, $y$, $z$, whose values are as above specified.

### EXAMPLE III.

What is the area of a trigon whoſe ſide is 20?

Here $\cdot 433013 \times 20 \times 20 = 173 \cdot 2052 =$ the area required.

### PROBLEM VI.

*To find the Diameter and Circumference of a Circle, the one from the other.**

### RULE I.

As 7 is to 22, ſo is the diameter to the circumference.

As 22 is to 7, ſo is the circumference to the diameter.

RULE

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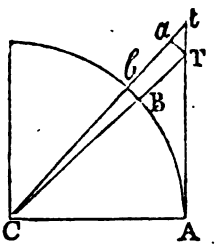
* DEMONSTRATION.

The proportion of the diameter of a circle to the circumference, is beſt found from the tangent of ſome arc, as follows.

Let AT be the tangent of the arc AB, and draw $ct$ indefinitely near CT, and $Ta$ perpendicular to $ct$. Put $r = CA$, $t = AT$, and $s =$ the arc AB. Then $Tt = \dot{t}$, $B\dot{b} = \dot{s}$, and $CT = \sqrt{rr + tt}$.




In the ſimilar $\triangle$s CAT, $Tat$, we have $CT : CA :: T\dot{t} : T\dot{a} = \frac{r\dot{t}}{CT}$; and in the ſimilar $\triangle$s $CB\dot{b}$, $CT\dot{a}$, we have $CT : T\dot{a} :: CB : B\dot{b}$, or $\dot{s} = \frac{CB \times T\dot{a}}{CT} = \frac{rr\dot{t}}{CT^2} = \frac{rr\dot{t}}{rr + tt} = \dot{t} \times (1 - \frac{t^2}{r^2} + \frac{t^4}{r^4} - \frac{t^6}{r^6} + \frac{t^8}{r^8}$ &c.) And the fluent is $s = t \times (1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8}$ &c.) = the length of the arc AB. Then, by taking AB = any given arc, whoſe tangent can be found in terms of the radius, the ſeries will then become known; and being repeated as often as AB is contained in the whole circumference, we ſhall have the length of the circumference in terms of the diameter.

This

RULE II.

As 113 is to 355, ſo is the diam. to the circumf.
As 355 is to 113, ſo is the circumf. to the diameter.

RULE III.

As 1 is to 3·1416, ſo is the diam. to the circumf.
As 3·1416 is to 1, ſo is the circumf. to the diam.



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This ſeries was firſt given, for this purpoſe, by Mr. JAMES GREGORY, in a letter of the 15th of February 1671, ſent to Mr. COLLINS, and inſerted in the *Commerc. Epiſtol.*—If the arc be aſſumed greater than half a quadrant, the ſeries will diverge, and become of no uſe here; becauſe the tangent will, in that caſe, be greater than the radius: but in every other caſe the ſeries will converge, and become a proper expreſſion for the arc.

Thus, if AB be the $\frac{1}{8}$ part of the circumference, or 45°, its tangent will be equal to the radius, and the ſeries will become $r \times (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$ &c) = the arc of 45°; and therefore $8r \times (1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9}$ &c) will be the whole circumference.

Hence the circumferences of circles are to one another as their radii, or as their diameters; and, conſequently, the circumference of a circle whoſe diameter is 1, being multiplied by the diameter of any other circle, will produce the circumference of this circle; which is our rule.

The laſt ſeries is the ſimpleſt form that the general ſeries will admit of, although it does not converge ſo quickly as may be required; and to find others converging quicker we muſt take the arc AB leſs. Thus,

Suppoſe AB = an arc of 30°, or the $\frac{1}{12}$ part of the circumference; then its tangent $t$ is $= r\sqrt{\frac{1}{3}}$. Or, to make the forms ſtill ſimpler, let us ſuppoſe $r = 1$; then $t = \sqrt{\frac{1}{3}}$, and the ſeries will become $a = \sqrt{\frac{1}{3}} \times (1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \frac{1}{9 \cdot 3^4}$ &c) = the arc of 30°; which converges ſo faſt, that it may be very quickly ſummed; and being multiplied by 12, will give the whole circumference. And this is Doctor HALLEY's method.

And by taking ſtill ſmaller arcs, ſeries may be found, in the ſame manner, converging much quicker; but then they will be more compounded with ſurd quantities.

As the famous quadrature of the late Mr. JOHN MACHIN, *Profeſſor of Aſtronomy in Greſham College*, is extremely expeditious, and but little known, I ſhall take this opportunity of explaining

EXAMPLE I.

If the diameter of a circle be 7, what is the circumference?

3·1416 × 7 = 21·9912 the circumference, = 22 nearly.

Therefore the diameter is to the circumference, nearly as 7 to 22.

EX-

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plaining it as follows.—Since the chief advantage consists in taking small arcs, whose tangents shall be numbers easy to manage, Mr. Machin very properly considered, that since the tangent of 45° is 1; and that the tangent of any arc being given, the tangent of double that arc can easily be had; if there be assumed some small simple number for the tangent of an arc, and then the tangent of the double arc be continually taken, until a tangent be found nearly equal to 1, the tangent of 45°; by taking the tangent answering to the small difference between 45° and this multiple, there would be had two very small tangents, viz. the tangent first assumed, and the tangent of the difference between 45° and the multiple arc; and that, therefore, the lengths of the arcs corresponding to these two tangents being calculated, and the arc belonging to the tangent first assumed being as often doubled as the multiple directs, the result increased or diminished by the other arc, would be the arc of 45°, according as the multiple arc should be below or above it.

Having thus thought of his method, by a few trials he was lucky enough to find a number, and perhaps the only one, proper for this purpose, viz. knowing that the tangent of $\frac{1}{4}$ of 45° is nearly $= \frac{1}{5}$, he assumed $\frac{1}{5}$ as the tangent of an arc: then, since if $t$ be the tangent of an arc, the tangent of the double arc will be $\frac{2t}{1-tt}$, the radius being 1; the tangent of an arc double to that of which $\frac{1}{5}$ is the tangent, will be $\frac{\frac{2}{5}}{1-\frac{1}{25}} = \frac{10}{24} = \frac{5}{12}$, and the tangent of the double of this last is $\frac{\frac{10}{12}}{1-\frac{25}{144}} = \frac{10 \times 12}{144-25} = \frac{120}{119}$; which, being very nearly equal to 1, shews that the arc which is equal to 4 times the first, is very near 45°. Then, since the tangent of the difference between 45° and an arc whose tangent is T, is $\frac{T-1}{T+1}$, we shall have the tangent of the difference between

45°

EXAMPLE II.

What is the diameter of the circle whose circumference is 355?

$355 \div 3{\cdot}1416 = 112{\cdot}9997 +$, the diameter, $= 113$ nearly.

And therefore the diameter is to the circumference nearly as 113 to 355.

EX-

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45° and the arc whose tangent is

$\frac{120}{119}$, equal to $\frac{\frac{120}{119} - 1}{\frac{120}{119} + 1} = \frac{120 - 119}{120 + 119} = \frac{1}{239}$.

Now, by calculating, from the general series, the arcs whose tangents are $\frac{1}{5}$ and $\frac{1}{239}$, which may be very quickly done, by reason of the smallness and simplicity of the numbers, and taking the latter arc from 4 times the former, the remainder will be the arc of 45°. And this is Mr. Machin's famous quadrature of the circle.

But it was by means of Dr. Halley's method that he found the circumference of a circle, whose diameter is 1, to be 3·141592653589793·3846, 2643383279, 5028841971, 6939937510, 5820974944, 5923071864, 0628620899, 8628034825, 3421170679 +, true to above 100 places of figures.

Or, by substituting the above numbers in the general series, we get the series $(\frac{16}{5} - \frac{4}{239}) - \frac{1}{3}(\frac{16}{5^3} - \frac{4}{239^3}) + \frac{1}{5}(\frac{16}{5^5} - \frac{4}{239^5})$ &c. equal to the semicircumference whose radius is 1, or the whole circumference whose diameter is 1. And this is the series published by Mr. JONES, and which he acknowledges he received from Mr. MACHIN.

But because the arc whose tangent is $\frac{1}{5}$, is $= 2$ times the arc whose tangent is $\frac{1}{70}$, minus the arc to tangent $\frac{1}{239}$; (for $\frac{\frac{2}{70}}{1 - \frac{1}{70 \cdot 70}} = \frac{20}{99} =$ tang. of twice arc to tang. $\frac{1}{70}$, and $\frac{\frac{20}{99} - \frac{1}{5}}{1 + \frac{4}{99 \cdot 5}} = \frac{1}{239}$ = tang. of dif. between the arcs whose tangents are $\frac{20}{99}$ and $\frac{1}{5}$); therefore 8 times arc to tang. $\frac{1}{70}$ − 4 times arc to tang. $\frac{1}{239}$ − arc to tang. $\frac{1}{239}$ = arc of 45°, or whose tang. is 1. Which is much easier than Machin's way. And various other ways may easily be discovered from the same principles.

If instead of $t$, in the series $t(1 \times - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6}$ &c) be substituted its value $\frac{rr}{\tau}$, we shall have $\frac{rr}{\tau} \times (1 - \frac{r^2}{3\tau^2} + \frac{r^4}{5\tau^4} - \frac{r^6}{7\tau^6}$ &c) for the length of the arc whose co-tangent is $\tau$, and radius $r$.

Or,

EXAMPLE III.

What is the circumference of the earth, ſuppoſing it to be perfectly round, and that its diameter is 7958 miles? Anſ. 25000·8528 miles.

EX-

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Or, if we write $\frac{rs}{c}$ inſtead of $t$, we ſhall have

$$\frac{rs}{c} \times (1 - \frac{s^2}{3c^2} + \frac{s^4}{5c^4} - \frac{s^6}{7c^6} \&c)$$

for the length of the arc whoſe ſine is $s$ and coſine $c$.

And, farther, by writing $rr - ss$ for $cc$ in this laſt ſeries, we obtain $s \times (1 + \frac{s^2}{2.3r^2} + \frac{2s^4}{2.4.5r^4} + \frac{3.5s^6}{2.4.6.7r^6} + \frac{3.5.7s^8}{2.4.6.8.9r^8} \&c)$ for the arc whoſe ſine is $s$.

Again, by ſubſtituting $\sqrt{2rv - vv}$, the value of $s$, inſtead of it, in the laſt ſeries, we get

$$\sqrt{2rv} \times (1 + \frac{v}{2.3.2r} + \frac{3v^2}{2.4.5.2^2r^2} + \frac{3.5v^3}{2.4.6.7.2^3r^3} \&c)$$

for the length of the arc whoſe verſed ſine is $v$.—Or, by writing $d$, the diameter, inſtead of $2r$, the value of the arc will be expreſſed by $\sqrt{dv} \times (1 + \frac{v}{2.3d} + \frac{3v^2}{2.4.5d^2} + \frac{3.5v^3}{2.4.6.7d^3} \&c)$.

Much after the ſame manner the arc may be expreſſed by the ſecants, co-ſecants, or co-ſines. And if for any of the letters, in the general ſeries, be ſubſtituted any of their particular values, as was done for the tangent, we may, by thoſe means, obtain ſeveral different particular forms of infinite ſeries, whoſe ſums will be expreſſed by means of the circumferences of any given circles: Thus, if, in the laſt ſeries but one, $s$ be taken $= r$, or the arc be ſuppoſed to be a quadrant, we ſhall obtain the infinite ſeries $r + \frac{r^3}{2.3r^2} + \frac{3r^5}{2.4.5r^4} + \frac{3.5r^7}{2.4.6.7r^6} \&c$, or $r \times (1 + \frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7} \&c)$, whoſe ſum will be equal to $\frac{1}{4}$ of the circumference of the circle whoſe radius is $r$.

And, if from this ſeries be taken the former ſeries, expreſſed by $s$, there will reſult

$$\frac{r - s}{1} + \frac{1}{2.3} \cdot \frac{r^3 - s^3}{r^2} + \frac{3}{2.4.5} \cdot \frac{r^5 - s^5}{r^4} + \frac{3.5}{2.4.6.7} \cdot \frac{r^7 - s^7}{r^6} \&c,$$

for the length of the arc whoſe coſine is $s$.

EXAMPLE IV.

If the circumference of the earth be 25000 miles, what is the diameter? Anſ. 7957·744 miles.

PROBLEM VII.

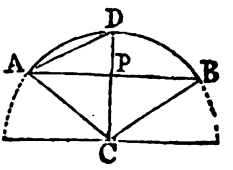
*To find the Length of any Arc of a Circle*

RULE I.*

As 180 is to the number of degrees in the arc,
So is 3·1416 times the radius, to its length.
Or as 9 is to the number of degrees in the arc,
So is ·0174533 times the radius, to its length.

EXAMPLE.

Required the length of the arc ADB, whoſe chord AB is 6, the radius being 9.




By trigonometry, 9 = AC : 3 = AP :: 1 = rad. of the tables : $\frac{3}{9}$ = ·3333333 the ſine of the ∠ ACP, or of the arc AD, to the radius 1; and the degrees, in the table of ſines, anſwering to this ſine, are 19·4712206, the double of which is 38·9424412, the degrees in the whole arc ADB.

Then, by the rule, 38·9424412 × ·0174533 × 9 = 6·117063, the length of the arc required.

RULE

* DEMONSTRATION.

For, when the radius is 1, half the circumference is 3·14159265 &c, and therefore $\frac{3\cdot14159265\ \&c}{180\ \text{degrees}}$ = ·0174532925199 &c is the length of an arc of 1 degree; hence $r$ × ·01745 &c = the length of 1 degree to the radius $r$, and, conſequently, ·01745 &c × $r$ × number of degrees in any arc, will be the length of that arc. *Q. E. D.*

RULE II.

* If $d$ denote the diameter of the circle, and $v$ the verſed ſine of half the arc; the arc will be expreſſed by $2\sqrt{dv} \times (1 + \frac{v}{2.3d} + \frac{3v^2}{2.4.5d^2} + \frac{3.5v^3}{2.4.6.7d^3}$ &c), or by $2d\sqrt{q} + \frac{q}{2.3}$ A $+ \frac{3.3q}{4.5}$ B $+ \frac{5.5q}{6.7}$ C &c. putting $q = \frac{v}{d}$, and A, B, C, &c, for the firſt, ſecond, third, &c, terms.

EXAMPLE.

Let there be taken here the ſame example as before.

Then PC $= \sqrt{CA^2 - AP^2} = \sqrt{9^2 - 3^2} = \sqrt{72} = 6\sqrt{2}$, and PD $=$ DC $-$ CP $= 9 - 6\sqrt{2} = .51471862 = v$; alſo $.51471862 \div 18 = .02859548 = \frac{v}{d} = q$.

Hence A $= 2d\sqrt{q} =$ 6.087672

B $= \frac{q}{2.3}$ A $=$ 29013

C $= \frac{3.3q}{4.5}$ B $=$ 373

D $= \frac{5.5q}{6.7}$ C $=$ 6

the ſum 6.117064 $=$ the arc ABD as before, nearly.

RULE III.

† If $r$ be the radius, and $s$ the ſine or half chord AP of the arc, then the length of the arc ADB will be

$$2s \times (1 + \frac{s^2}{3.2r^2} + \frac{3s^4}{5.2.4r^4} + \frac{3.5s^6}{7.2.4.6r^6} \text{ \&c,})$$

or $2s + \frac{q}{2.3}$ A $+ \frac{3.3q}{4.5}$ B $+ \frac{5.5q}{6.7}$ C $+ \frac{7.7q}{8.9}$ D &c,

where $q$ is $= \frac{ss}{rr}$; and A, B, C, &c, are the preceding terms.

EX-

* This rule is proved in page 122. † This likewiſe is proved in page 122.

EXAMPLE.

Taking the ſame example as in the firſt rule,

| | | | |
|---|---|---|---|
| $2s$ | = A = | | 6·000000 |
| $\frac{ss}{2.3rr}$ A | = B = | | 0·111111 |
| $\frac{3.3ss}{4.5rr}$ B | = C = | | 5555 |
| $\frac{5.5ss}{6.7rr}$ C | = D = | | 367 |
| $\frac{7.7ss}{8.9rr}$ D | = E = | | 28 |
| $\frac{9.9ss}{10.11rr}$ E | = F = | | 2 |
| | | the ſum | 6·117063 |

is the arc ADB.

RULE IV.

From 8 times the chord of half the arc, ſubtract the chord of the whole arc, and $\frac{1}{3}$ of the remainder will be the length of the arc, nearly.

That is, $\frac{8\text{AD} - \text{AB}}{3} = \text{ADB}$ nearly.*

EX-

* DEMONSTRATION.

For ſince, by the laſt rule, the arc AD $= A = s + \frac{s^3}{6r^2} + \frac{3s^5}{40r^4}$ &c.

And $c$ = the chord AD $= \sqrt{ss + xx} = \sqrt{s^2 + (r - CP)^2} =$

$$\sqrt{s^2 + (r - \sqrt{rr - ss})^2} = s + \frac{s^3}{8r^2} + \frac{7s^5}{128r^4} \text{ \&c.}$$

Therefore $A - c = \frac{s^3}{24r^2} + \frac{13s^5}{640r^4}$ &c.

But $\frac{c - s}{3} = \frac{s^3}{24r^2} + \frac{7s^5}{384r^4}$ &c.

Therefore $A - c - \frac{c - s}{3} = A - \frac{4c - s}{3} = \frac{s^5}{480r^4}$ &c.

And $2A = \frac{8c - 2s}{3}$ nearly, = the arc ADB, as in the rule. And this is Huygen's theorem.

EXAMPLE.

Taking, again, the same example; we shall have, as in the example to rule 2, PD $= 9 - 6\sqrt{2}$, and hence DA $= \sqrt{AP^2 + PD^2} = \sqrt{3^2 + (9 - 6\sqrt{2})^2} = 3\sqrt{18} - 12\sqrt{2} = 3{\cdot}043836$.

Then, by the rule, $\frac{3{\cdot}043836 \times 8 - 6}{3} = 6{\cdot}116896$ the length of the arc, nearly the same as before.

RULE V.

* Divide 3 times the versed sine of the half arc by the difference between the said versed sine and 3 times the diameter; multiply the square root of the quotient

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* DEMONSTRATION.

By the second rule, half the arc is $\sqrt{dv} \times (1 + \frac{v}{2.3d} + \frac{3vv}{2.4.5dd}$ &c).

Let there be assumed an expression which, when expanded into a series, shall be of the same form with the series above, and affected with indeterminate coefficients; then from the comparison of as many of the first terms as there are indeterminate quantities assumed, will be found the values of the said quantities in given numbers; which being written for them in the primary assumed expression, will make that expression an approximate value of the arc.

Thus, assume $d\sqrt{\frac{v}{gd - bv}}$

This in a series is $\sqrt{dv} \times (\frac{1}{\sqrt{g}} + \frac{bv}{2dg\sqrt{g}} + \frac{3bbvv}{2.4ddgg\sqrt{g}}$ &c); the two first terms of which compared with those of the series above, give these two equations, $\frac{1}{\sqrt{g}} = 1$, and $\frac{b}{g\sqrt{g}} = \frac{1}{3}$; hence $g = 1$, and $b = \frac{1}{3}$; and therefore the assumed expression $d\sqrt{\frac{v}{gd - bv}}$ becomes $d\sqrt{\frac{v}{d - \frac{1}{3}v}} = d\sqrt{\frac{3v}{3d - v}}$ = half the arc nearly; the double of which will be the whole arc, as in rule 5.

tient by 2 times the diameter; and the product will be the arc nearly.

That is, $4\text{CD}\sqrt{\frac{3\text{DP}}{6\text{CD}-\text{DP}}} = \text{ADB}$ nearly.

EXAMPLE.

Taking, ſtill, the ſame example; we find, as in the example to the laſt rule, $\text{DP} = 9 - 6\sqrt{2} = .5147186$.

Then, by the rule, $36\sqrt{\frac{.5147186\times 3}{54-.5147186}} = 36\times .1699137 = 6.116893$ the arc nearly the ſame as before.

RULE VI.

† Divide 5 times the verſed ſine of the half arc, by the difference between 5 times the diameter and 3 times the ſaid verſed ſine; multiply the ſquare-root of the quotient by 5 times the diameter; and to the product add 4 times the root of the product of the ſaid verſed ſine and diameter; and $\frac{2}{9}$ of the ſum will be the arc very nearly.

That is, if $d$ be the diameter, and $v$ the verſed ſine PD, then $5d\sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv}\times\frac{2}{9}$ will be the arc ADB very near.

EX-

† Again, to find another, ſtill nearer, approximation, let there be introduced another indeterminate quantity; thus, ſuppoſe $d\sqrt{\frac{v}{gd-bv}} + k\sqrt{dv} = \sqrt{dv}\times(k + \frac{1}{\sqrt{g}} + \frac{bv}{2dg\sqrt{g}} + \frac{3bbvv}{2.4ddgg\sqrt{g}})$ = the ſemi-arc; then, comparing the firſt three terms with thoſe of the given ſeries, we have theſe three equations, $k + \frac{1}{\sqrt{g}} = 1$, $\frac{b}{g\sqrt{g}} = \frac{1}{3}$, and $\frac{bb}{gg\sqrt{g}} = \frac{1}{5}$; whence $g = \frac{81}{25}$, $b = \frac{3}{1}\frac{4}{2}\frac{1}{5}$, and $k = \frac{4}{9}$; and by writing theſe values in the aſſumed quantity, we obtain

$$\tfrac{5}{9}d\sqrt{\frac{v}{d-\frac{3}{5}v}} + \tfrac{4}{9}\sqrt{dv} = \tfrac{1}{9}\times(5d\sqrt{\frac{5v}{5d-3v}} + 4\sqrt{dv})$$

for half the arc, the double of which will be rule 6.

EXAMPLE.

Taking, here, the ſame example as in the former rules, we ſhall have $5d\sqrt{\frac{5v}{5d-3v}}+4\sqrt{dv}\times\frac{2}{9}=6\cdot11706396$ = the arc very nearly, it being true to the ſixth or ſeventh place of decimals.

PROBLEM VIII.

*To find the Area of a Circle.*

* RULE I.

Multiply half the circumference by half the diameter, and the product will be the area.

*Note.* This rule will, alſo, ſerve for any ſector of a circle.

RULE

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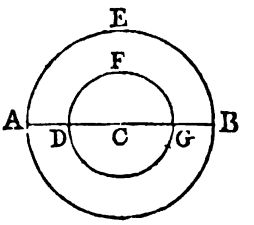
* DEMONSTRATION OF RULE I.

For, a circle may be conſidered as a polygon of an infinite number of ſides, the circumference being the perimeter, and the radius the perpendicular.

*Otherwiſe.*




Put $r$ = the radius AC, $c$ = the whole circumference AEBA, or any part of it, and $x$ = the radius CD of a circle continually expanded. Then $\frac{c}{r}\dot{x}x$ will expreſs the fluxion of the whole circle or ſector whoſe circumference is $c$; and, conſequently $\frac{cxx}{2r}$ = the area CDF; and $\frac{1}{2}cr$ = the area CAE of the whole circle or ſector accordingly. *Q. E. D.*

* RULE II.

Multiply the ſquare of the diameter by ·7854, and the product will be the area.

EXAMPLES.

1. What is the area of a circle whoſe diameter is 1, and circumference 3·1416?

$\frac{3\cdot1416\times1}{4} = \cdot7854$ the area.

Ex. 2. What is the area of a circle whoſe diameter is 7?

By the ſecond rule. $7\times7\times\cdot7854 = 38\cdot4846$ the area.

By the firſt rule. $7\times3\cdot1416 =$ the circumference.

K Then

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* DEMONSTRATION.

To prove the ſecond rule, let it be firſt obſerved, that all circles, being ſimilar figures, are as the ſquares of their diameters or circumferences: which, alſo, appears thus; by rule 1, circles are as the rectangles of their diameters and circumferences, but the circumference is as the diameter, by the notes to the laſt problem; therefore the circles are as the ſquares of either of them. Then, by rule 1, the area of a circle whoſe diameter is 1, is $\frac{1\times3\cdot14159\ \&c}{2\times2=4} = \cdot78539\ \&c$, whence $1^2 : d^2$ (the ſquare of any diameter) :: ·78539 &c : $\cdot78539d^2$, the circle whoſe diameter is $d$. *Q. E. D.*

COROLLARY.

If D be the diameter, C the circumference, and A the area of any circle; and $p = 3\cdot14159\ \&c$; from this prob. and prob. 6, we may deduce theſe equations:

1. $D = \frac{C}{p} = \frac{4A}{C} = 2\sqrt{\frac{A}{p}}$.

2. $C = pD = \frac{4A}{D} = 2\sqrt{pA}$.

3. $A = \frac{pD^2}{4} = \frac{C^2}{4p} = \frac{DC}{4}$.

4. $p = \frac{C}{D} = \frac{4A}{DD} = \frac{CC}{4A}$.

Then $\frac{7\times3\cdot1416\times7}{4} = 7\times7\times\cdot7854$ the area, the ſame as before.

Ex. 3. What is the area of the circle whoſe diameter is 1·13 chains, and circumference 3·550008 chains? Anſ. 1·00287726 ſquare chains.

Ex. 4. How many ſquare yards are in a circle whoſe diameter is $3\frac{1}{2}$ feet? Anſ. 1·06901$\frac{2}{3}$.

Ex. 5. How many ſquare feet are in a circle whoſe circumference is 10·9956 yards?

Anſ. 86·19266.

## PROBLEM IX.

### *To find the Area of any Sector of a Circle.*

### RULE I.

Multiply the radius, or half the diameter, by half the arc of the ſector, and the product will be the area, as in the whole circle.*

### EXAMPLE I.

What is the area of a ſector whoſe radius is 10 and arc 20?

By the rule, $\frac{20\times10}{2} = 100$ the area.

EX-

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* The demonſtration of this rule is contained in that of the laſt problem.

Putting $r$ = the radius of a circle, $d$ = the diameter, A = the area of a ſector of it, $a$ = the length of the arc of the ſector, $b$ = the degrees in $\frac{1}{2}a$, $s$ = half the chord of the arc $a$, or the ſine of $\frac{1}{2}a$, and $v$ = the verſed ſine of $\frac{1}{2}a$; then, by multiplying the radius by half the arc, as found by the ſeveral rules of the laſt problem but one, we ſhall obtain the following values of the area of the ſector.

EXAMPLE II.

To find the area of the ſector whoſe radius is 9, and the chord of its arc 6.

By the 7th problem the length of the arc is 6·117063.

Therefore $\frac{6\cdot117063\times9}{2} = 27\cdot5267835$ the area required.

RULE II.

As 360 is to the degrees in the arc of the ſector, ſo is the whole area of the circle, to the area of the ſector.

*Note.* For a ſemicircle take one half, for a quadrant one quarter, &c, of the whole circle.*

EXAMPLE I.

What is the area of a ſemicircle, and of a quadrant, the diameter being 10?



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1. $A = \frac{1}{2}ar = \cdot01745329\,brr.$

2. $A = r\sqrt{dv}\times\left(1+\frac{v}{2\cdot3d}+\frac{3v^2}{2\cdot4\cdot5d^2}+\frac{3\cdot5v^3}{2\cdot4\cdot6\cdot7d^3}\ \&c\right).$

3. $A = rs\times\left(1+\frac{s^2}{2\cdot3r^2}+\frac{3s^4}{2\cdot4\cdot5r^4}+\frac{3\cdot5s^6}{2\cdot4\cdot6\cdot7r^6}\ \&c\right).$

4. $A = \frac{4\sqrt{ss+vv}-s}{3}\,r = \frac{4\sqrt{2rv}-s}{3}\,r = \frac{4\sqrt{ss+vv}-s}{3}\times\frac{ss+vv}{2v}$, nearly.

5. $A = rd\sqrt{\frac{3v}{3d-v}}$, nearly.

6. $A = \frac{r}{9}\times\left(5d\sqrt{\frac{5v}{5d-3v}}+4\sqrt{dv}\right)$ nearly.

It is evident that the area of the ſector might be expreſſed in ſeveral other manners; ſuch as by the tangent, coſine, &c, of its ſemi-arc; but the forms above given, are the moſt uſeful ones.

* This rule is too evident to need any formal proof.

First, $\cdot 7854 \times 10 \times 10 = 78 \cdot 54 =$ the whole circle.
Then $78 \cdot 54 \div 2 = 39 \cdot 27 =$ the ſemicircle.
And $78 \cdot 54 \div 4 = 19 \cdot 635 =$ the quadrant.

EXAMPLE II.

To find the area of a ſector whoſe arc or angle contains 18 degrees, to a diameter of 3 feet?

First, $\cdot 7854 \times 3 \times 3 = 7 \cdot 0686 =$ the whole circle.

Then $360 : 18 :: 7 \cdot 0686 : \frac{7 \cdot 0686 \times 18}{360} = \frac{7 \cdot 0686}{20} =$ $\cdot 35343$ the area of the ſector required.

EXAMPLE III.

To find the area of a ſector whoſe radius is 9, and the chord of its arc 6.

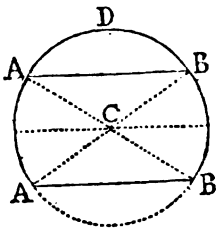
By the example to rule 1 of prob. 7, the degrees in the arc were found to be $38 \cdot 9424412$. But $\cdot 785398 \times 9 \times 9 \times 4 =$ the area of the circle.

Therefore $360 : 38 \cdot 9424412 :: \cdot 785398 \times 9 \times 9 \times 4 : \frac{38 \cdot 9424412 \times \cdot 785398 \times 9 \times 9 \times 4}{360} = 38 \cdot 9424412 \times \cdot 0785398 \times 9 = 27 \cdot 52678388$ the area of the ſector required.

PROBLEM X.

*To find the Area of the Segment of a Circle.*

RULE I.




By the laſt problem, find the area of the ſector ACB having the ſame arc with the ſegment ADB required.

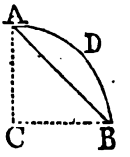
Find alſo the area of the triangle ABC, formed by the chord of the ſegment and the two radii of the ſector.

Then the ſum or difference of theſe areas will be that

that of the segment, according as it is greater or less than a semicircle.

### EXAMPLE I.

What is the area of a segment whose arc is a quadrant, or contains 90 degrees, the diameter being 18 feet?




First, $\cdot 7854 \times 18 \times 18$ is the whole circle.

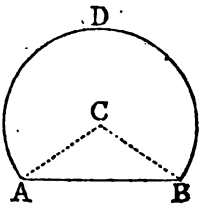
And $\frac{\cdot 7854 \times 18 \times 18}{4} = \cdot 7854 \times 9 \times 9 =$ the quadrant or sector CADB.

But $\frac{AC \times CB}{2} = \frac{9 \times 9}{2} = \cdot 5 \times 9 \times 9 =$ the triangle ACB.

Therefore $\cdot 7854 \times 9 \times 9 - \cdot 5 \times 9 \times 9 = \cdot 2854 \times 9 \times 9 = 23 \cdot 1174 =$ the area of the segment required.

### EXAMPLE II.

What is the area of a segment of a circle, whose arc contains 280°, the diameter being 50?




First, $\cdot 7854 \times 50 \times 50 = 1963 \cdot 5 =$ the whole circle.

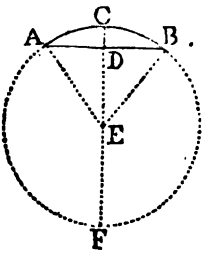
And $360 : 280 :: 1963 \cdot 5 : \frac{1963 \cdot 5 \times 280}{360} = \frac{1963 \cdot 5 \times 7}{9} = 1527 \cdot 16666 =$ the sector CADBC.

Again, the angle $ACB = 360 - 280 = 80°$. And the triangle $ACB = AC \times CB \times \frac{1}{2}$ sine $\angle ACB = 25 \times 25 \times \frac{1}{2}$ s. of $80° = 25 \times 25 \times \cdot 4924039 = 307 \cdot 7524375$.

Wherefore the sum $1834 \cdot 9191 =$ the segment ADBA required.

EXAMPLE III.

Required the area of the ſegment whoſe chord is 20, and height, or verſed ſine of half its arc, 2.




Since $AD^2 = DC \times DF$, therefore $DF = \frac{AD^2}{DC} = \frac{100}{2} = 50$. And the diameter $CF = FD + DC = 50 + 2 = 52$. Alſo $DE = EC - CD = \frac{1}{2}CF - CD = 26 - 2 = 24$.

Therefore the triangle $ABE = AD \times DE = 10 \times 24 = 240$.

Again, by trigonometry, $26 = AE : 10 = AD :: 1 = \text{s.} \angle D : \frac{10}{26} = \frac{5}{13} = .38461538 = \text{s.} \angle AED$, or of the arc $AC = 22.6198614$ degrees; the double of which, or 45.2397228, are the degrees in the arc ACB.

But $.785398 \times 52 \times 52 =$ the whole circle.

Therefore $360 : 45.2397228 :: .785398 \times 52 \times 52 : \frac{45.2397228 \times .785398 \times 26 \times 26}{90} = 266.8786995 =$ the ſector AEB.

Whence $266.8786995 - 240 = 26.8786995 =$ the ſegment ACBA required.

*Otherwiſe.*

Having found the triangle $= 240$, as above; we ſhall have, by rule 6 of the laſt problem, the ſector $= (\frac{5 \times 52}{4}\sqrt{\frac{5 \times 2}{5 \times 52 - 3 \times 2}} + \sqrt{52 \times 2}) \times \frac{2 \times 52}{9} = (65\sqrt{\frac{5}{127}} + 2\sqrt{26}) \times \frac{104}{9} = (\frac{65}{127}\sqrt{635} + 2\sqrt{26}) \times \frac{104}{9} = 266.87868$.

Then the difference is $26.87868 =$ the ſegment, nearly the ſame as before.

RULE

RULE II.

Let $d$ be the diameter, and $v$ the verſed ſine, or the height of the ſegment;

Then $2v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2} - \frac{v^3}{72d^3}$ &c), or $2v\sqrt{dv} \times (\frac{2}{3} - \frac{3v}{5.2d}\text{A} - \frac{5v}{7.4d}\text{B} - \frac{7.3v}{9.6d}\text{C} - \frac{9.5v}{11.8d}\text{D}$ &c) = the ſegment; A, B, C, &c, being the firſt, ſecond, third, &c, terms*.



* DEMONSTRATION.

By the laſt problem, the value of the ſector CADB (fig. in page 132) is $\frac{1}{2}d\sqrt{dv} \times (1 + \frac{v}{2.3d} + \frac{3v^2}{2.4.5d^2} + \frac{3.5v^3}{2.4.6.7d^3}$ &c). But $\pm\frac{1}{2}d \mp v$ = the altitude of the triangle ABC, and its baſe $\text{AB} = 2\sqrt{dv - vv} = 2\sqrt{dv} \times (1 - \frac{v}{2d} - \frac{v^2}{2.4d^2} - \frac{3v^3}{2.4.6.d^3}$ &c); and therefore the area of the triangle ABC $= (\pm\frac{1}{2}d \mp v) \times \sqrt{dv} \times (1 - \frac{v}{2d} - \frac{v^2}{2.4d^2} - \frac{3v^3}{2.4.6d^3}$ &c); which being added to, or ſubtracted from, the ſector, gives

$$2v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{4.7d^2} - \frac{3v^3}{4.6.9d^3} - \frac{3.5v^4}{4.6.8.11d^4}\ \&c)$$

for the value of the ſegment.

*Otherwiſe.*

Putting $y$ = the ſemi-chord of the ſegment, and the other letters as before. We have the fluxion of the ſegment $\dot{y}v = \dot{v}\sqrt{dv - vv} = \dot{v}\sqrt{dv} \times (1 - \frac{v}{2d} - \frac{v^2}{2.4d^2} - \frac{3v^3}{2.4.6d^3}$ &c); the fluent of which gives $v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{4.7d^2} - \frac{3v^3}{4.6.9d^3}$ &c) for the value of half the ſegment, as before. *Q. E. D.*

*Corol.*

EXAMPLE.

Take here the laſt example, in which $d = 52$, and $v = 2$; then $\frac{v}{d} = \frac{1}{26}$; and the proceſs will be thus:

$$+ A = \frac{4v\sqrt{dv}}{3} = \tfrac{8}{3}\sqrt{104} = + 27{\cdot}1947707$$

$$- B = \frac{3v}{5.2d} A = \frac{3}{10.26} A = - \cdot3137858$$

$$- C = \frac{5v}{7.4d} B = \frac{5}{7.4.26} B = - \cdot0021551$$

$$- D = \frac{7.3v}{9.6d} C = \frac{7}{3.6.26} C = - \cdot0000322$$

$$- E = \frac{9.5v}{11.8d} D = \frac{9.5}{11.8.26} D = - \cdot0000007$$

the ſum of the negative terms $- \cdot3159738$

taken from the firſt term, leave $26{\cdot}8787969$ = the area required.

RULE

---

*Corol.* 1. Let the chord of half the arc of the ſegment be denoted by $c$; then $d = \frac{cc}{v}$, by which exterminating $d$ out of the above ſeries, we have $cv \times (\frac{2}{3} - \frac{v^2}{5c^2} - \frac{v^4}{4.7c^4} - \frac{3v^6}{4.6.9c^6} - \frac{3.5v^8}{4.6.8.11c^8}$ &c) for the value of the ſemi-ſegment.

*Corol.* 2. Let the ſupplemental verſed ſine be denoted by v, viz. $v = d - v$; then $d = v + v$, by which exterminating $d$ out of the ſame ſeries, we have

$2v\sqrt{vv} \times (\frac{1}{1.1.3} + \frac{v}{1.3.5v} - \frac{v^2}{3.5.7v^2} + \frac{v^3}{5.7.9v^3} - \frac{v^4}{7.9.11v^4}$ &c) for the value of the ſaid ſemi-ſegment. And this ſeries, which is rule 3, converges very quickly in ſmall ſegments.

*Corol.* 3. By ſuppoſing the ſeveral indeterminate quantities in theſe ſeries to be expreſſed by any given numbers, we may obtain an endleſs variety of infinite ſeries, whoſe ſums will be given. And, in particular, if the diameter be 1, and ſuppoſing the verſed ſines to be equal to the radius, or the diameter, we ſhall obtain the following ſeries; in which $n$ denotes ·785398 &c, the

RULE III.

If $v$ denote the verſed ſine, as before, and v the verſed ſine of the ſupplement, or $v = d - v$; then $\frac{4}{3}v\sqrt{vv} + \frac{v}{5v}A - \frac{1v}{7v}B + \frac{3v}{9v}C - \frac{5v}{11v}D$ &c, will be = the ſegment; where A, B, C, &c, denote the preceding terms.

EXAMPLE.

Taking, again, the ſame example; we have $v = 2$, and $v = 52 - 2 = 50$. Hence $\frac{v}{v} = \frac{2}{50} = .04$, and

$$A = \tfrac{4}{3}v\sqrt{vv} = + 26.66666666\tfrac{2}{3}$$
$$B = \frac{v}{5v}A = + 21333333\tfrac{1}{3}$$
$$C = \frac{v}{7v}B = - 1219047\tfrac{16}{21}$$
$$D = \frac{3v}{9v}C = + 1625\tfrac{8}{21}$$
$$E = \frac{5v}{11v}D = - 29\tfrac{12}{21}$$
$$F = \frac{7v}{13v}E = + \tfrac{13}{21}$$

ſum of the affir. terms + 26.88001626
ſum of the negat. terms − 0.00121934$\frac{1}{3}$

the difference is 26.8787969$1\frac{2}{3}$ the area.

RULE

---

the area of the whole circle whoſe diameter is 1, or $\frac{1}{4}$ of the circumference of the ſame circle.

1. $n = \sqrt{2} \times (\frac{2}{3} - \frac{1}{5.2} - \frac{1}{4.7.2^2} - \frac{1.3}{4.6.9.2^3} - \frac{1.3.5}{4.6.8.11.2^4}$ &c).

2. $n = 2 \times (\frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \frac{1.3.5}{4.6.8.11}$ &c).

3. $n = 2 \times (\frac{1}{1.1.3} + \frac{1}{1.3.5} - \frac{1}{3.5.7} + \frac{1}{5.7.9} - \frac{1}{7.9.11} +$ &c).

Several other values of $n$ may be aſſigned from the corollaries in pages 53 and 54, and from pages 118, &c.

RULE IV.*

If $c$ denote the coſine of half the arc of the ſegment, or the difference between the radius and verſed ſine, the radius being $r$, and $q = \frac{c}{r}$; then the area of the ſegment will be equal to the difference between the ſemicircle and this ſeries, viz.

$2rc \times (1 - \frac{1}{2.3}q^2 - \frac{1}{2.4.5}q^4 - \frac{1.3}{2.4.6.7}q^6 - \frac{1.3.5}{2.4.6.8.9}q^8$ &c);

or $2rc - \frac{1}{2.3}q^2 A - \frac{1.3}{4.5}q^2 B - \frac{3.5}{6.7}q^2 C - \frac{5.7}{8.9}q^2 D$ &c:

where A, B, C, &c, are the 1ſt, 2d, 3d, &c, terms.

And this rule will be proper to uſe when the ſegment is large.

EX-

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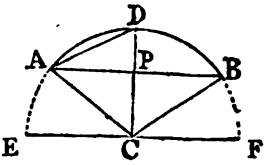
* DEMONSTRATION.




Put $x =$ CP, and $r =$ CD or CA, alſo AP $= y$. Then $r^2 = x^2 + y^2$, and $y = \sqrt{r^2 - x^2}$; and the fluxion of the area ABFE, or $2yx$, is $2\dot{x}\sqrt{r^2 - x^2} =$

$2r\dot{x} \times (1 - \frac{1}{2} \cdot \frac{x^2}{r^2} - \frac{1}{2.4} \cdot \frac{x^4}{r^4} - \frac{1.3}{2.4.6} \cdot \frac{x^6}{r^6}$ &c); and the fluent is $2rx \times (1 - \frac{1}{2.3} \cdot \frac{x^2}{r^2} - \frac{1}{2.4.5} \cdot \frac{x^4}{r^4} - \frac{1.3}{2.4.6.7} \cdot \frac{x^6}{r^6}$ &c); which is the area of the zone ABFE; and this being taken from the ſemicircle EADBF, will leave the area of the ſegment ADB, as in the rule. And this will converge very faſt when $x$ is ſmall, or the ſegment large.

*Corol.* When $x$ becomes equal to $r$, the zone becomes a ſemicircle, and then $2r^2 \times (1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{1.3}{2.4.6.7}$ &c) is the area of the ſemicircle, to the radius $r$; and conſequently the ſum of the ſeries is

$$n = 1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{1.3}{2.4.6.7} - \frac{1.3.5}{2.4.6.8.9} - \text{&c.}$$

where $n = .78539$ &c, the area of the circle whoſe diameter is 1; being one more value of $n$, to be added to thoſe found in page 137.

EXAMPLE.

Suppoſe the verſed ſine be 24, or the coſine 2; the radius being 26, as before.

Here $r = 26$, and $c = 2$; then $q = \frac{c}{r} = \frac{1}{13}$, & $q^2 = \frac{1}{169}$.

Hence $+ A = 2rc = +104.0000000$

$- B = \frac{1}{2 \cdot 3} q^2 A = - 0.1025642$

$- C = \frac{1 \cdot 3}{4 \cdot 5} q^2 B = - 910$

$- D = \frac{3 \cdot 5}{4 \cdot 7} q^2 C = - 2$

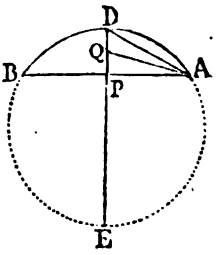
the ſum of the neg. terms — 0.1026554
taken from the firſt, leaves 103.0973446 for the ſeries;
this taken from the ſemicir. 1061.8494892

leaves the area of the ſeg. 958.7521446

RULE V.

To the chord of the whole arc, add $\frac{4}{3}$ of the chord of half the arc, multiply the ſum by the verſed ſine, or height of the ſegment, and $\frac{4}{10}$ of the product will be the area of the ſegment nearly.

That is, $(c + \frac{4}{3}c) \times \frac{4}{10}v$, or $(c + \frac{4}{3}\sqrt{dv}) \times \frac{4}{10}v$, or $(c + \frac{4}{3}\sqrt{\frac{1}{4}c^2 + v^2}) \times \frac{4}{10}v$, or $(\sqrt{dv - vv} + \frac{2}{3}\sqrt{dv}) \times \frac{4}{5}v =$ the area nearly: where $d =$ DE the diameter, $c =$ BA the chord of the whole arc, $c =$ AD the chord of the half arc, and $v =$ DP the height of the ſegment.*




EX-

* DEMONSTRATION.

The ſegment is $= 2v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v}{28d}$ &c) by rule 3.

Suppoſe

EXAMPLE.

Taking the ſame example as in the firſt three rules, in which $c = 20$, and $v = 2$.

Then $(c + \frac{4}{3}\sqrt{\frac{1}{4}cc + vv}) \times \frac{4}{10}v = 26{\cdot}877998 =$ the area, nearly the ſame as before.

RULE VI.

Draw AQ ſo, that DP : PQ :: 5 : $\sqrt{10}$, then the ſegment ADBA will be nearly equal to $\frac{4}{3}$DP × AQ $= \frac{4}{3}v\sqrt{dv - \frac{3}{5}vv} = \frac{4}{3}v\sqrt{cc - \frac{3}{5}vv} = \frac{4}{3}v\sqrt{\frac{1}{4}cc + \frac{2}{5}vv}$. Where $d =$ the diameter, $v =$ DP, $c =$ DA, and $c =$ AB, as in the laſt rule.*

EX-

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Suppoſe it $= 2\text{DP} \times (m \times \text{PA} + n \times \text{AD}) = 2v \times (m\sqrt{dv - vv} + n\sqrt{dv})$
$= 2v\sqrt{dv} \times (m\sqrt{1 - \frac{v}{d}} + n) = 2v\sqrt{dv} \times (\begin{matrix}+m \\ +n\end{matrix} - \frac{mv}{2d} - \frac{mv^2}{8d^2}$ &c)

Then let the coefficients of the correſponding terms be equated, and we ſhall have $m + n = \frac{2}{3}$, and $\frac{m}{2} = \frac{1}{5}$; whence $m = \frac{2}{5}$, and $n = \frac{2}{3} - m = \frac{2}{3} - \frac{2}{5} = \frac{4}{15}$.

Then, by ſubſtituting theſe values of $m$ and $n$ in the aſſumed quantity, we ſhall have $2\text{DP} \times (m \times \text{PA} + n \times \text{AD}) = 2\text{DP} \times (\frac{2}{5}\text{PA} + \frac{4}{15}\text{AD}) = \frac{4}{15}\text{DP} \times (2\text{PA} + \frac{4}{3}\text{AD})$, which is the rule. Or it is $= \frac{4}{15}\text{DP} \times (3\text{PA} + 2\text{AD})$. And this is one of Sir I. NEWTON's rules, publiſhed by JONES, WALLIS, and COLSON.

* DEMONSTRATION OF RULE VI.

By rule 2 the ſegment is $= 2v\sqrt{dv} \times (\frac{2}{3} - \frac{v}{5d} - \frac{v^2}{28d^2}$ &c).

Suppoſe it equal to $2v\sqrt{mdv - nvv} = 2v\sqrt{dv} \times \sqrt{m - \frac{nv}{d}} =$
$2v\sqrt{dv} \times (m^{\frac{1}{2}} - \frac{nv}{2m^{\frac{1}{2}}d} - \frac{n^2v^2}{8m^{\frac{3}{2}}d^2}$ &c). Then, by equating the coefficients,

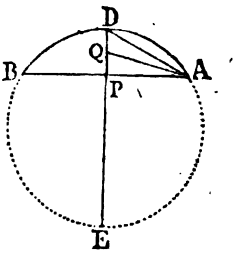
EXAMPLE.

Taking, again, the ſame example;

We have $\frac{4}{3}v\sqrt{\frac{1}{4}cc+\frac{2}{5}vv}=\frac{8}{3}\sqrt{101\frac{3}{5}}=26{\cdot}87915$ = the area nearly.

RULE VII.

If AQ biſect DP, the ſegment ADBA will be very nearly equal to $(4QA+AD)\times\frac{4}{15}DP=(2\sqrt{cc+vv}+c)\times\frac{4}{15}v=(2\sqrt{\frac{1}{4}dv-3vv}+\sqrt{dv})\times\frac{4}{15}v=(2\sqrt{cc+vv}+\sqrt{\frac{1}{4}cc+vv})\times\frac{4}{15}v$; where the letters denote the ſame quantities as in the laſt rule.*




EX-

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coefficients, we have $\sqrt{m}=\frac{2}{3}$, and $\frac{n}{\sqrt{m}}=\frac{2}{5}$; whence $m=\frac{4}{9}$, and $n=\frac{4}{15}$.

Then, by ſubſtituting theſe values, we get $2v\sqrt{mdv-nvv}=2v\sqrt{\frac{4}{9}dv-\frac{4}{15}vv}=\frac{4}{3}v\sqrt{dv-\frac{3}{5}vv}$, as in the rule.

*Corol.* But, now, to find what line the quantity $\sqrt{dv-\frac{3}{5}vv}$ repreſents; I conſider that it is greater than AP, which is $=\sqrt{dv-vv}$, and therefore I ſuppoſe it to be ſome line AQ meeting DP between D and P.

Then $PQ=\sqrt{QA^2-AP^2}=\sqrt{dv-\frac{3}{5}vv-dv+vv}=v\sqrt{\frac{2}{5}}=\frac{v\sqrt{10}}{5}$.

Wherefore if there be made DP : PQ :: 5 : $\sqrt{10}$, and QA be drawn; the ſegment will be nearly $\frac{4}{3}$ DP × AQ. And this is another of Sir I. NEWTON's Rules.

* DEMONSTRATION OF RULE VII.

Suppoſe the ſegment $2v\sqrt{dv}\times(\frac{2}{3}-\frac{v}{5d}$ &c) to be = 2 PD ×

(*m*

EXAMPLE.

Taking, ſtill, the ſame example;

We ſhall have $(2\sqrt{cc+vv}+\sqrt{\frac{1}{4}cc+vv})\times\frac{4}{15}v = 26{\cdot}8786888 =$ the area very nearly.

RULE VIII.

From the ſquare of the chord of half the arc, ſubtract $\frac{5}{7}$ of the ſquare of the verſed ſine of the ſaid half arc; to 7 times the root of the difference, add $\frac{4}{3}$ of the ſaid chord; multiply the ſum by the ſaid verſed ſine, and $\frac{4}{25}$ of the product will be the area very near.

That is, $(7\sqrt{cc-\frac{5}{7}vv}+\frac{4}{3}c)\times\frac{4}{25}v = (7\sqrt{dv-\frac{5}{7}vv}+\frac{4}{3}\sqrt{dv})\times\frac{4}{25}v = (7\sqrt{\frac{1}{4}cc+\frac{2}{7}vv}+\frac{4}{3}\sqrt{\frac{1}{4}cc+vv})\times\frac{4}{25}v =$ the area very near: where the letters are the ſame as in the laſt rules.*

EX-

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$(m\times QA+n\times AD) = 2v\times(m\sqrt{dv-\frac{3}{4}vv}+n\sqrt{dv}) = 2v\sqrt{dv}\times(m\sqrt{1-\frac{3v}{4d}}+n) = 2v\sqrt{dv}\times(\begin{matrix}+m\\+n\end{matrix}-\frac{3mv}{8d}$ &c).

Then, equating the coefficients, we get $m+n=\frac{2}{3}$, and $\frac{3m}{8} = \frac{1}{5}$; whence $m = \frac{8}{15}$, and $n=\frac{2}{15}$.

Therefore $2PD\times(m\times QA+n\times AD) = 2PD\times(\frac{8}{15}QA+\frac{2}{15}AD) = \frac{4}{15}PD\times(4QA+AD)$, as in the rule. And this likewiſe is a rule given by Sir I. NEWTON.

* DEMONSTRATION.

The ſegment is $= 2v\sqrt{dv}\times(\frac{2}{3}-\frac{v}{5d}-\frac{v^2}{28d^2}$ &c).

Suppoſe it $= 2v\times(m\sqrt{dv-nvv}+p\sqrt{dv}) = 2v\sqrt{dv}\times(p+m\sqrt{1-\frac{nv}{d}}) = 2v\sqrt{dv}\times(\begin{matrix}+p\\+m\end{matrix}-\frac{mnv}{2d}-\frac{mn^2v^2}{8d^2}$ &c).

Then, equating the like coefficients, we have $m+p=\frac{2}{3}$, $\frac{mn}{2}$ $=\frac{1}{5}$, and $\frac{mnn}{8}=\frac{1}{28}$; whence $n=\frac{5}{7}$, $m=\frac{14}{25}$, and $p=\frac{8}{75}$.

Wherefore

EXAMPLE.

Taking again, the same example as before, where $c = 20$, and $v = 2$;

We have $(7\sqrt{\frac{1}{4}cc + \frac{2}{7}vv} + \frac{4}{3}\sqrt{\frac{1}{4}cc + vv}) \times \frac{4}{25}v = 26{\cdot}8787998 =$ the area very exact.

RULE IX.

1. Divide the versed sine, or height, by the diameter.

2. Find the quotient in the column of versed sines in the table of circular segments, at the end of the volume.

3. Multiply the number immediately on the right of the versed sine, in the table, by the square of the diameter, and the product will be the area.*

*Note.*

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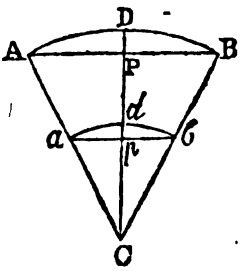
Wherefore $2v \times (m\sqrt{dv - nvv} + p\sqrt{dv}) = \frac{4}{25}v \times 7\sqrt{dv - \frac{5}{7}vv} + \frac{4}{3}\sqrt{dv}$), as in the rule.

*Corol.* Suppose $\sqrt{dv - \frac{5}{7}vv} =$ AQ [*see last fig.*]; then PQ $=$ $\sqrt{QA^2 - AP^2} = \sqrt{dv - \frac{5}{7}vv - dv + vv} = v\sqrt{\frac{2}{7}} = \frac{v\sqrt{14}}{7}$.

Wherefore if AQ divide DP in Q, so that $7 : \sqrt{14} :: DP : PQ$: then $(7\,QA + \frac{4}{3}AD) \times \frac{4}{25}DP$ will be nearly equal to the segment ADBA. And this rule is not only new and very simple, but also the most exact of any approximation that I know of for this purpose.

* DEMONSTRATION.

The rule is founded on this property, That segments whose versed sines are as the diameters, will be to each other as the squares of the diameters; which is thus proved.




Let ADBA, *adba*, be two similar segments, cut off from the similar sectors ADBCA, *adbca*, by the chords AB, *ab*. And draw CD to bisect them.

Then, by similar triangles, CA : C*a* :: CA $-$ DP (CP) : C*a* $-$ *dp* (C*p*) :: DP : *dp*.



That

*Note.* When the quotient, arising from the verſed ſine divided by the diameter, has a remainder, or fraction, after the fourth place of decimals; having taken the area anſwering to the ſaid four firſt figures, ſubtract it from the next following area, multiply the remainder by the ſaid fraction, and add the product to the firſt area; and the ſum will be the area for the whole quotient.—This method is uſed when accuracy is required; but for common uſe, the fraction may be omitted.

EXAMPLE.

If the diameter be 52, and the verſed ſine 2; what is the area?

Firſt $\frac{2}{52} = \frac{1}{26} = \cdot 0384\frac{8}{13}$, the tabular verſed ſine.

Then againſt ·0384 ſtands ·00991672, and the difference between this area and the next ·00995517 is ·00003845, which multiplied by $\frac{8}{13}$ produces ·00002366, which added to ·00991672 gives ·00994038 = the area belonging to the verſed ſine $\cdot 0384\frac{8}{13}$.

Wherefore ·00994038 × 52 × 52 = 26·87878752 = the area, nearly the ſame as before.

PRO-

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That is, ſimilar ſegments have their verſed ſines as their radii or diameters.

Again, ſince ſimilar ſectors are as the ſquares of the diameters, as well as ſimilar triangles as the ſquares of their like ſides, we ſhall have CA² : *ca*² :: ſector CADB : ſector *cadb* :: △ CAB : △ *cab* :: ſegment ADBA (ſector CADB − △CAB) : ſegment *adba* (ſector *cadb* − △ *cab*).

Now, putting $d$ = any diameter, and $v$ = verſed ſine. We ſhall have $d : v ::$ 1 (diameter in the table) : $\frac{v}{d}$ = the verſed ſine of a ſimilar ſegment in the table, whoſe area let be called $a$.

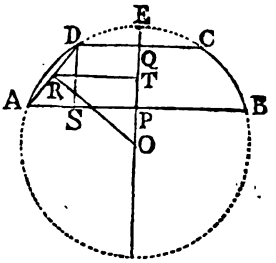
Then $1^2 : dd :: a : add$ = the area of the ſegment whoſe height is $v$ and diameter $d$, as in the rule.

## PROBLEM XI.

*To find the Area of the Zone or Space* ABCDA *included between two parallel Chords* AB, CD, *and the two Arcs* AD, CB.

### RULE I.

Find the area of each ſegment AEBA, DECD, and their difference will be that of the zone required.




### EXAMPLE I.

Suppoſe the greater chord AB be 96, the leſs chord DC 60, and the diſtance PQ between them 26; required the area.

Draw AD, and let DS be ⊥ AP, OR ⊥ AD, and RT ⊥ PQ; then QT = TP, becauſe AR = RD; and TR = $\frac{1}{2}$AP + $\frac{1}{2}$DQ. But, in the ſimilar triangles DAS, ROT, it will be DS : AS :: RT : TO = $\frac{RT \times AS}{DS} = \frac{AP + DQ \times AP - DQ}{2PQ} = \frac{78 \times 18}{52} = 27$. And hence OP = OT − TP = 27 − 13 = 14.

Then OE = OA = $\sqrt{AP^2 + PO^2} = \sqrt{48^2 + 14^2} =$ 50 = the radius.

Whence PE = OE − OP = 50 − 14 = 36
QE = OE − OQ = 50 − 40 = 10 the two verſed ſines.

And $\frac{36}{100} = .36$

$\frac{10}{100} = .10$ the two tab. verſed ſines. Againſt which, in the table, ſtand the areas .25455055, and .04087528.

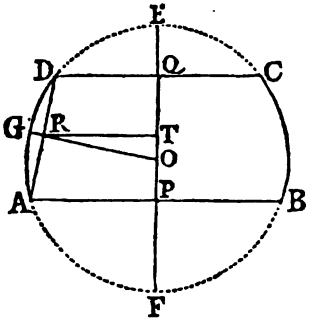
Then $.25455055 \times 100 \times 100 = 2545.5055$

$.04087528 \times 100 \times 100 = 408.7528$

their difference 2136.7527 is the area of the zone required.

EXAMPLE II.

Suppoſe the greater chord AB to be 20, the leſs DC 15, and their diſtance PQ $17\frac{1}{4}$. Required the area of the zone ABCD.




Here, as in the laſt example,

$$TO = \frac{PA + DQ \times AP - DQ}{2PQ} = \frac{35 \times 5}{2 \times 2 \times 35} = \frac{5}{4}.$$

Which being leſs than TP ($8\frac{3}{4}$) half the diſtance of the chords, it ſhews that the center falls within the zone; and then it is evident that the whole circle diminiſhed by the ſum of the two ſegments AFB, DEC, will give the zone.

Now, $OP = TP - TO = 8\frac{3}{4} - \frac{5}{4} = 7\frac{1}{2}$. Hence $AO = \sqrt{OP^2 + PA^2} = \sqrt{\left|\frac{15}{2}\right|^2 + 10^2} = \frac{25}{2}$ = the radius; therefore the diameter is 25. And $OQ = QT + TO = 8\frac{3}{4} + 1\frac{1}{4} = 10$.

Whence

Whence QE = OE − OQ = $12\frac{1}{2} - 10 = 2\frac{1}{2}$,
PF = OF − OP = $12\frac{1}{2} - 7\frac{1}{2} = 5$.

Then $\frac{2\frac{1}{2}}{25} = \frac{5}{50} = \cdot 1$,

and $\frac{5}{25} = \frac{1}{5} = \cdot 2$ the tab. verſed ſines.

Againſt which ſtand the areas ·04087528,
and ·11182380.

Then ·04087528×25×25 = the ſegment DEC,
·11182380×25×25 = the ſegment AFB,

whoſe ſum is ·15269908×25².

But ·7853981 6×25² = the whole circle.

Their dif. is ·62369908×25² = 395·436925 the zone ABCD required.

## RULE II.

To the area of the trapezoid ADQP, add that of the ſmall ſegment AGDKA, and double the ſum will be the area of the zone ABCD required.—And this rule is eaſier than the other, as only one ſegment is to be found.

## EXAMPLE.

Suppoſe the ſame dimenſions as in the laſt example.

Then, drawing the radius ORG, we ſhall have OR = $\sqrt{OT^2 + TR^2} = \sqrt{(\frac{1}{2}DQ + \frac{1}{2}PA)^2 + OT^2} = \sqrt{(\frac{35}{4})^2 - (\frac{5}{4})^2}$
$= \frac{5}{4}\sqrt{1^2 + 7^2} = \frac{5}{4}\sqrt{50} = \frac{25}{4}\sqrt{2} = 8{\cdot}838834765$.
Hence GR = OG − OR = 12·5 − 8·838834765 = 3·661165 the verſed ſine.

Therefore $\frac{3{\cdot}661165}{25} = \cdot 1464466$ the tabular verſed ſine; the tabular area for which is ·07134954.

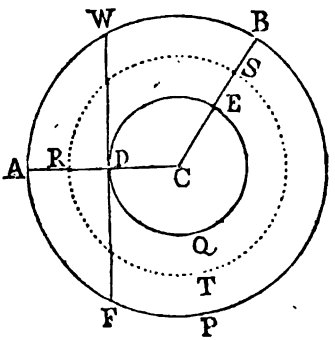
Then ·07134954 × 25 × 25 = 44·5934625, the ſegment AGDRA. But

$(\frac{1}{2}DQ+\frac{1}{2}AP)\times PQ=\frac{35}{4}\times\frac{35}{2}=\frac{1225}{8}=153{\cdot}125$, the trapezoid APQD.

Their ſum is - - - - - - - 197·7184625, the ſemi-ſeg. APQDGA. The double 395·4369250, is the zone ABCD, the ſame as before.

## PROBLEM XII.

*To find the Area of the Ring included between the Circumferences* ABPA, DEQD *of two concentric Circles.*




### RULE I.

Multiply the ſum of the diameters by the difference of the diameters, and the product again by ·7854, for the area of the ring required.*

EX-

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* DEMONSTRATION.

For, ſince the ring is evidently equal to the difference of the two circles, if the diameters be called D, *d*, and ·785398 &c. = *a*; the ring will be $= aD^2-ad^2=a\times(D+d)\times(D-d)$. *Q. E. D.*

*Corol.* 1. Whence, if there be made DW perpendicular to the radius CDA, the ring will be equal to the circle whoſe radius is DW;

EXAMPLE.

If the diameters of two concentric circles be 10 and 6, what will be the area of the ring included between their circumferences?

$\cdot 7854 \times (10+6) \times (10-6) = \cdot 7854 \times 16 \times 4 =$ 50·2656, the area required.

RULE II.

Multiply half the sum of the circumferences by half the difference of the diameters, and the product will be the area.*

EXAMPLE.

Taking the same example as before, we shall have

First $3 \cdot 1416 \times \begin{matrix} 10 = 31 \cdot 4160 \\ 6 = 18 \cdot 8496 \end{matrix}$ the circumferences.

Then $\frac{31 \cdot 416 + 18 \cdot 8496}{2} \times \frac{10-6}{2} = \frac{50 \cdot 2656}{2} \times \frac{4}{2} =$ 50·2656, the area the same as before.

*Note.* This rule will serve, also, for any part ABEDA of the ring included between the parts AD, BE, of two radii, by using half the sum of their intercepted arcs for half the sum of the circumferences.



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DW; or equal to the circle whose diameter is FW, a tangent to the less circle, and terminated by the greater.

*Corol.* 2. Or the ring is equal to an ellipse whose axes are $D + d$ and $D - d$. As will appear by comparing the above rule with the rule for an ellipse to be hereafter given.

* DEMONSTRATION OF RULE II.

For the circumferences c, $c$, are equal to $4aD$, $4ad$; therefore $a \times (D + d) = \frac{1}{4}c + \frac{1}{4}c$; which, substituted in the last rule, gives $a \times (D+d) \times (D-d) = (\frac{1}{4}c + \frac{1}{4}c) \times (D-d) = (\frac{1}{4}c + \frac{1}{4}c) \times (\frac{1}{2}D - \frac{1}{2}d)$, as in the rule. *Q. E. D.*

And it is evident that the same rule will serve for a part of the ring cut off by two radii, using the lengths of the included curves for c and $c$.

Thus, if the length of the arc AB be 15; then, by reaſon of the ſimilarity of the arcs AB, DE, it will be $10 : 6 :: 15 : 3 \times 3 = 9$, the arc DE.

Whence $\frac{1}{2}(15+9) \times \frac{1}{2}(10-6) = 12 \times 2 = 24$, the area of the part ABEDA.

### RULE III.

Multiply the perpendicular breadth of the ring, that is, the difference of the radii, by the circumference RST (or part RS for the part ABEDA) having the ſame center with, and equally diſtant from the bounding arcs.*

### EXAMPLE.

Taking, ſtill, the ſame example; it will be

Firſt $AD = CA - CD = 5 - 3 = 2$, the diſtance of the circumferences.

And $CR = CD + DR = CD + \frac{1}{2}AD = 3 + 1 = 4$, the radius of the middle arc.

Hence $3{\cdot}1416 \times 8 = 25{\cdot}1328 =$ the middle circumference.

Therefore $25{\cdot}1328 \times 2 = 50{\cdot}2656$, the area the ſame as before.

And for the part ABEDA, we have $CA : CR :: AB : SR = \frac{15 \times 4}{5} = 3 \times 4 = 12$.

Then $AD \times RS = 2 \times 12 = 24$, the area.

PRO-

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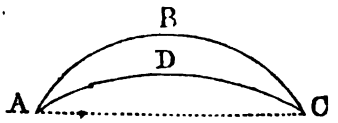
* DEMONSTRATION.

For this circumference, being equally diſtant from the other two, will be equal to half their ſum. Wherefore &c.

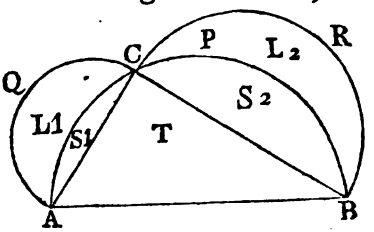
*Corol.* Hence the whole ring, or any part of it ABEDA, included between two radii, is equal to a parallelogram on the ſame baſe AD, and whoſe altitude is equal to RS the middle circumference.

## PROBLEM XIII.

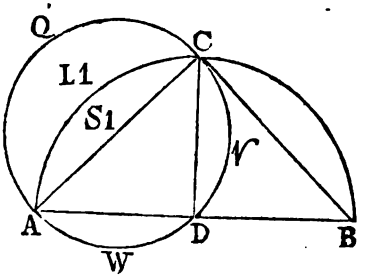
*To find the Areas of Lunes.*




Lunes, *lunulæ*, or little moons, are ſpaces included between the interſecting arcs of two excentric circles, as the lune ABCDA; which is, evidently, equal to the difference of the ſegments ABCA, ADCA.




If ABC be a triangle right-angled at C, and if ſemicircles be deſcribed on the three ſides as diameters; then the triangle T (ABC) will be equal to the ſum of the two lunes L 1, L 2.—For the greateſt ſemicircle is equal to the ſum of the other two; from the greateſt ſemicircle take the ſegments S 1, S 2, and there will remain the △ T; from the two leſs ſemicircles take alſo the ſame two ſegments S 1, S 2, and there will remain the two lunes L 1, L 2; wherefore T = L 1 + L 2.



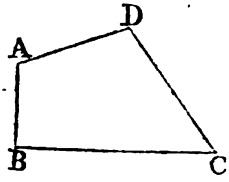
Whence, if the two ſides AC, CB, of the triangle be equal to each other; the two lunes will, alſo, be equal, and each lune LI equal to the △ ACD; and therefore the ſegment SI = ſemicircle AQCA − △ ACD = double the ſeg. AWDA, or double the ſeg. DVCD.

## *SECT. II.

### A PROMISCUOUS COLLECTION OF QUESTIONS CONCERNING AREAS.

#### QUESTION I.

IN the trapezium ABCD are given AB = $6\frac{1}{2}$, BC = $15\frac{3}{5}$, CD = 12, and DA = 9, alſo B a right angle; to find the area of the trapezium.




First $\sqrt{AB^2 + BC^2} = \sqrt{(\frac{13}{2})^2 + (\frac{78}{5})^2} =$ 16·9 = AC; and $\sqrt{\frac{37\cdot9}{2} \times \frac{4\cdot1}{2} \times \frac{13\cdot9}{2} \times \frac{19\cdot9}{2}} = \frac{1}{4}\sqrt{37\cdot9 \times 4\cdot1 \times 13\cdot9 \times 19\cdot9} = \frac{1}{4}\sqrt{42982\cdot4279} =$ 51·8305098 = the area of the triangle ADC.

And

* Some of the queſtions in this ſection are taken from other books; but the methods of ſolution are, generally, different from thoſe uſed in the books from which they were taken, they being there moſtly ſolved by an analytical proceſs. And I have conſtructed thoſe of which the conſtructions do not appear to be ſelf-evident.

And $\frac{6\frac{1}{2} \times 15\frac{3}{4}}{2} = \frac{13}{4} \times \frac{78}{4} \times \frac{1014}{20} = 50{\cdot}7$, the area of the triangle ABC.

The sum is 102·5305098, the area of the trapezium required.

QUEST. 2. To find the area of a trapezium, the length of its sides being as in the margin, and the sum of the two opposite angles B and D equal to 180 degrees.

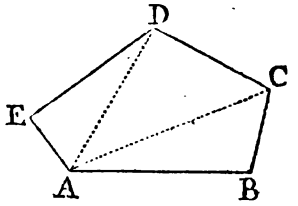
| | Chains |
|---|---|
| AB | = 15·6 |
| BC | = 13·2 |
| CD | = 10·0 |
| DA | = 26·0 |

By rule 5 of prob. 3 of sect. 1. (32·4 being half the perimeter)

$\sqrt{(32{\cdot}4-15{\cdot}6)\times(32{\cdot}4-13{\cdot}2)\times(32{\cdot}4-10)\times(32{\cdot}4-26)} = \sqrt{16{\cdot}8 \times 19{\cdot}2 \times 22{\cdot}4 \times 6{\cdot}4} = 215{\cdot}04$ square chains $= 21{\cdot}504$ acres $= 21$ a. 2 r. 0·64 perches, the area required.

*Note.* A construction of this problem may be seen in Simpson's Select Exercises, page 135.

QUEST. 3. In the pentangular field ABCDE are given AB = 14, BC = 7, CD = 10, DE = 12, EA = 5, and the diagonal AC = 17 chains, also E a right angle; required the area.




First AD $= \sqrt{DE^2 + EA^2} = \sqrt{12^2 + 5^2} = 13$.

Then AE $\times \frac{1}{2}$ED $= 5 \times 6 = 30 = \triangle$ ADE.

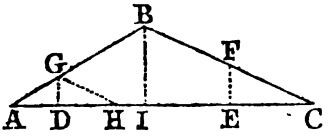
And $\sqrt{20 \times 3 \times 7 \times 10} = 10\sqrt{42} = 64{\cdot}807407 = \triangle$ ADC.

Also $\sqrt{19 \times 2 \times 5 \times 12} = 2\sqrt{570} = 47{\cdot}749345 = \triangle$ ABC.

The sum of all three is 142·556752 square chains = 14 a. 1·0227 r. the area required.

QUEST.

QUEST. 4. Given the baſe AC = 32, AD = 5, EC = 9, the perpendicular EF = 4, and the perpendicular DG = 3; required the area of the triangle ABC.

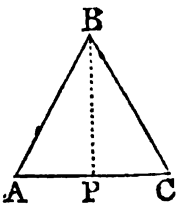



Draw GH parallel to BC. Then, by the ſimilar triangles GDH, FEC, we have FE 4 : EC 9 :: GD 3 : DH $\frac{27}{4}$; and hence AH = $\frac{47}{4}$. Again, from the ſimilar triangles AGH, ABC, we have AH $\frac{47}{4}$ : AC 32 :: DG 3 : the perpendicular IB = $\frac{384}{47}$.

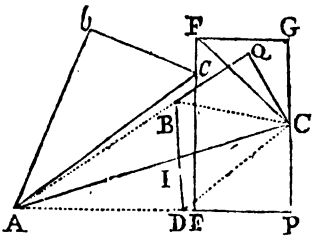
Hence $\frac{1}{2}$AC × IB = $16 \times \frac{384}{47} = 130\frac{34}{47} = 130{\cdot}7234\frac{2}{47}$ is the area of the triangle required.

QUEST. 5. What is the ſide of that equilateral triangle, whoſe area coſt as much paving at 8d a foot, as the palliſading the three ſides did at a guinea a yard?

The ſides are 7s. a foot, and the area $\frac{2}{3}$s. a foot. And that the produces may be equal, the quantities muſt be inverſely as the prices; but, by rule 2, page 113, $\frac{1}{4}$BC$^2\sqrt{3}$ is the area; therefore $\frac{2}{3} : 7 :: 3BC : \frac{1}{4}BC^2\sqrt{3}$ $:: \frac{3 \times 4}{\sqrt{3}} : BC = \frac{3 \times 4 \times 7 \times 3}{2\sqrt{3}} = 42\sqrt{3}$ $= 72{\cdot}7461339$, the ſide required.




QUEST. 6. Surveying a quadrangular field, I found the four ſides to be 10, 9, 7, and 6 chains, in a ſucceſſive order: I likewiſe, at the two extremes of the longeſt ſide, took the bearings of the oppoſite angles, which were N. E. by E. and N. W. Hence the content of the field is required.




From the given bearings of the angles C and D, from A and B, it appears that the diagonals AC, BD, make the angle AID at their interſection of 7 points or $78\frac{3}{4}$ degrees; therefore, by rule 3, problem 3, ſection 1, we have $\frac{1}{4}(10^2+7^2-9^2-6^2) \times \frac{1}{4}$ tang. $78\frac{1}{4}^{\circ} = 32 \times \frac{1}{4} \times 5{\cdot}0273395 = 8 \times 5{\cdot}0273395 =$ 40·218716 ſquare chains $=$ 4a. 3·499456 perches, the content required.*

QUEST.

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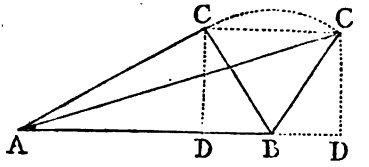
* CONSTRUCTION.

With two of the given lines *Ab* 10, and *bc* 6, make a right angle *b*; and with the other two complete the trapezium *Abc*D. In the perpendicular E*c* produced, take EF $=$ 8·937492 $=$ the double of the area 40·218716 divided by AD 9; and with the center F and radius equal to $6\frac{1}{3}$ a fourth proportional to DA, A*b*, *bc*, deſcribe an arc meeting, in C, another arc deſcribed with the center D and radius D*c*; then join D, C, and with the other two given lines, $=$ *cb* and *b*A, complete the trapezium ABCD, and it is done.

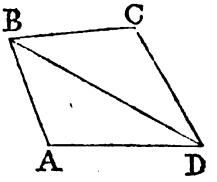
For, having drawn A*c*, AC, and CF, and let fall the perpendiculars, CP upon AD, CQ upon AB, and FG upon PCG; ſince $AD^2 + DC^2 + 2AD \times DP = AC^2 = AB^2 + BC^2 + 2AB \times BQ$, and $AD^2 + Dc^2 + 2AD \times DE = Ac^2 = Ab^2 + bc^2$, by taking theſe latter quantities from the former, it follows that $2AD \times EP = 2AD \times DP - 2AD \times DE = 2AB \times BQ$, and conſequently BQ : FG (EP) :: DA : AB :: BC : CF by the conſtruction; whence the triangles CQB, CGF are ſimilar, and therefore CQ : CG :: CB : CF :: (by the conſtruction) AD : AB; and hence CQ $\times$ AB $=$ AD $\times$ CG; and, by adding CP $\times$ AD to each we have CP $\times$ AD $+$ CQ $\times$ AB $=$ double the area ABCD $=$ CP $\times$ AD $+$ CG $\times$ AD $=$ EF $\times$ AD $=$ (by the conſtruction) double the given area.

QUEST. 7. Given two ſides of an obtuſe-angled triangle, which are 20 and 40 poles; required the third ſide, that the triangle may contain juſt an acre of land.




Now the area $160 \div 20$ ($\frac{1}{2}$AB) $= 8 =$ the perpendicular CD; but $\sqrt{CB^2 - CD^2} = \sqrt{20^2 - 8^2} =$ DB; hence $AD = AB \pm BD = 40 \pm \sqrt{20^2 - 8^2}$ and $AC = \sqrt{AD^2 + DC^2} = \sqrt{8^2 + 40^2 + 20^2 - 8^2 \pm 80\sqrt{(20^2 - 8^2)}} = 4\sqrt{125 \pm 20\sqrt{21}} = 4 \times \sqrt{105 \pm 2\sqrt{5}} = 58{\cdot}87634686$ and $23{\cdot}09925922$, either of which may be the ſide required.

QUEST. 8. Given the four ſides AB $= 9$, BC $= 10$, CD $= 11$, and DA $= 12$, and the angle BDC $= 30°$, formed by the diagonal BD and the ſide DC; required the area of the trapezium.




Firſt, by trigonometry, BC : CD :: $\frac{1}{2}$ = S. $\angle$BDC $= \frac{11}{20} = {\cdot}55$ the ſine of the angle DBC $= 33° 22'$; whence $180° - 30° - 33° 22' = 180° - 63° 22' = 116° 38' = \angle C$; and S. $\angle$BDC : S. $\angle C$ :: BC : BD $= 10 \times 8938936 \times 2 = 17{\cdot}877872$.

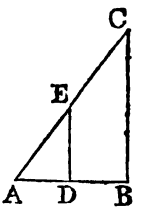
Then BC $\times$ CD $\times \frac{1}{2}$ S. $\angle C = 10 \times 11 \times {\cdot}4469468 = 49{\cdot}164148 =$ the area of the triangle BCD.

And

And $\sqrt{19{\cdot}438936 \times 10{\cdot}438936 \times 7{\cdot}438936 \times 1{\cdot}561064} = 48{\cdot}54337$ = the area of the triangle ABD.

And their sum = 97·707518, is the area of the whole trapezium.

QUEST. 9. If from the right-angled triangle ABC, whose base is 12, and perpendicular 16 feet, be cut off, by a line DE parallel to the perpendicular, a triangle whose area is 24 square feet; what are the sides of this triangle?




The triangles ABC, ADE, are similar, and the areas of similar triangles are as the squares of their like sides; but $12 \times 8 = 96$ is the area of the triangle ABC, therefore

$$\sqrt{96} : \sqrt{24} :: \begin{matrix} 16 : 16\sqrt{\frac{24}{96}} = 16\sqrt{\frac{1}{4}} = 16 \times \frac{1}{2} = 8 = \text{ED} \\ 12 : 12\sqrt{\frac{24}{96}} = 12\sqrt{\frac{1}{4}} = 12 \times \frac{1}{2} = 6 = \text{DA}' \end{matrix}$$

and $\sqrt{\text{AD}^2 + \text{DE}^2} = \sqrt{6^2 + 8^2} = \sqrt{100} = 10 = \text{AE}.$

QUEST. 10. If from a triangle whose sides are 13, 14, 15, be cut off by a line drawn parallel to the longest side, an area of 24; what are the lengths of the sides including that area?

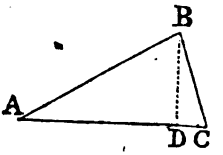
First, $\frac{13+14+15}{2} = \frac{42}{2} = 21$ = half the sum of the sides; from which each side being taken, leaves 6, 7, 8; therefore $\sqrt{21 \times 6 \times 7 \times 8} = \sqrt{3^2 \times 7^2 \times 4^2} = 3 \times 4 \times 7 = 84$ = the area of the given triangle. Then, as in the last,

$$\sqrt{84} : \sqrt{24} :: \begin{matrix} 13 : 13\sqrt{\frac{2}{7}} = \frac{13}{7}\sqrt{14} \\ 14 : 14\sqrt{\frac{2}{7}} = 2\sqrt{14} \\ 15 : 15\sqrt{\frac{2}{7}} = \frac{15}{7}\sqrt{14} \end{matrix}$$

the 3 sides including the area cut off.

QUEST. 11. Given the area = 144, the base AC = 24, and one angle BAC = 30°, at the base; to find the sides AB, BC.

First,




Firſt, $144 \div 12\ (\frac{1}{2}AC) = 12 =$ the perpendicular BD.

Then, as $\frac{1}{2} =$ S. $\angle A$ 30° : 1 = S. $\angle D$ :: 12 = DB : 24 = AB.

But $AD = \sqrt{AB^2 - BD^2} = \sqrt{24^2 - 12^2}$, and $DC = AC - AD = 24 - \sqrt{24^2 - 12^2}$.

Therefore $BC = \sqrt{BD^2 + DC^2} = 24\sqrt{2 - \sqrt{3}} = 12 \times (\sqrt{6} - \sqrt{2}) = 12{\cdot}423314184$.

Or, ſince AB is = AC, the $\angle C$ will be = the $\angle B = \frac{1}{2}(180 - 30) = 75°$; hence $\angle DBC = 15°$; and therefore as radius 1 : ſec. 15° = 1·0352762 :: BD 12 : 12 × 1·0352762 = 12·4233144 = BC.

*Note.* It is pretty evident, that this triangle will be conſtructed by drawing AB to make with the given baſe AC an angle of 30°, and meeting with a line drawn parallel to, and at the diſtance of, the perpendicular BD from AC in B the vertex of the triangle.

QUEST. 12. Given the area 1012, and ratio of the ſides of a triangle, viz. AC to CB, as 4 to 3, and AB to CB as 3 to 2; to find the ſides.

It is evident that the ſides AB, AC, BC, are in the ratio of the three numbers 9, 8, 6, reſpectively; and therefore a triangle whoſe ſides are 9, 8, 6, will be ſimilar to the triangle propoſed; but ſimilar triangles are as the ſquares of their like ſides, and $\sqrt{\frac{23}{2} \times \frac{5}{2} \times \frac{7}{2} \times \frac{11}{2}} = \frac{1}{4}\sqrt{8855}$ is the area of the triangle

angle whoſe ſides are 9, 8, 6; therefore $\frac{1}{2}\sqrt[4]{8855}$ : $\sqrt{1012}$ ::

$8 : 32\sqrt[4]{\frac{23\times 11}{35}} = \frac{32}{35}\sqrt[4]{10847375} = 52{\cdot}47024 = \text{AC}$,

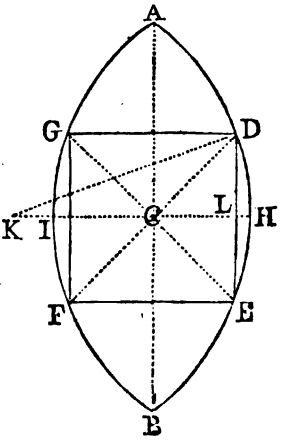
$6 : 24\sqrt[4]{\frac{23\times 11}{35}} = \frac{24}{35}\sqrt[4]{10847375} = 39{\cdot}35268 = \text{CB}$,

$9 : 36\sqrt[4]{\frac{23\times 11}{35}} = \frac{36}{35}\sqrt[4]{10847375} = 59{\cdot}02902 = \text{BA}$.

QUEST. 13. The diſtance of the centers of two circles, whoſe diameters are each 50, being given equal to 30; it is required to find the area of the ſpace incloſed by their circumferences, and that of a ſquare inſcribed in the ſaid ſpace?

*Conſtruction.*

Having deſcribed the circumferences including the common ſpace AHBI, IH being the common part of both the diameters, biſect IH in C, and through C draw FCD, GCE, each making half a right angle with IH, and meeting the arcs in D, E, F, G, the four angular points of the ſquare.



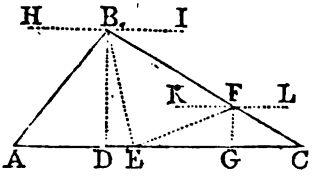

*Calculation.*

If K be the center of one of the circles, then KH—KC = 25—15 = 10 = CH = CI is the height of each ſegment; and $\frac{\text{CH}}{2\text{KH}} = \frac{10}{50} = {\cdot}2$ = the tab. verſed ſine; to which, in the table of circular ſegments, correſponds the area ·11182380; therefore ·1118238× 50×50×2 = 559·119 = the two ſegments, or common ſpace AHBIA.

Again,

Again, drawing KD, we ſhall have, as KD 25 : KC 15 :: s. ∠C 135° or 45° : s. ∠KDC = 25° $6\frac{1}{4}'$; hence 180° − 135° − 25° $6\frac{1}{4}'$ = 180° − 160° $6\frac{1}{4}'$ = 19° $53\frac{3}{4}'$ = ∠K; then as radius : s. ∠K :: KD 25 : DL = 8·507778, the double of which is DE = 17·015556 = the ſide of the ſquare. And therefore 17·015556 × 17·015556 = 289·529 is the area of the ſquare.

QUEST. 14. Given the baſe AC = 15, the area 45, and the ratio of AB to BC as 2 to 3; to find the ſides AB, BC, of the triangle?




Firſt, the area 45 ÷ $7\frac{1}{2}$ ($\frac{1}{2}$AC = 6 = the perpendicular BD. Then take AE : EC :: AB : BC, that is 5 = 2 + 3 : AC 15 :: $\begin{cases} 2 : 6 = \text{AE}, \\ 3 : 9 = \text{EC}; \end{cases}$ and ſuppoſe BF = BA, as alſo the lines BE, EF, and the perpendicular FG to be drawn; and we ſhall have the two triangles ABE, EBF, equal to each other in every reſpect; but

$$\tfrac{1}{2}\text{BD} \times \left\{ \begin{matrix} \text{AE} \\ \text{EC} \end{matrix} \right\} = 3 \times \left\{ \begin{matrix} 6 = 18 = \triangle \text{ABE}, \\ 9 = 27 = \triangle \text{EBC}; \end{matrix} \right.$$

their difference is 9 = △EFC;

or △EFC = △ABC − ABFE = △ABC − 2△ABE = 45 − 36 = 9; hence 9 (△EFC) ÷ $4\frac{1}{2}$ ($\frac{1}{2}$EC) = 2 = the perpendicular FG; but EG = $\sqrt{\text{EF}^2 - \text{FG}^2}$ = $\sqrt{6^2 - 2^2} = \sqrt{32}$, and GC = EC − EG = $9 - \sqrt{32}$; whence, by ſimilar triangles, FG : GC :: BD : DC = $27 - 3\sqrt{32}$, and, conſequently, AD = AC − DC = $3\sqrt{32} - 12$; wherefore, laſtly, AB = $\sqrt{\text{BD}^2 + \text{DA}^2}$ = $\sqrt{6^2 + (3\sqrt{32} - 12)^2} = 3\sqrt{2^2 + (\sqrt{32} - 4)^2}$ = $6\sqrt{13 - 8\sqrt{2}} = 7{\cdot}79146$; and, as 2 : 3 :: AB : BC = $9\sqrt{13 - 8\sqrt{2}} = 11{\cdot}68719$.

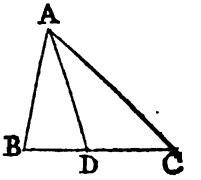
From

From the foregoing ſolution the following conſtruction is evident.

Having drawn HI parallel to AC, and at ſuch diſtance from it as is expreſſed by the perpendicular BD, found from the given area and baſe, and taken AE to EC in the given proportion of AB to BC; with the center E and radius EA deſcribe an arc meeting in F, with a line KL drawn parallel to AC, and at ſuch diſtance from it as is denoted by a third proportional to EC, EC − EA, and BD; then through F draw CB meeting HI in B, and join B, A.

QUEST. 15. Given the ſegments BD = 10, and DC = 14, of the baſe BC, made by a line AD biſecting the vertical angle A, and the ſum of the ſides BA + AC = 48; to find the area.

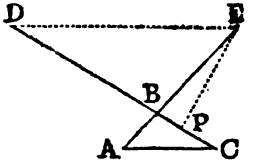
If a line biſect any angle of a triangle and cut the oppoſite ſide, it is known that the ſegments of that ſide are to each other, as the other ſides adjacent to them.




Therefore as BC 24 : BA + AC 48 :: $\begin{cases} \text{BD } 10 : 20 = \text{BA}, \\ \text{DC } 14 : 28 = \text{CA}; \end{cases}$

hence the area is $\sqrt{36 \times 8 \times 12 \times 16} = 6 \times 2 \times 2 \times 4\sqrt{2 \times 3} = 96\sqrt{6} = 235{\cdot}15100154$.

QUEST. 16. Having given the continuations of the two ſides of the vertical angle of a given triangle, to find the area of the triangle formed by the ſaid continuations, and a line connecting their extremities.

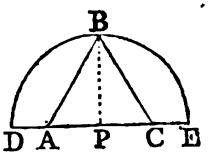



That is, Given AB = 2, BC = 3, AC = 4; and, AE = 7, CD = 10, or BE = 5, BD = 7; required DE and the area of the triangle BDE.

Firſt, $\sqrt{\frac{9}{2}\times\frac{1}{4}\times\frac{3}{2}\times\frac{5}{2}}=\frac{3}{4}\sqrt{15}=$ the area of the triangle ABC. But ſince the vertical angles at B are equal, and triangles, having one angle in each equal, are as the rectangles of the ſides about the equal angles; we ſhall have $6:35::\frac{3}{4}\sqrt{15}:\frac{35}{8}\sqrt{15}=16{\cdot}944302=$ the area of the triangle BDE.

Then, having drawn EP perpendicular to BC, $\frac{35}{8}\sqrt{15}\div\frac{7}{2}(\frac{1}{2}DB)=\frac{5}{4}\sqrt{15}=EP$; but $\sqrt{BE^2-EP^2}=\sqrt{25-\frac{15}{16}\times25}=\sqrt{\frac{25}{16}}=\frac{5}{4}=BP$, and $DB+BC=7+\frac{5}{4}=\frac{33}{4}=DP$; hence $\sqrt{DP^2+PE^2}=\sqrt{(\frac{33}{4})^2+15\times(\frac{5}{4})^2}=\frac{1}{4}\sqrt{33^2+15\times5^2}=\frac{1}{4}\sqrt{1464}=\frac{1}{2}\sqrt{366}=9{\cdot}565563235=DE$.

QUEST. 17. Given the area 100 of the equilateral triangle ABC, whoſe baſe falls on the diameter, and vertex in the middle of the arc of a ſemicircle; required the diameter DE of the ſemicircle.



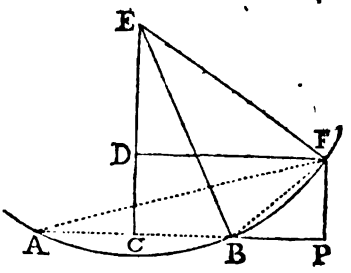

Since the area of an equilateral triangle is equal to the ſquare of the ſide drawn into $\frac{1}{4}\sqrt{3}$, we ſhall have the ſide $AB=\sqrt{\frac{4\times100}{\sqrt{3}}}$, and the half of it $AP=\sqrt{\frac{100}{\sqrt{3}}}=\frac{10}{\sqrt[4]{3}}$; whence the radius of the circle or perpendicular of the triangle BP is $=$ area $\div AP=100\div\frac{10}{\sqrt[4]{3}}=10\sqrt[4]{3}$; the double of which is $20\sqrt[4]{3}=26{\cdot}32148026$, the diameter required.

*Otherwiſe.* Since $AP=BP\sqrt{\frac{1}{3}}$, therefore $AP\times BP=BP^2\sqrt{\frac{1}{3}}=100$ the area; hence $BP\sqrt[4]{\frac{1}{3}}=10$, and $2BP=20\sqrt[4]{3}$, the diameter.

QUEST. 18. The area of a triangular meadow is 100 perches or ſquare poles, and the length of one of its ſides is 40 poles, which is alſo a mean proportional

tional between the other two sides; from which it is required to find the unknown fences.

*Construction.*



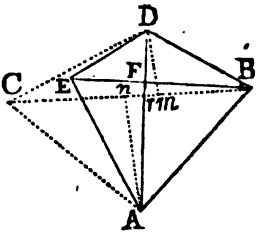

Make AB = 40 = the base, or given side; on the middle C of which raise CD perpendicular and equal to $\frac{200}{40} = 5$, the perpendicular height of the triangle. Then, since the rectangle of two sides of a triangle drawn into the sine of their included angle, is equal to double the area, we shall have $\frac{200}{40 \times 40} = \frac{1}{8}$ for the sine of the vertical angle; therefore draw BE making the angle CBE equal to the complement of that angle, and meeting CD produced in E; so will E be the center of the circle circumscribing the triangle. Therefore, with the center E and radius EB or EA, describe an arc ABF meeting DF, drawn parallel to AB, in F; then draw AF, FB, and it is evident that AFB will be the triangle required.

*Calculation.*

Draw FP perpendicular to AB produced. Then, since $\frac{1}{8}$ is the s. of $\angle$E, it will be as $\frac{1}{8}$ : 1 s. $\angle$C :: CB 20 : 160 = BE; or thus, by geometry, $\frac{AF \times FB}{2FP} = \frac{40 \times 40}{10} = 160 = BE$. Again $CE = \sqrt{EB^2 - BC^2} = \sqrt{160^2 - 20^2} = 20\sqrt{63} = 60\sqrt{7}$; hence $DE = CE - CD = 60\sqrt{7} - 5$, and $DF = \sqrt{EF^2 - ED^2} = 5\sqrt{15 + 24\sqrt{7}}$; but $BP = DF - CB = 5\sqrt{15 + 24\sqrt{7}} - 20$, and therefore $BF = \sqrt{FP^2 + BP^2} = 10\sqrt{8 + 6\sqrt{7} - 2\sqrt{15 + 24\sqrt{7}}} = 24{\cdot}80833$, and consequently $FA = \frac{AB^2}{BF} = \frac{40 \times 40}{24{\cdot}80833} = 64{\cdot}49447$.

QUEST. 19. The four sides of a field, whose diagonals are equal to each other, are known to be 25, 35, 31, and 19 poles, in a successive order; from whence the content of the field is required.

*Construction.*




Make AB and AC perpendicular to one another, and each equal to one of the given sides, as 35; with the centers B, C, and radii equal to the two remaining opposite sides 25 and 31, describe two arcs, intersecting in D; draw AD, and BE perpendicular and equal to it; draw AE and DE; and ABDE will be the trapezium.*

*Calculation.*

Draw BC meeting AD in $r$, upon which let fall the perpendiculars A$n$, D$m$. Then $BC = \sqrt{AB^2 + AC^2} = 35\sqrt{2}$, and $An = Bn = \frac{1}{2}BC = \frac{35\sqrt{2}}{2} = \frac{35}{\sqrt{2}}$; again,

$$Bn = \frac{BD^2 + BC^2 - CD^2}{2BC} = \frac{25^2 + 2 \times 35^2 - 31^2}{70\sqrt{2}} = \frac{2114}{70\sqrt{2}} = \frac{151\sqrt{2}}{10};$$

and $nm = Bn - Bm = \left(\frac{35}{2} - \frac{151}{10}\right) \times \sqrt{2} = \frac{12}{5}\sqrt{2}$; but

---

* For, by the construction, AB, BD are the two sides 35, 25, and the diagonal BE = AD; so that we have only to shew that AE, ED are equal to the other two sides, 31, 19.—Now, by the construction the angles CAD, ABE, being each the complement of BAD, are equal to each other; and, by the construction CA and AD, AB and BE, about those equal angles, are also equal; and therefore the remaining sides CD, AE are likewise equal, that is AE = 31.—But, by the property of all triangles, $AB^2 - AE^2 = BF^2 - FE^2 = BD^2 - DE^2$; and, therefore, $DE = \sqrt{BD^2 + AE^2 - AB^2} = \sqrt{25^2 + 31^2 - 35^2} = 19$. *Q. E. D.*

but $Dm = \sqrt{BD^2 - Bm^2} = \sqrt{25^2 - 2(\frac{151}{10})^2} = \frac{1}{10}\sqrt{250^2 - 2\times 151^2} = \frac{1}{10}\sqrt{16898}$; and, by similar triangles, $An + Dm : nm :: AB : nr = \frac{420\sqrt{2}}{175+\sqrt{8449}}$; hence $Ar =$

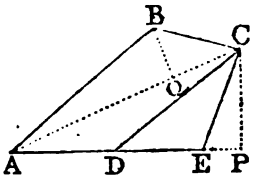
$\sqrt{An^2 + nr^2} = \frac{\sqrt{\frac{1}{2}\times 35^2\times(175+\sqrt{8449})^2 + 2\times 420^2}}{175+\sqrt{58449}}$;

and, by similar triangles, $nr : nm :: Ar : AD = \sqrt{793+7\sqrt{8449}} = BE$.—Then, becauſe AD is equal and perpendicular to BE, by rule 2 prob. 3 we obtain the area $= AD \times BE \times \frac{1}{2}$ ſine $\angle F = \frac{1}{2} BE^2 = \frac{1}{2}\times(793+7\sqrt{8449}) = 718{\cdot}214547$ ſquare poles $= 4$ac. 1ro. 38·214547 perches.

*Otherwiſe.*

Having found $Bm = \frac{151}{10}\sqrt{2}$; we ſhall have, as $BD : Bm :: 1 : \frac{151}{250}\sqrt{2} = {\cdot}604\sqrt{2} = {\cdot}85418499$ the coſine of the $\angle DBm = 31° 20'$; but $\angle ABm = 45°$, conſequently $\angle DBA = 76° 20'$; then as $AB+BD$ 60 : $AB-BD$ 10 :: tang. $\frac{1}{2}\angle BDA + \frac{1}{2}\angle BAD = 51° 50'$ : tang. $\frac{1}{2}\angle BDA - \frac{1}{2}\angle BAD = 11° 58'$, hence $51° 50' - 11° 58' = 39° 52' = \angle BAD$; laſtly, s. $\angle BAD$ : s. $\angle ABD$ :: $BD : DA = 37{\cdot}897 = BE$, which is nearly equal to $\sqrt{793+7\sqrt{8449}} = 37{\cdot}90025$, the value above found.

QUEST. 20. The trapezium ABCD is to be meaſured; and, becauſe of ſome obſtructions from wood, water, &c, or for want of proper inſtruments, there can be taken only the following meaſures: viz. AB 58·5 chains, BC 27·3, CD 50, and DA 32 chains; alſo in AD, continued, there is meaſured any diſtance DE 27·5, and the diſtance from E to C is 32·5 chains; required the area.




Draw the diagonal AC, and on AD, produced, let fall the perpendicular CP.

Becauſe $DC^2$ is $= DE^2 + EC^2 + 2DE \times EP$, we ſhall have $EP = \frac{DC^2 - EC^2 - DE^2}{2DE} = \frac{50^2 - 32{\cdot}5^2 - 27{\cdot}5^2}{55} = 12{\cdot}5$; but $PC = \sqrt{CE^2 - EP^2} = \sqrt{32{\cdot}5^2 - 12{\cdot}5^2} = 30$, therefore $\frac{1}{2}AD \times PC = 16 \times 30 = 480$ the area of the triangle ACD.

Again, $AP = AD + DE + EP = 32 + 27{\cdot}5 + 12{\cdot}5 = 72$; therefore $AC = \sqrt{AP^2 + PC^2} = \sqrt{72^2 + 30^2} = 78$.

Alſo, letting fall the perpendicular BQ, ſince $AB^2 = AC^2 + BC^2 - 2AC \times CQ$, we ſhall have $CQ = \frac{AC^2 + BC^2 - AB^2}{2AC} = 21{\cdot}84$; and hence $BQ = \sqrt{BC^2 - CQ^2} = 16{\cdot}38$; therefore $\frac{1}{2}AC \times QB = 39 \times 16{\cdot}38 = 638{\cdot}82$ = the triangle ABC.

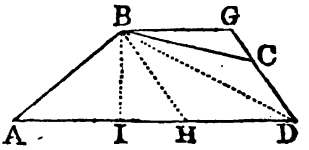
Whence $480 + 638{\cdot}82 = 1118{\cdot}82$ ſquare chains $= 111{\cdot}882$ acres $=$ 111 ac. 3 r. 21·12 perches is the whole area required.

QUEST. 21. To find the length of a chord cutting off any part, as ſuppoſe $\frac{1}{3}$, from a circle whoſe diameter is 289.

The area of the whole tabular circle, or of the circle whoſe diameter is 1, is ·78539816; the $\frac{1}{3}$ of which is ·26179939 nearly, which is a ſegment ſimilar to the ſegment to be cut off. Now, the verſed ſine anſwering to this area, in the table of circular ſegments, is ·36753395, which taken from the whole diameter

diameter 1 leaves ·63246605 for the other part of the diameter; then, becauſe the ſemi-chord is a mean proportional between the two ſegments of the diameter, we ſhall have $2\sqrt{\cdot 63246605 \times \cdot 36753395} = \cdot 96426162$ = the chord of the tabular ſimilar ſegment. Therefore as 1 : 289 :: ·96426162 : 278·6716 = the chord line required.

QUEST. 22. In the unpaſſable field ABCD there are meaſured the ſides AB 29, AD 59, and DC 15 chains: alſo, by drawing BG parallel to AD till it meet with DC, produced, in G, BG meaſures 23; and GC, 10 chains; required the content.




Draw BH parallel to GD: then is BGDH a parallelogram, and conſequently BH = GD = 25, and HD = BG = 23; hence AH = AD − HD = 36.

But, letting fall the perpendicular BI, $IH = \frac{AH^2 + BH^2 - AB^2}{2AH} = 15$, and $BI = \sqrt{BH^2 - IH^2} = 20$. Whence $(AD + BG) \times \frac{1}{2}BI = 82 \times 10 = 820$ is the area of the trapezoid ABGD.

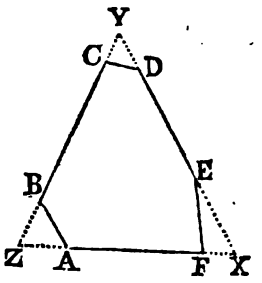
Again, drawing the diagonal BD, $HD \times \frac{1}{2}BI = 23 \times 10 = 230$ = the triangle BHD = the triangle BGD; but the triangles BGD, BGC, being of the ſame height, are to each other as their baſes GD, GC; that is, DG : GC :: triangle BDG : triangle BCG = $\frac{2}{5} \times 230 = 2 \times 46 = 92$.

Whence, by taking the triangle BGC from the trapezoid ABGD, we have 820 − 92 = 728 ſquare chains = 72a. 3r. 8 perches = the area required.

*Note.* It is evident that this figure will be conſtructed, by firſt forming the triangle ABH, and joining the parallelogram BGDH to it.

QUEST. 23. To find the area of the hexagonal field ABCDEF, the sides BC, DE, AF, being produced to meet in the points X, Y, Z; there are given the following dimensions, ZB 39, BC 118, CY 25, YD 30, DE 100, EX 65, XF 23, FA 109, and AZ 37 chains.

First, by prob. 2 sect. 1, the area of the triangle ZYX is 14196 square chains.



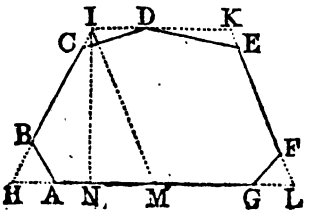

But, it appears from rule 2 of the same problem, that the areas of triangles, having one angle common, are as the rectangles of the sides including the common angle; therefore

ZX×ZY = 169×182 : 37×39 :: 14196 : 666 = △ZBA,
YZ×YX = 182×195 : 25×30 :: 14196 : 300 = △CYD,
XY×XZ = 195×169 : 65×23 :: 14196 : 644 = △EXF,

whose sum 1610
taken from 14196 the △ZYX
leaves 12586 the area
of the hexagon ABCDEF required.

QUEST. 24. It is required to find the area of the heptagon ABCDEFG inscribed in the trapezoid HIKL, the dimensions being as below:

Viz. HB 34, BC 85, CI 17, ID 42, DK 66, KE 13, EF 91, FL 26, LG 29, GA 160, and AH 33.




First,

Firſt, drawing IM parallel to KL, and letting fall the perpendicular IN, IM will be = KL = 130, HM = HL − IK = 114; hence NM = $\frac{IM^2 + HM^2 - HI^2}{2\,HM}$ = 50, and IN = $\sqrt{IM^2 - NM^2}$ = 120.

Hence $\frac{1}{2}$IN × (IK + HL) = 60 × 330 = 19800 = the trapez. IKLH.

But a trapezoid is to a triangle, having each one angle equal, as the rectangle under the ſum of the parallel ſides and the other ſide including that angle, in the trapezoid, is to the rectangle under the ſides including the angle, in the triangle.

Or, As that one of the two ſides in the trapezoid, about the angle, which is not one of the parallel ſides, is to half the diſtance of the parallel ſides, ſo is the rectangle under the two ſides of the triangle about the angle, to its area.* Therefore

136 : 60 :: 33 × 34 : 495 = △HBA,
136 : 60 :: 17 × 42 : 315 = △CID,
130 : 60 :: 66 × 13 : 396 = △DKE,
130 : 60 :: 26 × 29 : 348 = △GFL,

the ſum is 1554
which taken from 19800 the trapezoid,
leaves 18246 ſquare chains
= 1824a. 2 r. 16 perches.

*Note.* This is conſtructed as the 22d.

QUEST.

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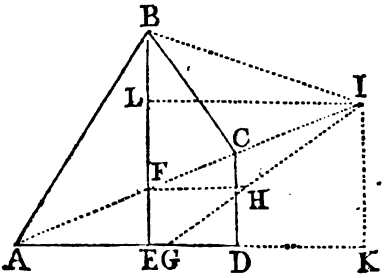
* For AH × HB : △BAH :: (radius : $\frac{1}{2}$s. ∠H ::) HI : $\frac{1}{2}$IN :: (by equal mult.) HI × $\overline{IK + HL}$ : $\frac{1}{2}$IN × $\overline{IK + HL}$ = the trapezium.
In the ſame manner, GL × LF : △GLF :: LK : $\frac{1}{2}$IN :: LK × $\overline{IK + HL}$ : $\frac{1}{2}$IN × $\overline{IK + HL}$.
But the angles I and K are the ſupplements of the angles H and L, and have, therefore, the ſame ſines; and, conſequently, the rule will be the ſame for them.

QUEST. 25. In the quadrangular field ABCD, if AD be 46·8, DC 19·5, the perpendicular BE 44·8 chains, and D a right angle; alſo if AB be to BC as 8 to 5; it is required to find the content of the trapezium ABCD, and the length of the ſides AB, BC.




*Conſtruction.*

Draw AD = 46·8, and DC = 19·5 making a right angle D with it; join A, C; take AF : FC :: 8 : 5, or AF : AC :: 8 : 13; make AG = AF, and FH parallel to it and = to FC; draw GHI meeting AC produced in I; with the center I and radius IF, deſcribe an arc meeting in B, a line drawn parallel to, and at the diſtance of, the perpendicular 44·8 from AD; and B will be the other point of the figure required.*

*Calculation.*

Firſt, $AC = \sqrt{AD^2 + DC^2} = 50{\cdot}7$, and, by the conſtruction, 13 : 8 :: AC : 31·2 = AF; hence FC

---

* DEMONSTRATION.

Becauſe, by ſimilar triangles, AI : AG or AF :: FI : FH or FC, we have, by diviſion, AI : FI :: FI : CI, or joining IB, AI : BI :: BI : CI; but the ∠I, included by theſe proportional lines, is common to both the triangles ABI, CBI, therefore thoſe triangles are ſimilar, and conſequently AB : BC :: AI : BI or FI :: AG : FH :: 8 : 5.

FC = AC − AF = 19·5. Then, by similar triangles, 11·7 (= AG − FH or AF − FC) : 31·2 (= AF or AI − FI) :: 19·5 (= FH or FC) : 52 = FI or BI; whence AI or AF + FI = 83·2. Then, by similar triangles again, (drawing IK perpendicular to AD produced) as AC 50·7 : AI 83·2 ::

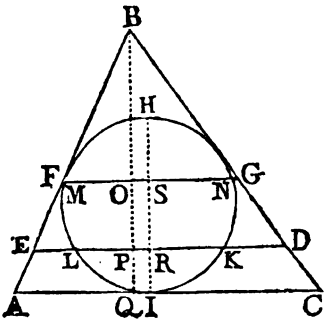
$$\begin{cases} \text{AD } 46{\cdot}8 : 76{\cdot}8 = \text{AK}, \\ \text{DC } 19{\cdot}5 : 32 = \text{KI}, \end{cases}$$

whence, (drawing IL parallel to KE), LB or BE − KI = 12·8, and EK or LI = $\sqrt{\text{BI}^2 - \text{BL}^2} = 50{\cdot}4$; and hence AE = AK − EK = 26·4, and AB = $\sqrt{\text{AE}^2 + \text{EB}^2}$ = 52; also 8 : 5 :: AB 52 : 32·5 = BC.

But $\frac{1}{2}$ AE × EB = 591·36 triang. ABE,
And $\frac{1}{2}$ ED × (BE + CD) = 655·86 trapez. EBCD,

Their sum is - - - 1247·22 sq. chains
or 124 ac. 2 r. 35·52 p.
the area of the quadrangle ABCD required.

QUEST. 26. Given the three sides AB 13, AC 14, and BC 15, of a triangle ABC, divided into three equal parts by the two lines FG, ED, both parallel to AC; it is required to find the areas of the two segments MHN, KIL, and zone LMNK, into which the inscribed circle is cut by those lines.




First,

First, $\sqrt{21\times8\times7\times6}=\sqrt{7^2\times3^2\times4^2}=7\times3\times4=84$ the area of the triangle ABC.

Then $84\div7\ (\frac{1}{2}AC)=12$ is the perpendicular BQ; and $84\div21\ (\frac{1}{2}AB+\frac{1}{2}AC+\frac{1}{2}BC)=4$ is the radius, or 8 is the diameter HI of the inſcribed circle.

Now, the triangles BFG, BED, BAC, are ſimilar; but ſimilar triangles are as the ſquares of their like dimenſions; alſo thoſe triangles are to one another as the numbers 1, 2, 3, reſpectively; therefore

$$\sqrt{3} : BQ\ 12 :: \begin{cases} \sqrt{1} : 12\sqrt{\frac{1}{3}} = 4\sqrt{3} = BO, \\ \sqrt{2} : 12\sqrt{\frac{2}{3}} = 4\sqrt{6} = BP; \end{cases}$$

Hence $BQ-BP=12-4\sqrt{6}=PQ=$ the verſed ſine IR,
and $BQ-BO=12-4\sqrt{3}=OQ=$ the verſed ſine IS;
and $IH-IS=4\sqrt{3}-4=$ the verſed ſine SH.

Then, dividing the verſed ſines HS, IR, by the diameter, we have $\frac{4\sqrt{3}-4}{8}=\frac{\sqrt{3}-1}{2}=\cdot366025404$,

and $\frac{12-4\sqrt{6}}{8}=\frac{3-\sqrt{6}}{2}=\cdot275255128$;

the correſponding tabular verſed ſines; to which, in the table of circular ſegments, belong the areas

$\cdot26034449$,
and $\cdot17577983$,
whoſe ſum $\cdot43612432$,
taken from $\cdot78539816$,

leaves $\cdot34927384$,

the area correſponding to the zone. Then each of theſe being multiplied by 64, the ſquare of the diameter, gives the ſeveral areas following, viz.

$$64\times\begin{cases} \cdot26034449 = 16\cdot66204736 = \text{the ſegment LIK}, \\ \cdot17577983 = 11\cdot24990912 = \text{the ſegment MHN}, \\ \cdot34927384 = 22\cdot35352576 = \text{the zone MLKN}. \end{cases}$$

PART

# PART III.

# MENSURATION OF SOLIDS.

## GENERAL DEFINITIONS.

1. A Solid, or body, is a figure extended in every direction. It is commonly ſaid to conſiſt of length, breadth, and thickneſs, which are three of its extenſions; of which the direction of each is perpendicular to thoſe of the other two.

2. The meaſure of a ſolid is called its ſolidity, capacity, or content.

3. By the menſuration of ſolids then are determined the ſpaces included by contiguous ſurfaces; and the ſum of the meaſures of theſe including ſurfaces, is the ſurface or ſuperficies of the body.

4. Solids are meaſured by cubes, whoſe ſides are inches, feet, yards, or any other aſſigned quantity; and hence the ſolidity of a body is ſaid to be ſo many cubic inches, feet, yards, &c, as will fill its capacity or ſpace, or another of an equal magnitude

5. The leaſt ſolid meaſure is the cubic inch, other cubes being taken from it according to the proportion in the following table.

*Table*

## *Table of Solid Meaſure.*

| Cubic inches | cubic feet | cubic yards | cub. poles | c. furl. | c. mile |
|---|---|---|---|---|---|
| 1728 | 1 | | | | |
| 46656 | 27 | 1 | | | |
| 7762392 | 4492$\frac{1}{8}$ | 166$\frac{2}{3}$ | 1 | | |
| 946793088000 | 287496000 | 10648000 | 64000 | 1 | |
| 2543580610560000 | 147197952000 | 5451776000 | 32768000 | 512 | 1 |

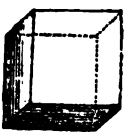
## SECTION I.

### OF PRISMS, PYRAMIDS, AND THE SPHERE, WITH THE PARTS INTO WHICH SOME OF THEM MAY BE CUT BY PLANES.

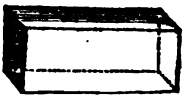
1. A Priſm is a ſolid, or body, whoſe ends are any plane figures, which are equal and ſimilar; and its ſides are parallelograms.

A priſm is called a triangular priſm, when its ends are triangles; a ſquare priſm, when its ends are ſquares; a pentagonal priſm, when its ends are pentagons; and ſo on.

2. A cube is a ſquare priſm, having ſix ſides, which are all ſquares. It is like a die, having its ſides perpendicular to one another.



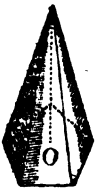
3. A parallelopipedon is a ſolid having ſix rectangular ſides, every oppoſite pair of which are equal and parallel.



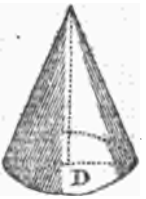
4. A cylinder is a round priſm; having circles for its ends.

5. A pyramid is a ſolid having any plane figure for a baſe, and its ſides are triangles whoſe vertices meet in a point at the top, called the vertex of the pyramid.

The pyramid takes names according to the figure of its baſe, like the priſm; being triangular, or ſquare, or hexagonal, &c.




6. A cone is a round pyramid; having a circular baſe.




7. A ſphere is a ſolid bounded by one continued convex ſurface, every point of which is equally diſtant from a point within, called the center.—The ſphere may be conceived to be formed by the revolution of a ſemicircle about its diameter, which remains fixed.

8. The axis of a ſolid, is a line drawn from the middle of one end, to the middle of the oppoſite end; as between the oppoſite ends of a priſm. Hence the axis of a pyramid, is the line from the vertex to the middle of the baſe, or the end on which it is ſuppoſed to ſtand. And the axis of a ſphere, is the ſame as a diameter, or a line paſſing through the center, and terminated by the ſurface on both ſides.

9. When the axis is perpendicular to the baſe, it is a right priſm or pyramid; otherwiſe it is oblique.

10. The height or altitude of a ſolid, is a line drawn from its vertex or top, perpendicular to its baſe.—This is equal to the axis in a right priſm or pyramid;

pyramid; but in an oblique one, the height is the perpendicular ſide of a right-angled triangle, whoſe hypotenuſe is the axis.

11. Alſo a priſm or pyramid is regular or irregular, as its baſe is a regular or an irregular plane figure.

12. The ſegment of a pyramid, ſphere, or any other ſolid, is a part cut off the top by a plane parallel to the baſe of that figure.

The ſection made by the plane, is a plane ſimilar to the baſe of the figure; and every ſection of a ſphere is a circle. If the ſection be made through the center, it is a great circle of the ſphere, having the ſame diameter with the ſphere; if not, it is a little circle.

13. A fruſtum, truncus, or trunk, is the part remaining at the bottom after the ſegment is cut off.

If a fruſtum be cut by a plane diagonally paſſing through the extremity of one ſide at the leſs end, and through the extremity of the oppoſite ſide at the greater end, the two parts into which it is cut, are called ungulas or hoofs; the greater hoof being that including the greater end; and the leſs, that including the leſs.

14. A zone of a ſphere, is a part intercepted between two parallel planes; and it is the difference between two ſegments. When the ends, or planes, are equally diſtant from the center, on both ſides, the figure is called the middle zone.

15. The ſector of a ſphere, is compoſed of a ſegment leſs than a hemiſphere or half ſphere, and of a cone having the ſame baſe with the ſegment, and its vertex in the center of the ſphere.

16. A circular ſpindle, is a ſolid generated by the revolution of a ſegment of a circle about its chord, which remains fixed.

17. A

17. A wedge is a ſolid having a rectangular baſe, and two of the oppoſite ſides ending in an acies or edge.

18. A priſmoid is a ſolid having for its two ends any diſſimilar parallel plane figures of the ſame number of ſides, and all the upright ſides of the ſolid trapezoids.—If the ends of the priſmoid be bounded by diſſimilar curves, it is ſometimes called a cylindroid.

19. An ungula, or hoof, is a part cut off a ſolid by a plane oblique to the baſe.

## PROBLEM I.

### *To find the Surface of a Priſm.*

#### GENERAL RULE.

It is evident, that, if the area of each ſide and end be calculated ſeparately, the ſum of thoſe areas will be the whole ſurface of any priſm, whether right or oblique; or, indeed, of any other body whatever.* —But for a right priſm obſerve the following

#### PARTICULAR RULE.

Multiply the perimeter of the end by the height, and the product will be the ſum of the ſides, or upright ſurface.

If the ends of the priſm be regular plane figures, multiply the perimeter of the end by the ſum of the height of the priſm and the radius of the circle inſcribed in the end, and the product will be the whole ſurface.



* The ſurfaces of ſimilar priſms, and indeed of any other ſimilar bodies, are as the ſquares of their like lineal dimenſions. This follows from their being compoſed of ſimilar plane figures, alike placed.

EXAMPLE I.

What is the upright ſurface of a triangular priſm whoſe length is 20 feet, and the ends of its baſe each 18 inches?

Here $18 \times 3 = 54$ inches $= 4\frac{1}{2}$ feet is the perimeter of the baſe.

Therefore $4\frac{1}{2} \times 20 = 90$ ſquare feet is the upright ſurface.

Again, by rule 2 problem 4 ſection 1, we have $2 \times \frac{3}{2} \times \frac{3}{2} \times .433013 = 1.9485585 =$ the area of the two ends.

Therefore 91.9485585 is the whole ſurface.

EXAMPLE II.

What is the ſurface of a cube, the length of each of whoſe ſides is 20 feet?

Firſt, $20 \times 20 = 400$ is the area of one ſide.

And $400 \times 6 = 2400$ is the whole ſurface required.

EXAMPLE III.

What muſt be paid for lining a rectangular ciſtern with lead at 2d a lb, the lead being 7 feet to the lb, ſuppoſing the length within ſide to be 3 feet 2 inches, the breadth 2 feet 8 inches, and height 2 feet 6 inches?

Firſt, $(38 + 32) \times 2 \times 30 = 70 \times 60 = 4200$ ſquare inches, the two ſides and two ends together.

Then, $38 \times 32 = 1216$ is the area of the bottom.

Therefore $4200 + 1216 = 5416$ ſquare inches $= 37\frac{11}{18}$ ſquare feet, is the whole area.

And $37\frac{11}{18} \times 7 = 263\frac{5}{18}$ lb, is the whole weight.

Therefore 1 lb : 2d :: $263\frac{5}{18}$ : 2l 3s $10\frac{5}{9}$d, the coſt.

EXAMPLE IV.

What is the convex ſurface of a round priſm, or cylinder, whoſe length is 20 feet, and the diameter of whoſe baſe is 2 feet?

Firſt, $3{\cdot}1416 \times 2 = 6{\cdot}2832$ is the circumference.

Therefore $6{\cdot}2832 \times 20 = 125{\cdot}664$ is the convex ſurface.

EXAMPLE V.

What is the whole ſurface of a cylinder whoſe length is 10 feet, and the circumference 3 feet?

Here $\frac{3}{2 \times 3{\cdot}1416} = {\cdot}477463$ is the radius of the end.

Theref. $10{\cdot}477463 \times 3 = 31{\cdot}43239$ is the whole ſurface.

PROBLEM II.

*To find the Solidity of a Priſm.*

Multiply the area of the baſe by the height, and the product will be the ſolidity.*



* For, if we conceive to be cut off from the priſm, by a plane parallel to the ends, a part whoſe height is equal to the lineal meaſuring unit; and then imagine both its ends to be divided, in a ſimilar manner, into as many ſquares as are expreſſed by the area of each, the ſide of each of thoſe ſquares being equal to the lineal meaſuring unit; and, laſtly, ſuppoſe planes to be drawn through the correſponding lines of diviſion; it is evident that the part cut off will be divided, by thoſe planes, into as many cubes as there are ſquares in each end, and alſo having the ſame dimenſion with thoſe ſquares, viz. the lineal meaſuring unit; and this number of cubes is the meaſure of the part.

But the magnitude of the whole priſm, or, indeed, of any other of an equal baſe, is to the magnitude of the part whoſe height is the lineal meaſuring unit, as the length of the whole is to that unit (1).

And

EXAMPLE I.

How many ſolid feet are in a ſquare priſm whoſe length is $5\frac{1}{2}$ feet, and each ſide of its baſe $1\frac{1}{3}$ feet?

$1\frac{1}{3} \times 1\frac{1}{3} = \frac{4}{3} \times \frac{4}{3} = \frac{16}{9}$ is the area of the end.
And $\frac{16}{9} \times 5\frac{1}{2} = \frac{16}{9} \times \frac{11}{2} = \frac{88}{9} = 9\frac{7}{9}$ ſolid feet, is the content.

EXAMPLE II.

How many ale gallons of water will the ciſtern hold, whoſe length, breadth, and height, are 3 feet 2 inches, 2 feet 8 inches, and 2 feet 6 inches?

$38 \times 32 \times 30 = 36480$ is the ſolidity in inches.
And $36480 \div 282 = 6080 \div 47 = 129\frac{17}{47}$, the ale gallons.

EXAMPLE III.

What is the capacity of a cylinder whoſe height, and the circumference of its baſe, are each 20 feet?

Firſt, $\frac{20}{3{\cdot}1416}$ = the diameter, and $10 \times \frac{10}{3{\cdot}1416} = \frac{100}{3{\cdot}1416} = \frac{25}{{\cdot}7854}$ is the area of the end.
Then $\frac{25}{{\cdot}7854} \times 20 = \frac{2500000}{3927} = 636{\cdot}61828$ is the content.

EX

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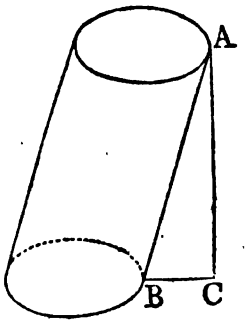
And therefore the meaſure of the whole is equal to that of the part as often repeated as there are lineal meaſuring units in the height; that is, equal to the baſe drawn into the height.

And that the rule is true for oblique priſms as well as right ones, will be evident by conceiving, according to the method of CAVALERIUS, a right and an oblique priſm, of equal baſes and heights, to be made up of an indefinite number of equal thin plates, all parallel to the baſe; for the priſms being both of a height, the one of them will require as many of ſuch plates to compoſe it as the other.

EXAMPLE IV.

What is the capacity of the oblique cylinder, whose axe, and the circumference of its base, are each 20 feet; the axe making an angle of 75° with the base?




As in the last example, the area of the base is $\frac{25}{\cdot 7854}$ or $\frac{125000}{3927}$.

But, as radius : sin. $\angle$ B 75° :: AB 20 : 19·318516 = AC, the height of the cylinder.

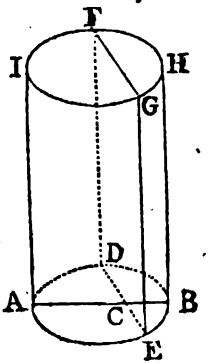
Therefore $\frac{125000}{3927} \times 19 \cdot 318516$ = 125000 × ·0049194082 = 614·92602 is the capacity required.

*Otherwise.*

As radius : sin. $\angle$ B :: 636·61828 (the capacity in the last example) : 636·61828 × ·9659258 = 614·92602 the capacity, the same as before.

EXAMPLE V.

Suppose the right cylinder whose length is 20 feet, and diameter 3 feet, be cut by a plane parallel to, and at the distance of, 1 foot from its axe: required the solidities of the two prisms into which the cylinder is cut.




$\frac{CB}{BA} = \frac{\frac{1}{2}}{3} = \frac{1}{6} = \cdot 16\dot{6}$ = the tab. versed sine; to which in the table of circular segments corresponds the area - - - - ·08604117
which taken from ·78539816
leaves the other seg. ·69935699



Then

Then multiplying each by $3^2$ or 9 gives

$9 \times \cdot08604117 = \cdot77437053 =$ ſeg. DBE,
$9 \times \cdot69935699 = 6\cdot29421291 =$ ſeg. DAE.

Hence

$20 \times \cdot77437053 = 15\cdot48741 =$ ſlice FGHBED,
$20 \times 6\cdot29421699 = 125\cdot88434 =$ ſlice FGIAED.

## PROBLEM III.

### *To find the Surface of a Right Pyramid.*

Multiply the perimeter of the baſe by the ſlant height, or length of the ſide, and half the product will, evidently, be the ſurface, or the ſum of the areas of all the triangles which form it.*

EX-

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* *Corol.* 1. Hence, becauſe, in a right cone, the circumference of the baſe into half the ſide, is equal to the curve ſurface, and into half the radius, is equal to the baſe; therefore as the radius of the baſe is to the ſide, ſo is the baſe to the curve ſurface.—And the ſame is true of any other pyramid whoſe baſe is a regular figure, viz. as the radius of the circle inſcribed in the baſe is to the ſlant height, ſo is the baſe to the ſurface.

*Corol.* 2. Let there be a cone and cylinder of the ſame baſe and altitude, the common altitude being equal to the radius of the baſe; then the baſe, the ſurface of the cone, and the ſurface of the cylinder, are to one another as the numbers 1, $\sqrt{2}$, and 2, and ſo are in continual proportion.

And the ſame is true of any other regular priſm and pyramid, of the ſame baſe and altitude, the altitude being equal to the radius of the circle inſcribed in the baſe.

Moreover, the curve ſurface of a cylinder, is equal to a circle whoſe radius is a mean proportional between the ſide of the cylinder and the diameter of its baſe. And the curve ſurface of a cone, is equal to the circle whoſe radius is a mean proportional between the ſide of the cone and the radius of its baſe.

EXAMPLE I.

What is the ſurface of a triangular pyramid, the ſlant height being 20, and each ſide of the baſe 3?

Firſt, $3 \times 3 = 9$ is the perimeter of the baſe.
Then $9 \times 10 = 90$ is the ſurface required.

EXAMPLE II.

Required the ſurface of a ſquare pyramid whoſe ſlant height is 20, and each ſide of the baſe 3.

Here $3 \times 4 = 12$ is the perimeter of the baſe.
Therefore $12 \times 10 = 120$ is the ſurface ſought.

EXAMPLE III.

Required the convex ſurface of a circular pyramid, or cone, whoſe ſlant ſide is 20, and the circumference of the baſe 9.

Here $10 \times 9 = 90$ is the convex ſurface required.

PROBLEM IV.

*To find the Surface of the Fruſtum of a Right Pyramid.*

Multiply the ſum of the perimeters of the ends by the ſlant height, and half the product will be the ſurface.*

EXAMPLE I.

How many ſquare feet are in the ſurface of the fruſtum of a ſquare pyramid, whoſe ſlant height is



* DEMONSTRATION.

For the ſurface is compoſed of a number of equal trapezoids, whoſe common height is equal to the ſlant height of the fruſtum, and the ſums of whoſe parallel ſides, make up the perimeters of the ends of the fruſtum.

10 feet, each ſide of the greater baſe being 3 feet 4 inches, and each ſide of the leſs 2 feet 2 inches?

Here $4 \times (40 + 26) = 4 \times 66 = 264$ is the ſum of the perimeters.

Then $264 \times 10 \times 6$ ſquare inches $= 11 \times 10 = 110$ feet, is the ſurface required.

EXAMPLE II.

If a ſegment of 6 feet ſlant height be cut off a cone whoſe ſlant height is 30 feet, and circumference of its baſe 10 feet; what will be the ſurface of the fruſtum?

As 30 = the height of the whole
: 6 = the height of the upper part
:: 10 = the bottom circumference
: 2 = the circumference at the top of the fruſtum.

Then $6 \times 24 = 144$ is the ſurface required.

PROBLEM V.

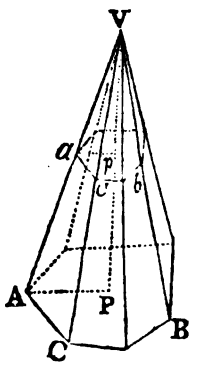
*To find the Solidity of a Pyramid.*

Multiply the baſe by the perpendicular height, and $\frac{1}{3}$ of the product will be the content.*

EX-

* Let VAB be a pyramid whoſe baſe is any plane figure, regular or irregular, VP a perp. to the baſe in P, and meeting the ſection $ab$, parallel to the baſe, in $p$; and put A for the area of the baſe AB, $a$ for that of the ſection $ab$, $b$ the height PV of the pyramid, and $x$ the height $Pp$ of the fruſtum $Aab$B.

Firſt, the parallel ſections AB, $ab$, are to each other as the ſquares of their diſtances, VP, V$p$, from the vertex. For, becauſe they are ſimilar plane figures, which are as the ſquares of their like ſides, we ſhall have $A : a :: AC^2 : ac^2$ : (by ſim. triangles) $AV^2 : aV^2$ :: (by ſimilar triangles again) $PV^2$ : $(bb)$ : $pV^2 = (\overline{b - x}|^2)$.




Hence

EXAMPLE I.

Required the ſolidity of a triangular pyramid whoſe height is 30, and each ſide of the baſe 3.

First, $3^2 \times .433013 = 3.897117$ = area of baſe.
Then $3.897117 \times 10 = 38.97117$ is the ſolidity.

EX-

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Hence we find $a = (b-x)^2 \times \frac{A}{bb}$, and $\dot{a}\dot{x} = \frac{A\dot{x}}{bb} \times (b-x)^2$, the fluent of which is $Ax \times (1 - \frac{x}{b} + \frac{xx}{3bb}) =$ the content of the fruſtum $AabB$.

*Corol.* 1. Since $\sqrt{A} : \sqrt{a} :: b : b-x$, it will be $\sqrt{A} - \sqrt{a} : \sqrt{A} :: x : b = \frac{x\sqrt{A}}{\sqrt{A} - \sqrt{a}}$, which being ſubſtituted for $b$ in the quantity $Ax \times (1 - \frac{x}{b} + \frac{xx}{3bb})$, expreſſing the content of the fruſtum, that content will become $\frac{1}{3}x \times (A + a + \sqrt{Aa})$.

*Corol.* 2. Or, if $s$ be a ſide, or any other line, in the greater end, and $s$ a ſimilar ſide or line in the leſs, ſince $s - s : s :: x : b$, hence $\frac{x}{b} = \frac{s-s}{s}$, which being ſubſtituted inſtead of it, gives $\frac{1}{3}Ax \times \frac{ss+ss+ss}{ss}$ for the value of the ſaid fruſtum.

*Corol.* 3. If $x = b$, then $a = 0$, and the general expreſſion becomes $\frac{1}{3}Ab =$ the whole pyramid; which is our rule.

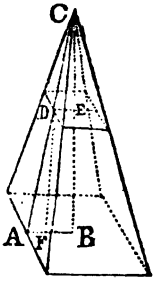
*Corol.* 4. If the baſe be a regular figure; let $s$ be one of its ſides, $s$ a ſide of the leſs end of the fruſtum, and $n$ the area of a ſimilar figure whoſe ſide is 1, to be found in the table in page 114: then $A = nss$, and $a = nss$; hence the ſolidity of the fruſtum will be $(ss + ss + ss) \times \frac{1}{3}nx = \frac{s^3 - s^3}{s-s} \times \frac{1}{3}nx$, and that of the pyramid $= \frac{1}{3}nss^3b$.

*Corol.* 5. If $f = \frac{1}{2}s + \frac{1}{2}s$ the half ſum of the ſides, and $d = \frac{1}{2}s - \frac{1}{2}d$ the half dif. then the fruſtum will be $= (3f^2 + d^2) \times \frac{1}{3}nx$.

*Corol.*

EXAMPLE II.

Required the ſolidity of the ſquare pyramid, each ſide of whoſe baſe is 30, and the ſlant height 25.




Firſt, $30 \times 30 = 900 =$ the baſe.

But $BC = \sqrt{AC^2 - AB^2} = \sqrt{25^2 - 15^2}$ $= \sqrt{5^2 \times 5^2 - 3^2 \times 5^2} = \sqrt{4^2 \times 5^2} =$ $4 \times 5 = 20$, the perpendicular height.

Then $300 \times 20 = 6000$ is the ſolidity.

EX-

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*Corol.* 6. If the baſe be a ſquare; then $n = 1$, and the laſt expreſſions become $\frac{1}{3}ssb$ for the whole pyramid, and $(ss + sz + zz) \times \frac{1}{3}x$, or $\frac{s^3 - z^3}{s - z} \times \frac{1}{3}x$, or $(3f^2 + d^2)\frac{1}{3}x$, for the fruſtum.

*Corol.* 7. If the baſe be a circle; then $n = .7854$, $.2618ssb =$ the cone, and $(ss + sz + zz) \times .2618x$, or $\frac{s^3 - z^3}{s - z} \times .2618x$, or $(3f^2 + d^2) \times .2618x =$ the fruſtum; where s is the diameter of the baſe, and $z$ that of the top.

*Corol.* 8. If $a = A$; then $(A + a + \sqrt{Aa}) \times \frac{1}{3}x$ becomes $3A \times \frac{1}{3}x = Ax =$ the priſm.

*Corol.* 9. Hence a priſm is to a pyramid of the ſame baſe and height, as 3 to 1, and to the fruſtum of the ſame baſe and height, as 1 to $1 - \frac{x}{b} + \frac{xx}{3bb}$, or as 3 to $1 + \frac{a}{A} + \sqrt{\frac{a}{A}}$, or as 3 to $\frac{ss + sz + zz}{ss}$ or $1 + \frac{z}{s} + \frac{zz}{ss}$.

*Corol.* 10. Similar pyramids are as the cubes of their like ſides. For the pyramid is as $Ab$, or as $b^3$, or as $vA^3$ becauſe A is as $bb$ by ſimilar triangles.

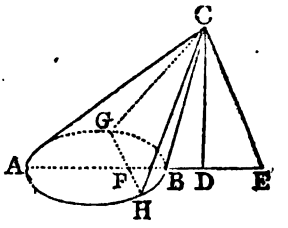
*Corol.* 11. All ſimilar ſolids are as the cubes of their like ſides. This follows from their being compoſed of ſimilar pyramids.

EXAMPLE III.

Wanting to meaſure an oblique cone; and having taken the circumference of the baſe equal to 40 feet, and the outward angle which the ſide of the cone, where it was ſhorteſt, made with the horizon, 85°; going thence in a direct line towards the part to which the vertex inclined, to the diſtance of 100 feet, I found there the angle of elevation of the vertex of the cone to be 50°: required the ſolidity.

The $\angle B + \angle E = 85° + 50° = 135°$
which taken from 180
leaves $\angle BCE$ 45

| | | |
|---|---|---|
| Then s. $\angle BCE$ | 45° | 9·8494850 |
| s. $\angle E$ | 50° - | 9·8842540 |
| BE | 100 - | 2·0000000 |
| BC | - - | 2·0347690 |
| And radius | - - | 10·0000000 |
| s. $\angle B$ | 85° - | 9·9983442 |
| BC | - - | 2·0347690 |
| DC | 107·9228 | 2·0331132 |




But $\frac{40}{3·1416} = \frac{10}{·7854} = 12·732396$ is the diameter.

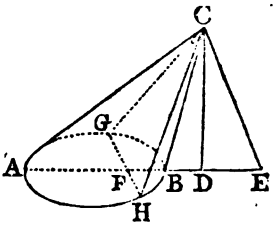
And $12·732396 \times 10 = 127·32396$ is the baſe.

Theref. $\frac{1}{3}DC \times 127·32396 = 42·44132 \times 107·9228$
$= 4580·386$ is the content of the cone.

EXAMPLE IV.

There is a cone whoſe perpendicular altitude is 40 feet, and the circumference of its baſe 30 feet, which is cut by a plane drawn from the vertex, and paſſing through the baſe at the diſtance of 2 feet from the center;

center; what is the ſolidity of the whole cone, and of the two pyramids, GHBC, GHAC, into which it is cut?




Here $\frac{30}{3\cdot1416} = 9\cdot549299$ is the diameter of the baſe.

And its half $4\cdot774648$ is the radius.

Therefore $4\cdot774648 - 2 = 2\cdot774648$ is the verſed ſine of the leſs ſegment of the circle.

Then, to find its area by the table of circular ſegments, $2\cdot774648 \div 9\cdot549299 = \cdot2905611$ is the tabular verſed ſine, to which anſwers the tabular ſegment $\cdot18955709$;

and this taken from $\cdot78539816$

leaves the other tab. ſeg. $\cdot59584107$.

Then each of theſe ſegments multiplied by the ſq. of the diam. give $17\cdot2856$ for the leſs baſe GBHG,

and $54\cdot3341$ for the greater baſe GAHG.

Then $\frac{4}{3\cdot0} \times 17\cdot2856 = 230\cdot474\frac{2}{3}$ = the leſs pyramid,

and $\frac{4}{3\cdot0} \times 54\cdot3341 = 724\cdot454\frac{2}{3}$ = the greater pyr.

their ſum is $954\cdot929\frac{1}{3}$ = the whole cone.

## PROBLEM VI.

### *To find the Solidity of the Fruſtum of a Pyramid.*

Add into one ſum the areas of the two ends and the mean proportional between them, multiply the ſum by the perpendicular height, and $\frac{1}{3}$ of the product will be the ſolidity.

That is, If A be the area of the greater end,

*a* that of the leſs, and *b* the height,

Then $(A + a + \sqrt{Aa}) \times \frac{1}{3}b$ will be the ſolidity.

*Note*

*Note* 1. If the ends be regular polygons, the particular rule for them will be eaſier, thus: Add together the ſquare of a ſide of each end of the fruſtum, and the product of thoſe ſides, multiply the ſum by the height, and the product by the tabular area anſwering to the particular figure of the ends, and $\frac{1}{3}$ of the laſt product will be the content.

Or, Divide the difference of the cubes of the ſaid ſides by their difference, and multiply the quotient by the height, and the tabular area, and take $\frac{1}{3}$ of the product.

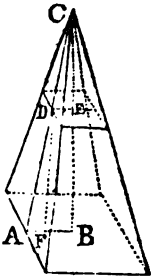
*Note* 2. If the ends be circles, the fruſtum will be that of a cone, and then multiply ·2618, namely $\frac{1}{3}$ of ·7854, by the height, and the product either by the quotient ariſing from the diviſion of the difference of the cubes of the diameters by the difference of the diameters, or by the ſum ariſing from the addition of the ſquare of each diameter and the product of the diameters, or by the ſum ariſing from the ſquare of the half difference of the diameters added to triple the ſquare of the half ſum.*

EXAMPLE I.

How many ſolid feet are there in a tree whoſe baſes are ſquares, each ſide of the one being 15 inches, and each ſide of the other 6, and the length along the ſide meaſures 24 feet?

Here $15 \times 15 = 225$, the greater baſe,
and $6 \times 6 = 36$, the leſs,
and $15 \times 6 = 90$, their mean.

their ſum is 351,
and $\frac{1}{3}$ of it is 117.




But,

* All theſe rules come from the demonſtration of prob. 5, and its corollaries.

But, AB − DE = $7\frac{1}{2} - 3 = 4\frac{1}{2}$ = AF, and $\sqrt{AD^2 - AF^2}$ $\sqrt{(24 \times 12)^2 - (4\frac{1}{2})^2} = 287{\cdot}9649$ inches = DF, the perpendicular.

Therefore $117 \times 287{\cdot}9649 = 33691{\cdot}8933$ inches $= 19{\cdot}49762$ feet is the ſolidity.

EXAMPLE II.

If a caſk, which is two equal conic fruſtums joined together at the baſes, have its bung diameter 28 inches, its head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold?

Here $20^2 \times {\cdot}7854 = 314{\cdot}16$ is the area at the end,
and $28^2 \times {\cdot}7854 = 615{\cdot}7536$ = area bung circle,
and $20 \times 28 \times {\cdot}7854 = 439{\cdot}824$ their mean propor.
their ſum is - - - 1369·7376,
and its third part is - 456·5792,
which multiplied by 40, the length of both fruſtums together, produces 18263·168 ſolid inches; which divided by 231, the inches in a wine gallon, thus,

$231 = 3 \times 7 \times 11$

3) 18263·168

7) 6087·7226

11) 869·6746

gives 79·0613 wine gallons.

Or $(20^2 + 28^2 + 20 \times 28) \times 40 \times {\cdot}2618 = 1744 \times 10{\cdot}472 = 18263{\cdot}168$, the ſolidity the ſame as before.

PROBLEM VII.

*To find the Solidity of a Wedge.*

To twice the length of the baſe add the length of the edge, multiply the ſum by the breadth of the baſe, and

and the product by the height of the wedge, and $\frac{1}{6}$ of the laſt product will be the ſolidity.

That is, uſing the letters in the demonſtration below, $(2L + l) \times \frac{1}{6} bb$ is the content.*

*Note.* If the length of the edge be equal to the length of the baſe, the wedge will be equal to half a priſm of the ſame baſe and height, or equal to half the product of the baſe and height.

EXAMPLE I.

How many ſolid feet are in a wedge whoſe baſe is 3 feet 4 inches long, and 10 inches broad; and each end is inclined to the baſe in an angle of 70°; the edge being 2 feet 6 inches long?

Firſt, FP $= \frac{1}{2}L - \frac{1}{2}l = 5$ inches.

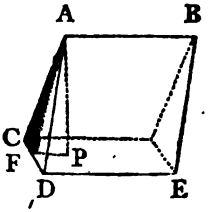
And as rad. : tang. $\angle$F 70° :: FP 5 : AP $=$ 13·737387 $= b$.

Then, $\frac{1}{6} bb \times (2L + l) =$ 10 × 110 × 2·2895645 $=$ 1100 × 2·2895645 $=$ 2518·52095 ſolid inches $=$ 1·45747 ſolid feet.

EX-

* DEMONSTRATION.

Put L = BC the length of the baſe,
$l$ = EF the length of the edge,
$b$ = AB the breadth of the baſe,
$b$ = EP the height of the wedge.



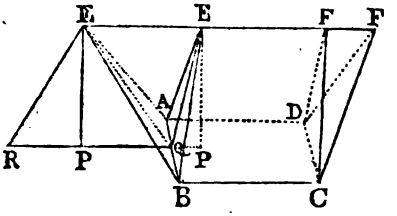

Then, ſince it is evident that, according as the edge is ſhorter or longer than the baſe, the wedge is greater or leſs than half a priſm of the ſame height and breadth with the wedge, and length equal to that of the edge, by a pyramid of the ſame height and breadth at the baſe alſo, and the length of whoſe baſe is equal to the difference of the lengths of the edge and baſe of the wedge; we ſhall have $\frac{1}{2} blb \pm \frac{1}{3} bb \times (\pm L \mp l) = \frac{1}{2} blb + \frac{1}{3} bb \times (L - l) = \frac{1}{6} bb \times (3l + 2L - 2l) = bb \times (2L + l)$. *Q. E. D.*

*Corol.* If $l = L$, the rule will become $\frac{1}{6} bb \times (3L) = \frac{1}{2} bbL$ $= \frac{1}{2}$ a priſm of the ſame baſe and height, as it ought.

*Scholium.* It is evident that, whether the two ends, or the two ſides of the wedge, be equally or unequally inclined to the baſe, it will make no difference in the rule.

EXAMPLE II.

If the length and breadth of the base of a wedge be 35 and 15 inches, and the length of the edge 55; what will be its solidity in feet, supposing the base to make an angle of 115° with one end of the wedge, and of 125° with the other.




Let EP be perpendicular to the base, ER parallel to the end DCF, and RPQ perpendicular to AB.

Then RQ = EF − BC = 55 − 35 = 20, and the angles at R and Q, or the supplements of the angles 115° and 125°, are 65° and 55°; and consequently the angle REQ = 60°. Hence,

As s. ∠E 60° : s. ∠Q 55° :: RQ : RE = 18·9175;
and s. ∠P 90° : s. ∠R 65° :: RE : EP = 17·14508,
the height of the wedge, or *h*.

Wheref. $\frac{1}{6}bh \times (2L + l) = \frac{1}{6} \times 15 \times 17{\cdot}14508 \times (70 + 55) = 8{\cdot}57254 \times 5 \times 125 = 5357{\cdot}8375$ solid inches = 3·1006 solid feet, the content required.

PROBLEM VIII.

*To find the Solidity of a Prismoid.*

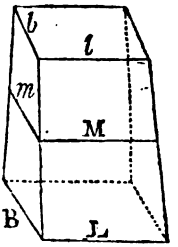
GENERAL RULE.

To the sum of the areas of the two ends, add four times the area of a section parallel to, and equally distant from, both ends; multiply the last sum by the height, and $\frac{1}{6}$ of the product will be the solidity.

PAR-

PARTICULAR RULE.*

Or, If the baſes be diſſimilar rectangles, take two correſponding dimenſions, and multiply each by the ſum of double the other dimenſion of the ſame end and the dimenſion of the other end correſponding to this laſt dimenſion; then multiply the ſum of the products by the height, and $\frac{1}{6}$ of the laſt product will be the ſolidity.




That is, if L, $l$ be one dimenſion of each correſponding to each other, and B, $b$ the other correſponding dimenſions, and $h$ the height.

Then $[(2L + l)B + (2l + L)b] \times \frac{1}{6}h$ = the ſolidity.

*Note.* Correſponding dimenſions are thoſe which are connected by a ſide of the ſolid, as is evident in the figure.



* DEMONSTRATION.

It is evident that the rectangular priſmoid is compoſed of two wedges, whoſe baſes are the two ends of the priſmoid, and whoſe heights are each equal to that of the priſmoid; therefore by the laſt problem, its ſolidity will be $= [(2L + l)B + (2l + L)b] \times \frac{1}{6}h$. Which is the particular rule.

*Corol.* 1. Since $\frac{1}{2}L + \frac{1}{2}l = M$, and $\frac{1}{2}B + \frac{1}{2}b = m$, are the length and breadth of a ſection parallel to, and equally diſtant from, each end; the above rule $[(2L + l)B + (2l + L)b] \times \frac{1}{6}h$ or $(2BL + Bl + 2bl + bL) \times \frac{1}{6}h$, will become $(BL + bl + 4Mm) \times \frac{1}{6}h$. That is, the ſum of the areas of the two ends and four times the ſection in the middle, multiplied by $\frac{1}{6}h$.

*Corol.* 2. This laſt rule will ſerve for any priſmoid, or cylindroid, of whatever figures the ends may be; inaſmuch as they may be conceived to be compoſed of an infinite number of rectangular priſmoids. Which is the general rule.

EXAMPLE I.

How many ſolid feet of timber are in a tree whoſe ends are rectangles, the length and breadth of the one being 14 and 12 inches, and their correſponding ſides of the other 6 and 4 inches; and the perpendicular length $30\frac{1}{2}$ feet?

Firſt, by the general rule,

$\left.\begin{array}{l}\frac{14+6}{2} = \frac{20}{2} = 10\\ \frac{12+4}{2} = \frac{16}{2} = 8\end{array}\right\}$ the dimenſions in the middle

$10 \times 8 \times 4 = 320$ = four times the middle area
$14 \times 12 = 168$ = area of the greater end
$6 \times 4 = 24$ = area of the leſs end

their ſum is 512 ſquare inches $= \frac{32}{9}$ ſquare feet.

Then $\frac{32}{9} \times \frac{30\frac{1}{2}}{6} = \frac{16 \times 30\frac{1}{2}}{9 \times 3} = \frac{162\frac{2}{3}}{9} = 18\frac{2}{27}$ feet, the ſolidity.

Secondly, by the particular rule,

Here $L = 14$, $B = 12$. $l = 6$, $b = 4$ inches, and $b = 30\frac{1}{2}$ feet; therefore $[(2L + l)B + (2l + L)b] \times \frac{1}{6}b = \frac{34 \times 12 + 26 \times 4}{6 \times 144} \times 30\frac{1}{2} = 18\frac{2}{27}$ feet, the ſame as before.

EXAMPLE II.

What is the capacity of a waggon whoſe inſide dimenſions are thus: at the top, the length and breadth $81\frac{1}{4}$, and 55 inches; at the bottom, the length 41 and breadth $29\frac{1}{2}$; and the perpendicular depth $47\frac{1}{4}$?

Here $[(2L + l)B + (2l + L)b] \times \frac{1}{6}b = (102 \times 55 + 81{\cdot}75 \times 29{\cdot}5) \times 47{\cdot}25 \times \frac{1}{3} = 8021{\cdot}625 \times 15{\cdot}75 = 126340{\cdot}59375$ cubic inches $= 448{\cdot}01629$ ale gallons, the content required.

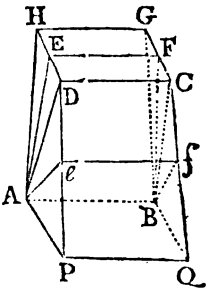
PRO-

## PROBLEM IX.

*To find the Solidity of the two Parts, called Ungulas or Hoofs, into which the Fruſtum of a Rectangular or a Square Pyramid, or a Rectangular Priſmoid, is cut by a Plane Inclined to its Baſe.*

### CASE I.

If the plane, paſſing through A and B, cut the end in EF, between GH and DC; it will cut off the wedge AEHGFB, whoſe baſe is EFGH, edge AB, and height the ſame with that of the fruſtum, or priſmoid; and the remaining part AEDCFBQP will be a priſmoid.




Then, by problem 7, find the content of the wedge ABFEHG, and that of the priſmoid ABQPDCFE by problem 8.

### EXAMPLE.

If the fruſtum of a ſquare pyramid be cut by a plane, paſſing through one ſide of the leſs end, and through the middle of the greater end; what are the contents of the two parts, ſuppoſing each ſide of the greater end to be 15 inches, each ſide of the leſs 6, and the ſlant height 24 feet?

Since, by example 1 problem 6, the perpendicular height is 287·9649 inches, therefore,

Firſt, $\frac{1}{6}(2 \times 15 + 6) \times 7\frac{1}{2} \times 287{\cdot}9649 = 45 \times 287{\cdot}9649 = 12958{\cdot}4205$ inches $= 7{\cdot}499086$ feet, the content of the wedge.

And $\frac{1}{6}[(30+6)\times 7\frac{1}{2}+(12+15)\times 6]\times 287{\cdot}9649$ $= \frac{1}{6}(18\times 15+27\times 6)\times 287{\cdot}9649 = 72\times 287{\cdot}9649$ $= 20733{\cdot}4728$ inches $= 11{\cdot}998536$ feet, the prismoid.

CASE II.

If the plane pass through DC, as well as AB, the solid is thereby divided into two wedges or hoofs ABCGHD, ABCQPD, whose two bases are the ends or bases of the solid.

And then the contents of the two parts will be found by problem 7.

EXAMPLE.

Let there be taken here the same figure as in the last example, to find the content of the two wedges into which it is cut.

First, $\frac{1}{6}(30+6)\times 15\times 287{\cdot}9649 = 90\times 287{\cdot}9649$ $= 25916{\cdot}841$ inches $= 14{\cdot}99817$ feet, the greater wedge.

And $\frac{1}{6}(12+15)\times 6\times 287{\cdot}9649 = 27\times 287{\cdot}9649$ $= 7775{\cdot}0523$ inches $= 4{\cdot}49945$ feet, the less wedge.

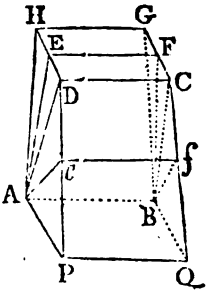
CASE III.

If the plane cut the side of the solid in *ef*; the part cut off ABQ*fe*P will be a wedge, whose base is AQ; and which being taken from the whole figure, will leave the content of the part AB*fe*DCGH.

EXAMPLE.

Let there be taken here the same figure, supposing the plane to cut the side PC in *ef* at the distance of 10 feet from PQ, which let be now supposed one side of the less end AQ; to find the solidity of the two parts.

First,




First, As PD 24 : *Pe* 10 :: 287·9649 the height of the whole ſolid : 119·98536 the height of the wedge A Q *e*.

And as PD 24 : *Pe* 10 :: DC — PQ 9 : *ef* — PQ = $\frac{15}{4}$ = 3·75.

Hence *ef* = 3·75 + 6 = 9·75 = the length of the edge.

Wherefore $\frac{1}{6}$ (12 + 9·75) × 6 × 119·98536 = $21\frac{3}{4}$ × 119·98536 = 87 × 29·99634 = 2609·68158 inches = 1·51023 feet, the ſolidity of the wedge A Q *e*.

which taken from 19·49762 the whole fruſtum found by prob. 8,

leaves 17·98739 for the content of the part A B *fe* D C H G.

## PROBLEM X.

*To find the Curve Surface of a Sphere, or of any Segment or Zone of it.*

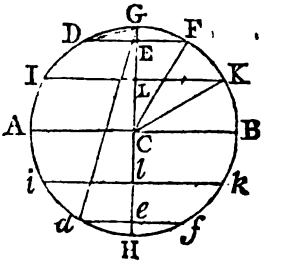
Multiply the circumference of the ſphere by the height of the part required, and the product will be the curve ſurface, whether it be ſegment, zone, hemiſphere, or the whole ſphere.*



* DEMONSTRATION.

Put *d* = the diameter A B or 2 C G, *x* = C E the height of the zone A D F B, *y* = D E, *z* = A D, *p* = 3·14159, and *s* = the ſurface required.

Then $s = 2pyz$; but $y : \frac{1}{2}d :: x : z$, or $2yz = dx$; therefore $s = pdx$, and conſequently $s = pdx$; viz. the product of the circumference of the ſphere and height of the zone; for $pd$ is = the circumference of the circle whoſe diameter is *d*.




*Corol.*

*Note.* The height of the whole ſphere is its diameter.

EXAMPLE I.

If the diameter or axe of the earth be $7957\frac{3}{4}$ miles, what is the whole ſurface, ſuppoſing it a perfect ſphere?

Firſt, $7957\frac{3}{4} \times 3{\cdot}141592 = 25000$ miles, very near, = the circumference.

Then $7957\frac{3}{4} \times 25000 = 198943750$ ſquare miles = the whole ſurface required. And the half is $99471875$ = the ſurface of the hemiſphere.

EX-

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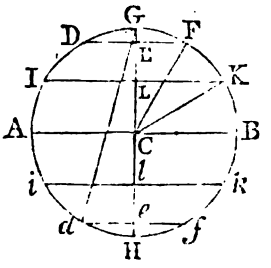
*Corol.* 1. When $x$ becomes = CG $= \frac{1}{2}d$, $s$ will be $= \frac{1}{2}pdd =$ the ſurface of the hemiſphere. And conſequently that of the whole ſphere is $pdd$, the product of the circumference and height or diameter.




*Corol.* 2. To or from $\frac{1}{2}pdd$, the ſurface of the hemiſphere, add or ſubtract $pdx$, that of the zone ADFB, and the remainder $pd \times (\frac{1}{2}d \pm x) =$ $pd \times$ GE will be the ſurface of the ſegment DGF, viz. the product of the circumference and height. So that the rule is general. *Q. E. D.*

*Corol.* 3. The ſurfaces of ſpheres, and alſo of their ſimilar parts, are to each other as the ſquares of their diameters. For, by exterminating the common given quantity $p$, they are as $d$, the diameter, into the height of the part; but the heights of ſimilar parts are as the diameters; therefore &c.

*Corol.* 4. The ſurfaces of any ſegments or zones of a ſphere are to each other, or to that of the whole ſphere, as their heights. For $pd$ is common to them all.

*Corol.* 5. Or the ſurface of any ſegment or zone of a ſphere, is as its height.

*Corol.*

EXAMPLE II.

To find the ſurface of the two frigid zones of the earth.

*Note.* The frigid zones are the two oppoſite ſegments DGF, *dHf*, in which each of the arcs DG, *d*H, or half the breadth of the zone, is $23\frac{1}{2}$ degrees.

Draw the radius CF; then the angle FCE $= 23\frac{1}{2}$ degrees, and its complement EFC $= 66\frac{1}{2}$ degrees; alſo the angle E $= 90$ degrees.

Then, as S.∠E : S.∠F :: FC $= 3978\frac{7}{8}$ : CE
$= 3648{\cdot}8675054.$
Which taken from CG — — $3978{\cdot}8750000,$
leaves the height GE — — $330{\cdot}0074946.$
And the circumference is 25000.

Hence $25000 \times 330{\cdot}0074946 = 8250187{\cdot}365$ = the ſurface of the ſegment DGF or frigid zone.



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*Corol.* 6. The ſurface of any ſegment or zone, is equal to 4 times a circle whoſe diameter is a mean proportional between its height and the diameter of the ſphere. For this circle is $= \frac{1}{4}p \times (\sqrt{p}dx)^2 = \frac{1}{4}p\,dx$.

*Corol.* 7. Hence the ſurface of a ſegment DGF is equal to 4 times the circle whoſe diameter is the chord DG drawn from the vertex to the extremity of the baſe; or equal to a circle whoſe radius is that chord. For DG $= \sqrt{GH \times GE} = \sqrt{dx}$.

*Corol.* 8. And hence the ſurface of a ſphere is equal to 4 times the area of a great circle of it; that is, 4 times the area of a circle of the ſame diameter with the ſphere. And conſequently the ſurface of an hemiſphere is double the area of its baſe.

*Corol.* 9. Or the ſurface of a ſphere is equal to a circle whoſe diameter is double to that of the ſphere.

*Corol.* 10. The ſurface of any ſegment or zone of a ſphere, is equal to the curve ſurface of a cylinder of the ſame height with it, and whoſe diameter is equal to that of the ſphere.

*Corol.* 11. Hence the ſurface of a whole ſphere, or of a hemiſphere, is equal to the curve ſurface of the circumſcribed cylinder.

EXAMPLE III.

To find the ſurface of the torrid zone IK*ki*, which extends to the diſtance of AI or A*i* $23\frac{1}{2}$ degrees on each ſide from AB the diameter.

Here the $\angle$ LKC $= 23\frac{1}{2}$ degrees.

Then, s. $\angle$ L : s. $\angle$ LKC :: CK : CL $= 1586{\cdot}572$.

And the height L*l* $=$ 2CL $= 3173{\cdot}144$.

Hence $25000 \times 3173{\cdot}144 = 79328600$ ſquare miles $=$ the ſurface of the torrid zone.

EXAMPLE IV.

Required the convex ſurface of each of the temperate zones IDFK, *idfk*, which are included between the frigid and torrid zones.

Here, by the 1ſt ex. CE $= 3648{\cdot}8675$

and by the 2d ex. CL $= 1586{\cdot}572$

their difference is LE $= 2062{\cdot}2955 =$ the height.

Therefore $25000 \times 2062{\cdot}2955 = 51557387\frac{1}{2}$ ſquare miles $=$ each temperate zone.

Hence the two frigid zones $= 16500375$

the two temperate zones $= 103114775$

the torrid zone $= 79328600$

their ſum is 198943750

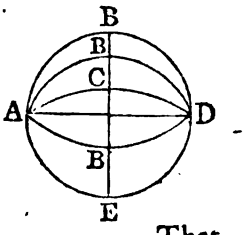
or the ſurface of the whole ſphere.

PROBLEM XI.

*To find the Lunar Surface* ABDCA, *included between two Great Circles* ABD, ACD *of a Sphere.*

RULE I.

Multiply the diameter AD by the breadth BC of the ſurface in the middle, that is, by the arc which meaſures the angle BAC of inclination of the two circles; and the product will be the ſurface ABDCA.




That

That is, if $d$ = AD the diameter,
and $a$ = the length of the arc BC.
Then $ad$ is the ſurface.

* RULE II.

As four right angles, to the ſurface of the ſphere;
Or, As one right angle, to a great circle of the ſphere;
So is the angle made by the two great circles,
To the ſurface included by them.

EXAMPLE.

Required the ſurface included by two great circles forming an angle of 25 degrees; the diameter of the ſphere being 10 feet.

1. By rule 1.

Firſt, $3{\cdot}1415926 \times 10 = 31{\cdot}415926$ is the circumf.
And $360 : 25$ (:: $72 : 5$ :: $144 : 10$):: $31{\cdot}415926 : 2{\cdot}181661$ the greateſt breadth of the ſurface.
Then $2{\cdot}181661 \times 10 = 21{\cdot}81661$ is the ſurface req.

2. By

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* DEMONSTRATION.

If the great circle BCE, whoſe poles are A and D, be conceived to be divided into an indefinite number of equal parts, and great circles be conceived to be drawn through the points of diviſion, and through A and D: it is evident that the whole ſurface will be divided, by the circles, into the ſame number of parts, ſimilar and equal to one another; and that, therefore, the ſurface included between any two of thoſe circles, will be as the number of parts, or as the arc of the circle BE included by them; wherefore as the whole circumference BE is to the arc BC, or as four right angles to the angle BAC, ſo is the ſurface of the ſphere to the lunar ſurface ABDCA; or alſo as one right angle is to the angle BAC, ſo is the area of a great circle of the ſphere ($\frac{1}{4}$ of the ſurface) to the lunar ſurface. Which is rule 2.

*Corollary.* If $d$ be the diameter, and $c$ the circumference of the ſphere, and $a$ the arc BC; then, from the proceſs above, $c : a :: cd$ (the ſurface of the ſphere) : $ad$ = the lunar ſurface. Which is rule 1.

2. By rule 2.

Here $31{\cdot}415926 \times 10 = 314{\cdot}15926$ is the surface of the sphere.

And $360 : 25 :: (144 : 10 ::) 314{\cdot}15926 : 21{\cdot}81661$ the surface required.

## PROBLEM XII.

*To find the Area of a Spherical Triangle; that is, the Spherical Surface included by the Arcs of Three Great Circles of the Sphere Interfecting one another.*

As 8 right angles or 720°,
To the surface of the sphere;
Or, As 2 right angles or 180°,
To a great circle of the sphere;
So is the excefs of the 3 angles above 2 right angles,
To the area of the triangle.*

That

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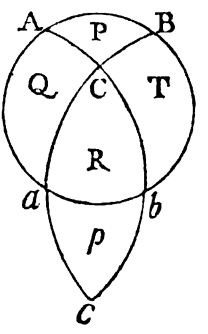
* DEMONSTRATION.




For having produced all the fides of the triangle ABC (P), till they interfect again, and form the triangle $p$, which by the principles of the fphere will be equal to the former triangle P; put $\frac{1}{2}s = \frac{1}{2}$ the furface of the fphere $= P + Q + R + T$. Then, by the laft problem,

$$\begin{cases} 180° : A :: \frac{1}{2}s : P + T, \\ 180 : B :: \frac{1}{2}s : P + Q, \\ 180 : C :: \frac{1}{2}s : P + R = p + R; \text{ hence} \end{cases}$$

$180 : A + B + C :: \frac{1}{2}s : 3P + T + Q + R = 2P + \frac{1}{2}s$,

and as $180 : A + B + C - 180 :: \frac{1}{2}s : 2P ::$

$\frac{1}{4}s : P = s \times \frac{A + B + C - 180}{720}$. Q. E. D.

*Corollary.* When $P = 0$, then $(A + B + C \text{ or}) s = 180$; but when $P = \frac{1}{2}pdd$, half the furface of the fphere, then $s - 180 = 360$, or $s = 540$: confequently $A + B + C$ is always between 180 and 540, that is, greater than 2, and lefs than 6, right angles.

That is, $pdd \times \frac{s-180}{720}$ = the area of the triangle;
putting $p = 3{\cdot}14159$,
$d$ = the diameter of the ſphere,
$s$ = the ſum of the 3 angles of the triangle.

EXAMPLE.

If the angles be 55, 60, and 85 degrees; what is the trilineal ſurface; ſuppoſing the diameter to be 10?

Here $\cdot 7853982 \times 102 = 78{\cdot}53982 = \frac{1}{4}$ of the ſurface of the ſphere. And $55 + 60 + 85 - 180 = 20$.

Then 180 : 20, or 9 : 1 :: 78·53982 : 8·72664, the area of the triangle required.

PROBLEM XIII.

*To find the Area of a Spheric Polygon, or to find the Spherical Surface included by Any Number of Interſecting Great Circles.*

As 8 right angles or 720°,
To the ſurface of the ſphere;
Or, As 2 right angles or 180°,
To a great circle of the ſphere;
So is the exceſs of all the angles, above the product of 180 and 2 leſs than the number of angles,
To the area of the ſpherical polygon.

That is, putting $n$ = the number of angles,
$s$ = the ſum of all the angles,
$d$ = the diameter of the ſphere,
$p = 3{\cdot}14159$ &c.

Then

Then $pd^2 \times \frac{s - 180 \times (n-2)}{720} =$ the area of the ſpherical polygon.*

EXAMPLE.

What ſurface is included by the intercepted arcs of five interſecting great circles of a ſphere, of 10 feet diameter; ſuppoſing the ſum of the angles formed by thoſe arcs to be 640 degrees.

Here the ſurface of the ſphere is $3{\cdot}1415926 \times 10^2 = 314{\cdot}15926$.

And $180 \times (n - 2) = 180 \times 3 = 540$; which taken from 640, leaves 100.

Then as $720 : 100 :: 314{\cdot}15926 : 43{\cdot}63323$, the area of the polygon required.

PRO-

* DEMONSTRATION.

For if the polygon be ſuppoſed to be divided into as many triangles as it has ſides, by great circles drawn to all the angles through any point within it, forming at that point the vertical angles of all the triangles. Then, by the laſt problem, it will be, in any one triangle,

As 720 : s :: ſum of its angles $-180$ : its area. Theref. by compoſ. as $720 : s :: s +$ all the vertical angles $- 180n$ : ſum of all the triangles or area of the polygon.

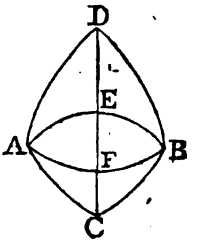
But all the vertical angles $= 360$ or $180 \times 2$. Therefore

As $720 : s :: s - 180 \times (n-2) : s \times \frac{s - 180 \times (n-2)}{720}$, the area of the polygon. *Q. E. D.*

*Corollary.* When the polygon is $= 0$, then $s$ is $= 180 \times (n-2)$; and when the polygon is $=$ the ſemi ſpheric ſurface, then $s = 180 \times (n-2) + 360 = 180 \times n$; conſequently $s$ the ſum of all the angles of any polygon, is always between $180 \times (n-2)$ and $180 \times n$, that is, leſs than $n$ times 2 right angles, but greater than $n - 2$ times 2 right angles, $n$ being the whole number of angles.

## PROBLEM XIV.

*To find the Surface* AEBF *included between Two Intersecting Little Circles of a Sphere.*




Through the poles D, C, and interſections A, B, of the two little circles AFB, AEB, draw the great circles AD, BD, AC, BC; and alſo the great circle DEFC.

By prob. 12 find the area of the triangle BCD, having firſt found the angles at B, C, and D, from the given ſides, by the principles of ſpherical trigonometry.

By prob. 10 find the ſurface of the ſegment cut off by the circle of which AFB is a part, thus, viz. As the diameter is to the verſed ſine of the arc DF, or DB, ſo is the ſurface of the ſphere to that of the ſegment.

Then as 4 right angles, to the ∠BDC,
So is that ſurface, to the part of it DBF.

In the ſame manner find the ſurface CBE.

Then from the ſum of DBF and CBE take the triangle DBC, and the remainder will be the part BEF; the double of which will be the whole AEBFA.

## PROBLEM XV.

*To find the Solidity of a Sphere or Globe.*

* RULE I.

Multiply the ſurface by $\frac{1}{3}$ of the radius, or by $\frac{1}{6}$ of the diameter; and the product will be the ſolidity.

RULE

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* DEMONSTRATION.

The ſphere may be conſidered as conſtituted of an infinite number of pyramids, whoſe baſes compoſe the ſpheric ſurface, and

RULE II.

Multiply the cube of the diameter by ·5236, and the product will be the ſolidity.

That is ($\frac{1}{6}$ of 3·1416 or) $\cdot 5236 d^3$ = the ſolidity.

EX-

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and all the vertices meeting in the center, their common height being equal to the radius of the ſphere. And conſequently the ſphere, or any ſpherical pyramid, being a part contained within right lines drawn from the ſurface to the center, is equal to a pyramid whoſe baſe is equal to the ſpherical ſurface, and height equal to the radius. And therefore the ſurface of the whole, or of any ſuch part, being drawn into $\frac{1}{3}$ of the radius, will give the ſolidity, as in rule 1.

*Corol.* 1. Since the ſurface of the ſphere is $= pdd$, we ſhall have $pdd \times (\frac{1}{3}$ of the radius or) $\frac{1}{6}d = \frac{1}{6}pd^3 = \cdot 523598775$ &c. $\times d^3$ (which is rule 2) $= \frac{2}{3}$ of a cylinder of the ſame diameter and height.

*Corol.* 2. If $b$ be the height, or verſed ſine, of any ſegment; then, ſince $pdb =$ its ſurface, $\frac{1}{4}pd^2b = \cdot 523598775$ &c. $\times d^2b$ will be the ſolidity of the ſpheric pyramid, or cone, whoſe baſe is the ſurface of the ſegment.

*Corol.* 3. Hence ſpheres and their ſimilar pyramids, and alſo any other ſimilar parts of them, are as the cubes of the diameters.

*Corol.* 4. If to or from $\frac{1}{6}pd^2b$, the ſpheric cone, be added or ſubtracted $\frac{1}{4}p \times (\mp\frac{1}{2}p \pm b) \times \frac{4}{3}b \times (d-b) = \mp\frac{1}{6}pd^2b \pm \frac{1}{2}pdb^2 \mp \frac{1}{3}pb^3$, the cone whoſe baſe is the ſame with the baſe of the ſegment, and whoſe vertex is in the center, the ſum or difference $\frac{1}{2}pdb^2 - \frac{1}{3}pb^3 = \frac{1}{3}pb^2 \times (\frac{3}{2}d - b)$, will be the ſpheric ſegment whoſe height is $b$, either greater or leſs than the hemiſphere, or of whatever magnitude $b$ is, not exceeding $d$. Or if $r$ be $=$ the radius of the ſegment's baſe, ſince $db = r^2 + b^2$, the ſegment will be $\frac{1}{2}pr^2b + \frac{1}{6}pb^3 = \frac{1}{6}pb \times (3r^2 + b^2)$.

*Corol.* 5. Hence the difference between two ſegments whoſe heights are H, $b$, and the radii of their baſes R, $r$, will give for the fruſtum or zone $\frac{1}{6}p \times (3R^2H + H^3 - 3r^2b - b^3)$; which, putting $a$ for the altitude of the fruſtum, and exterminating H and $b$ by means of the two equations $(R^2 + H^2)b = (r^2 + b^2)H$, and $a = H - b$, will become $\frac{1}{2}ap \times (R^2 + r^2 + \frac{1}{3}a^2)$.

EXAMPLE.

Suppoſing the earth to be ſpherical, and its diameter $7957\frac{3}{4}$ miles, what is its ſolidity?

1. By rule 1.

By exam. 1 of prob. 10 the ſurface is 198943750.

Then $198943750 \times 7957\frac{3}{4} \times \frac{1}{6} = 263857437760$ miles is the ſolidity.

2. By the 2d rule.

Here $\cdot 5236 d^3 = \cdot 5236 \times (7957\frac{3}{4})^3 = 263858149120$ miles = the ſolidity by this rule. The difference ariſing by taking the number $\cdot 5236$ rather too great.

PROBLEM XVI.

*To find the Solidity of the Segment of a Sphere.*

RULE 1.

To three times the ſquare of the radius of its baſe, add the ſquare of its height; multiply the ſum by the height, and the product by $\cdot 5236$, for the ſolidity.

That is, if $r =$ DE the radius of its baſe,

$h =$ GE the height;

Then $\cdot 5236h \times (3rr + hh) =$ the ſolidity of the ſegment DGF. By cor. 4 to the laſt problem. [See the figure in page 198.]

RULE

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*Corol.* 6. If one end of the fruſtum paſs through the center, then $R^2 = \frac{1}{4}d^2 = r^2 + a^2$, and the laſt theorem will become $ap \times (r^2 + \frac{2}{3}a^2) = ap \times (\frac{1}{4}d^2 - \frac{1}{3}a^2)$.

*Corol.* 7. Hence the middle zone, or the double of the laſt expreſſion, will be $2ap \times (r^2 + \frac{2}{3}a^2) = 2ap \times (\frac{1}{4}d^2 - \frac{1}{3}a^2)$; where $a$ is half its altitude, and $r$ half the diameter of each end. But if A be its whole altitude, and D the diameter of each end, thoſe theorems will become $\frac{1}{4}Ap \times (D^2 + \frac{2}{3}A^2) = \frac{1}{4}Ap \times (d^2 - \frac{1}{3}A^2) = \frac{1}{6}Ap \times (d^2 + 2r^2)$.

RULE II.

From three times the diameter of the ſphere, ſubtract twice the height of the fruſtum; multiply the difference by the ſquare of the height, and the product by ·5236, for the ſolidity.

That is,

If $d$ = GH the diam. of the ſph.
$b$ = GE the height of the fruſt. [*See fig. page* 198.]

Then $\cdot 5236b^2 \times (3d - 2b)$ = the ſolidity of DGF. By cor. 4 to the laſt problem.

EXAMPLE.

What is the ſolidity of each of the frigid zones of the earth; the axe being $7957\frac{3}{4}$ miles, and half the breadth, or arc DG, of the zone being $23\frac{1}{2}$ degrees?

By rule 2.

As 1 = tabular radius : $3978\frac{7}{8}$ = radius of the earth :: ·0829399 = tab. verſed ſine of $23\frac{1}{2}$ degrees : 330·0074946, the verſed ſine or height of the ſegment.

Then $\cdot 5236b^2 \times (3d - 2b) = \cdot 5236 \times 330 \cdot 0074946^2 \times 23213 \cdot 2350108 = 1323679710$, the content.

By rule 1.

As 1 : $3978\frac{7}{8}$ :: ·3987491 = tabular ſine of $23\frac{1}{2}$ degrees : 1586·57282526, the radius of the baſe.

Then $\cdot 5236b \times (3r^2 + b^2) = \cdot 5236 \times 330 \cdot 0074946 \times 7660544 \cdot 936 = 1323680299 \cdot 69$, the ſolidity.

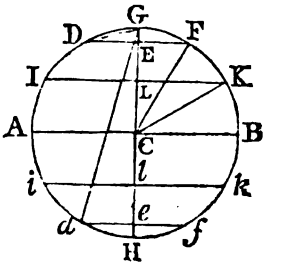
PRO-

## PROBLEM XVII.

*To find the Solidity of a Fruſtum or Zone of a Sphere.*

Add together the ſquares of the radii of the ends, and $\frac{1}{3}$ of the ſquare of their diſtance, or of the height; multiply the ſum by the ſaid height, and the product again by 1·5708 for the content.

That is, $(R^2 + r^2 + \frac{1}{3}b^2) \times \frac{1}{2}pb$ = the ſolidity of the fruſtum whoſe height is $b$, and the radii of its ends R and $r$, $p$ being 3·1416. By cor. 5 to prob. 15.




### EXAMPLE I.

What is the ſolidity of the fruſtum of a ſphere, the diameter of whoſe great end is 4 feet, the diameter of the leſs 3 feet, and the height $2\frac{1}{2}$ feet?

Here $(R^2+r^2+\frac{1}{3}b^2)\times 1\cdot5708b = (2^2+1\cdot5^2+\frac{1}{3}\times 2\cdot5^2) \times 1\cdot5708 \times 2\frac{1}{2} = 8\frac{1}{3} \times 3\cdot927 = 32\cdot725$, the ſolidity of the fruſtum required.

### EXAMPLE II.

What is the ſolidity of each temperate zone of the earth, extending from $23\frac{1}{2}$ degrees to $66\frac{1}{2}$ degrees of latitude, and the diameter of the earth being $7957\frac{3}{4}$ miles?

By the example to the laſt problem, the radius of the top is 1586·57282526.

And as 1 : $3978\frac{7}{8}$ :: ·9170601 = tabular ſine of $66\frac{1}{2}$ degrees : 3648·86750538, the radius of the baſe.

Alſo by example 4 prob. 10, the height is 2062·2955.

Then $(R^2 + r^2 + \frac{1}{3}b^2) \times 1{\cdot}5708b = 17249136 \times 2062{\cdot}2955 \times 1{\cdot}5708 = 55877778668$, the ſolidity of each temperate zone.

*Otherwiſe.*

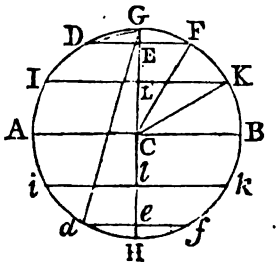
Since the radii of the ends of this zone, are the ſines of $23\frac{1}{2}$ and $66\frac{1}{2}$ degrees, which are complements the one of the other, the ſum of the ſquares of thoſe radii will be equal to the ſquare of the radius of the ſphere; and therefore $(\frac{1}{4}dd + \frac{1}{4}bb) \times 1{\cdot}5708b = 17249136 \times 2062{\cdot}2955 \times 1{\cdot}5708 = 55877778668$, the content, as before.

## PROBLEM XVIII.

*To find the Solidity of the Middle Zone of a Sphere.*

Multiply, either the ſum of the ſquare of the diameter of the end and $\frac{2}{3}$ of the ſquare of the height, or the difference between the ſquare of the diameter of the ſphere and $\frac{1}{3}$ of the ſquare of the height, by the height, and the product by ·7854 for the content.

That is, $(DD + \frac{2}{3}bb) \times {\cdot}7854b$ or $(dd - \frac{1}{3}bb) \times {\cdot}7854b$ is the content of the middle zone whoſe height is $b$, the diameter of each end D, and the diameter of the ſphere $d$. By cor. 6. to prob. 15.




EXAMPLE I.

Required the ſolidity of the middle zone of a ſphere whoſe top and bottom diameters are each 3 feet, and height 4 feet.

Here $(DD + \frac{2}{3}bb) \times .7854b = (3^2 + \frac{2}{3} \times 4^2) \times 4 \times .7854 = 59 \times 4 \times .2618 = 61.7848$, the content required.

EXAMPLE II.

What is the ſolidity of the torrid zone of the earth, which extends to $23\frac{1}{2}$ degrees on each ſide of the equator; the diameter of the earth being $7957\frac{3}{4}$ miles?

By the example to prob. 16, the ſine of $23\frac{1}{2}$ degrees, or $\frac{4}{7}$ the height of the zone, is 1586.57282526, and the whole height is $3173.14565052 = b$.

Then $(dd - \frac{1}{3}bb) \times .7854b = (7957.75^2 - \frac{1}{3} \times 3173.14565052^2) \times 3173.14565052 \times .7854 = 1494550811137$, the content.

SCHOLIUM.

From the laſt three problems we find that

| | | |
|---|---|---|
| The two frigid zones | = | 2647359420 |
| The two temperate zones | = | 111755557336 |
| The torrid zone | = | 1494550811137 |

whoſe ſum 263857997893 is the whole ſphere, nearly the ſame as found in prob. 15.

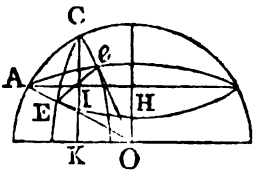
PROBLEM XIX.

*To find the Solidity of the Second Segment of a Sphere.*

DEFINITION.

A ſecond ſegment is a part cut off a ſegment by a plane perpendicular to the baſe.

RULE.




By prob. 14 find the curve ſurface of the ſecond ſegment AEC*e*, which being drawn into $\frac{1}{3}$ of the radius of the ſphere, will produce the content of the ſpheric ſector OEAC*e*; from which if there be taken the pyramid OEA*e*, whoſe baſe is the ſegment AE*e*, and height OH; and from the remainder be taken the pyramid OEC*e*, whoſe baſe is the ſegment CE*e*, and height KO, or HI; it is evident that the laſt remainder will be the ſecond ſegment AEC*e*: for theſe two pyramids and the ſecond ſegment compoſe the ſpheric pyramid.

PROBLEM XX.

*To find the Surface of a Circular Spindle, or of any Segment or Fruſtum of it.*

From the product of the height of the ſolid and radius of the revolving arc, ſubtract the product of the ſaid arc and central diſtance; multiply the remainder by 3·1416; and double the product will be the ſurface deſcribed by that arc; whether it be the whole or any part of the ſpindle.*

That

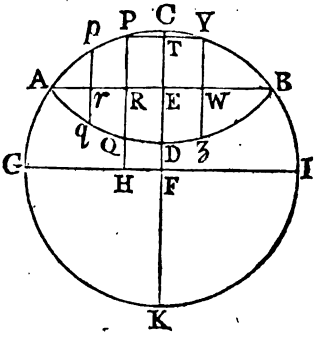
* DEMONSTRATION.

Put $z$ = the arc CP, $x$ = its ſine RE, $r$ and $c$ for the radius and central diſtance, and $p = 3{\cdot}14159$. Then the fluxion of the ſurface $\dot{s}$ is $= 2p\dot{z} \times RP = 2p \times (\dot{z}\sqrt{rr - xx} - c\dot{z})$; but, by the property of the circle, $\dot{z}\sqrt{rr - xx}$ is $= r\dot{x}$; therefore $\dot{s} = 2p \times (r\dot{x} - c\dot{z})$, and $s = 2p \times (rx - cz) = 2p \times (r \times RE - c \times PC)$.

*Corol.*

That is,

$r$ = FC the rad. of the arc,
$c$ = FE the central dist.
$h$ = $r$R the height,
$a$ = P$p$ its revolving arc,
$p$ = 3·1416.




Then $(hr - ac) \times 2p$ = the surface of that part $p$PQ$q$.

EXAMPLE.

Required the surface of a circular spindle whose greatest diameter is 30, and its length 40 inches.

Here, by the property of the circle, EK = $AE^2 \div EC = 400 \div 15 = 26\frac{1}{3}$; and $26\frac{2}{3} + 15 = 41\frac{2}{3}$ is the diameter CK, or $20\frac{5}{6}$ = the radius of the circle.

 And

---

*Corol.* 1. When RE = AE, the rule becomes $2p \times (r \times AE - c \times AC)$ for the surface of half the spindle, or $2p \times (r \times AB - c \times ACB)$ for that of the whole.

*Corol.* 2. If from the surface of the semi-spindle be taken that of the frustum, there will remain $2p \times (r \times AR - c \times AP)$ for that of the segment APQ: so that the rule is general.

*Corol.* 3. When E coincides with F, $c$ vanishes, and the spindle becomes a sphere; and then the theorem becomes barely $2prx$, the same with that before found for the sphere.

*Corol.* 4. From cor. 1, it appears that the radius is to the cosine of an arc, always in a greater proportion than that of the arc to its sine, or of the double arc to its chord. But those ratios approximate to an equality as the arc diminishes, till, when the arc vanishes, they become accurately equal, and the arc and chord vanish also in a ratio of equality.—Consequently, in small arcs, the arc is a fourth proportional to the cosine, sine, and radius, nearly. Thus, the sine and cosine of an arc of 1 degree are ·0174524 and ·9998477 to the radius 1; then ·9998477 : ·0174524 :: 1 : ·0174550 = the arc nearly, the true figures being ·0174533.

### PROBLEM XXII.

*To find the Content of the Middle Fruſtum or Zone of a Circular Spindle.*

From the ſquare of half the length of the ſpindle, take $\frac{1}{3}$ of the ſquare of half the length of the middle zone; and multiply the remainder by the ſaid half length of the zone; from the product ſubtract the product of the generating circular area, and central diſtance; then the remainder drawn into 2 times 3·14159 will be the content of the middle zone.

That is, putting

$l$ = ER half the length of the zone,
L = EA half the length of the ſpindle,
$c$ = FE the central diſtance,
$a$ = the generating area RPYW.

Then $[(LL - \frac{1}{3}ll)l - ac] \times 2p$ = the zone PYZQ.

### EXAMPLE.

If a caſk, in the form of the middle fruſtum of a circular ſpindle, have its head diameter 24, bung diameter 32, and length 40 inches; how many ale gallons will it hold?

Here CE − PR = 16 − 12 = 4 = TC, and $PT^2 \div TC = 400 \div 4 = 100 = TK$; hence 100 + 4 = 104 = the diameter, and 52 = the radius of the generating circle.

FC − CE = 52 − 16 = 36 = EF the central diſtance; and $\sqrt{AF^2 - FE^2} = \sqrt{52^2 - 36^2} = 4\sqrt{13^2 - 9^2} = 4\sqrt{88} = 8\sqrt{22}$ = AE half the length of the ſpindle.

PR × RW = 12 × 40 = 480 = the area RPTYW.

And the ſegment PCY, is 107·5185.

The

The ſum of theſe two is 587·5185 = the generating area RPCYW.

Then $[(LL - \frac{1}{3}ll)l - ac] \times 2p =$ $[(64 \times 22 - \frac{1}{3} \times (400 \times 20 - 36 \times 587{\cdot}5185] \times 2 \times 3{\cdot}14159 = 4342{\cdot}7879 \times 6{\cdot}28318 =$ 27286·5411256 the ſolidity in inches.

Then 27286·5411256 ÷ 282 = 96·7608 ale gallons.

## PROBLEM XXIII.

*To find the Content of the Segment of a Circular Spindle.*

From 3 times half the length of the ſpindle ſubtract the height of the ſegment, and multiply the remainder by the ſquare of the ſaid height; from $\frac{1}{3}$ of the product ſubtract double the product of the generating area and central diſtance; then the remainder drawn into 3·14159, will produce the content.

That is, $[(3EA - AR) \times \frac{1}{3}AR^2 - 2FE \times APR] \times p =$ the ſolidity of PAQ.

### EXAMPLE.

If the length of the whole ſpindle be 12, and its greateſt diameter 9; what will be the content of a ſegment of it whoſe height is 1?

$\frac{AE^2}{EC} = \frac{36}{4\frac{1}{2}} = \frac{72}{9} = 8 =$ EK; KE + EC = $8 + 4\frac{1}{2} = 12\frac{1}{2} =$ diam. and $6\frac{1}{4} =$ the radius of the generating circle; FC − CE = $6\frac{1}{4} - 4\frac{1}{2} = 1\frac{3}{4} =$ EF the central diſtance; $\sqrt{FP^2 - PT^2} = \sqrt{(6\frac{1}{4})^2 - 5^2} = \sqrt{(\frac{5}{4})^2 \times (5^2 - 4^2)} = 3\frac{3}{4} =$ TF, and CF − FT = $6\frac{1}{4} - 3\frac{3}{4} = 2\frac{1}{2} =$ TC.

By the table of circular ſegments, the half ſegment PCT is 8·736234375; to which adding RT = 5 × 2 = 10, makes 18·736234375 = the area RC.

Alſo

Alſo the half ſegment ACE is 19·886763281.

Their difference = 1·150528906 is the generating area APR.

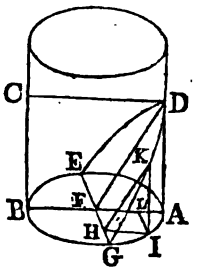
Therefore $[(3\,AE - AR) \times \frac{1}{3} AR^2 - 2\,FE \times APR] \times p = (17 \times \frac{1}{3} - 3\frac{1}{2} \times 1{\cdot}150528906) \times 3{\cdot}14159 = 1{\cdot}639815495 \times 3{\cdot}14159 = 5{\cdot}1516323$, the content required.

## PROBLEM XXIV.

*To find the Curve Surface of the Ungula* DEAGD *of a Cylinder.*

Put $b$ = the height AD,
$v$ = AF the verſ. ſine of AE,
$d$ = the diameter AB,
$a$ = the arc EAG of the baſe,
$s$ = the right ſine FG,
$c$ = the coſine of the $\frac{1}{2}$ arc.

Then $\frac{ds - ac}{v} \times b$ is the convex ſurface.*




That

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* DEMONSTRATION.

For, having drawn HI, IK parallel to FA and AD reſpectively, and joined the points H, K; ſince it is evident that the ſurface is generated by the motion of IK along the arc AIG, KI × the fluxion of IA will be the fluxion of the ſurface. Therefore put $z$ = AI, $x$ = its ſine IL, and $y$ = its coſine; then HI = $y - c$; and, by ſimilar triangles, FA : AD :: HI : IK = $\frac{b}{v} \times (y - c)$; and hence the fluxion of the ſurface, or $\dot{z} \times$ IK is $\frac{b}{v} \times (\dot{y}z - c\dot{z}) = \frac{b}{v} \times (\frac{1}{2}d\dot{x} - c\dot{z})$; the fluent of which is $= \frac{b}{v} \times (\frac{1}{2}dx - cz) =$ (when AI = AG) $\frac{b}{v} \times (\frac{1}{2}ds - \frac{1}{2}ac)$; the double of which is $\frac{b}{v} \times (ds - ac) =$ the whole convex ſurface DEAGD.

*Corol.*

That is, from the product of the diameter and sine, subtract the product of the arc and cosine, and multiply the difference by the height, and divide by the versed sine.

*Note* 1. When F is the center of the base; then $v = s = \frac{1}{2}d$, and $c = 0$; and then the theorem becomes $dh$, viz. the product of the diameter and height equal to the curve surface.

*Note* 2. When AF exceeds $\frac{1}{2}$AB, then $ac$ must be added.

EXAMPLE I.

Given the diameter AB 100, the height AD 140, and the versed sine AF 10: required the curve surface.

Here $d = 100$, $h = 140$, and $v = 10$:
therefore $\frac{1}{2}d - v = 50 - 10 = 40 = c$.
And $\sqrt{\frac{1}{4}dd - cc} = \sqrt{2500 - 1600} = \sqrt{900} = 30 = s$.

But $\frac{s}{\frac{1}{2}d} = \frac{30}{50} = \frac{3}{5} = .6$ is the sine reduced to the radius 1; to which, in a table of sines, belong 36° 52·268′ = 36·87113 degrees.

Then, by rule 1 for the length of a circular arc, ·01745329 × 36·87113 × 100 = 64·352252 is the arc $a$.

Whence $\frac{ds - ac}{v} \times h = (3000 - 2574{\cdot}09008) \times 14$
$= 425{\cdot}90992 \times 14 = 5962{\cdot}73888 =$ the convex surface required.

EX-

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*Corol.* 1. If F be the center; then $v = s = \frac{1}{2}d$, and $c = 0$; and then the theorem becomes barely $dh = 4$ times the triangle FDA.

*Corol.* 2. When AF exceeds $\frac{1}{2}d$, $c$ is negative, and then $- ac$ becomes $+ ac$.

*Corol.* 3. If F coincide with B; then $s = 0$, and $c = -\frac{1}{2}v$; and the theorem becomes $\frac{1}{2}ah =$ the surface of the half cylinder.

EXAMPLE II.

If the diameter and height be 100 and 140, as before, and the ſection be made through the center of the baſe, or $v = \frac{1}{2}d = 50$; what is the convex ſurface?

Here, by note 1, $dh = 100 \times 140 = 14000 =$ the convex ſurface required.

EXAMPLE III.

Suppoſing $d$ and $b$ ſtill the ſame, and $v = 90$; to find the convex ſurface.

Here $\frac{1}{2}d - v = 50 - 90 = -40 = c$, $s = 30$ the ſame as before, but it is here the ſine of the ſupplemental arc, which therefore is $180 - 36{\cdot}87113 = 143{\cdot}12887$ degrees. Hence $\cdot 01745329 \times 143{\cdot}12887 \times 100 = 249{\cdot}807013 =$ the arc $a$. Or the arc may be ſooner found by only ſubtracting the arc in the firſt example, viz. $64{\cdot}352252$, from $314{\cdot}159265$, the whole circumference.

Then, by note 2, $\frac{ds+ac}{v}b = \frac{14}{9}(3000 + 9992{\cdot}28052)$ $= \frac{14}{9} \times 12992{\cdot}28052 = 20210{\cdot}21414$ the convex ſurface required.

PROBLEM XXV.

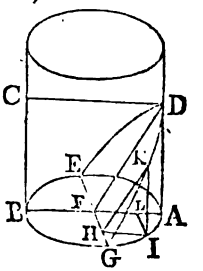
*To find the Solidity of the Hoof of a Cylinder.*

From $\frac{2}{3}$ of the cube of the right ſine, ſubtract the product of the baſe and coſine of half the arc of the baſe; then multiply the difference by the height, and divide by the verſed ſine, the quotient will be the

the ſolidity required. That is, putting, as before

$b$ = the height AD
$v$ = the verſed ſine AF
$s$ = the right ſine FG
$c$ = the coſine = $\frac{1}{2}$AB − AF
$b$ = the baſe, or area of the ſeg. GAEG;

Then $\frac{\frac{2}{3}s^3 - bc}{v} \times b$ = the ſolidity.*




*Note* 1. If F be the center, that is, if the baſe be equal to the ſemicircle, then $v = s$, and $c = 0$; and therefore $\frac{2}{3}bss = \frac{1}{6}ddb$ is the ſolidity in that caſe.

*Note* 2. If $v$ exceed $\frac{1}{2}d$, that is, if the baſe exceed the ſemicircle, then $c$ is negative, and $bc$ muſt be added.

EX-

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* DEMONSTRATION.

For, the fluxion of the ſolid is = the △ HIK drawn into the fluxion of LI, which fluxion will, therefore, be $\dot{x} \times \frac{b}{2v} \times$ HI² (uſing the ſame characters as in the demonſtration of the laſt problem) $= \frac{b\dot{x}}{2v} \times (y - c)^2 = \frac{b\dot{x}}{2v} \times (yy - 2cy + cc) = \frac{b\dot{x}}{2v} \times (\frac{1}{4}dd - xx - 2cy + cc) = \frac{b\dot{x}}{2v} \times (\frac{1}{4}dd - xx - dc + 2c \times \text{AL} + cc) = \frac{b\dot{x}}{2v} \times (ss - xx - 2c \times \text{FL})$; whoſe fluent, $\frac{b\dot{x}}{2v} \times (ss - \frac{1}{3}xx) - \frac{bc}{v} \times$ area FAIH, when I coincides with G, is $\frac{b}{2v} \times (\frac{2}{3}s^3 - bc)$, the double of which is $\frac{b}{v} \times (\frac{2}{3}s^3 - bc)$ = the content of the ſolid DEAGD required.

*Corol.* 1. If F fall in the center of the baſe; then $c = 0$, and $s = v = \frac{1}{2}d$; and the rule will be $\frac{1}{6}ddb$.

*Corol.* 2. If AF exceed FB, $c$ will be negative, and then $-bc$ will become $+bc$.

*Corol.* 3. If F fall in B; $s = 0$, and $c = -\frac{1}{2}v$; and then the theorem becomes $\frac{1}{2}bb$ = half the cylinder.

EXAMPLE I.

If the diameter AB be 50, the height AD 120, and the verſed ſine AF 10; what is the ſolidity of the hoof?

Or ſuppoſing a cylindric veſſel ABCD, containing a fluid, to be placed in ſuch a poſition that the ſurface of the fluid, diſpoſing itſelf parallel to the horizon, may cut the baſe in GE, leaving 40 inches of the diameter dry, and the ſide of the cylinder in D, 120 inches diſtant from the baſe; to find how many ale gallons are in it; the diameter of the baſe being 50 inches.

Here $b = 120$, $d = 50$, and $v = 10$. Then $\frac{1}{2}d - v = 25 - 10 = 15 = c$, and $\sqrt{\frac{1}{4}dd - cc} = \sqrt{25^2 - 15^2} = \sqrt{40 \times 10} = 20 = s$.

And, to find the baſe by the table of ſegments, $\frac{v}{d} = \frac{10}{50} = \cdot 2$; this being found in the column of verſed ſines, oppoſite to it is the area 1118238: hence $50 \times 50 \times \cdot 1118238 = 279 \cdot 5595 = b$ is the ſegment or baſe.

Then $\frac{\frac{2}{3}s^3 - bc}{v} b = 12 \times (\frac{2}{3} \times 8000 - 15 \times 279 \cdot 5595)$ $= 12 \times (5333\frac{1}{3} - 4193 \cdot 3925) = 12 \times 1139 \cdot 9408$ $= 13679 \cdot 2896 =$ the ſolidity in inches; which, divided by 282, the inches in a gallon, give 48·50939 ale gallons for the content.

EXAMPLE II.

Suppoſe the cylinder ſo placed, that the ſurface of the liquor may biſect the baſe, and riſe up the ſide to the ſame diſtance of 120 inches from the baſe: to find the content.

Here, by note 1, we have $\frac{1}{6}ddb = 50 \times 50 \times 20$ $= 50000$ ſolid inches $= 177 \cdot 3049645$ gallons, for the content in this caſe.

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EXAMPLE III.

Suppoſe, now, the ſame veſſel ſo placed, as that the ſurface of the liquor may leave only 10 inches of the diameter dry, ſtill riſing to the ſame diſtance of 120 inches along the ſide; to find the content.

Here the part of the cylinder's baſe left dry, is equal to the baſe in the firſt example, viz. 279·5595 which, therefore, taken from

$50 \times 50 \times .78539816 = 1963.4954$, the whole circle, leaves $1683.9359 = b$, the baſe of the ungula in this example.

Now $v = 40$, $c = -15$, and $s = 20$.

Whence $\frac{\frac{2}{3}s^3 - bc}{v} b = (\frac{2}{3} \times 8000 + 25259.0385) \frac{120}{40}$

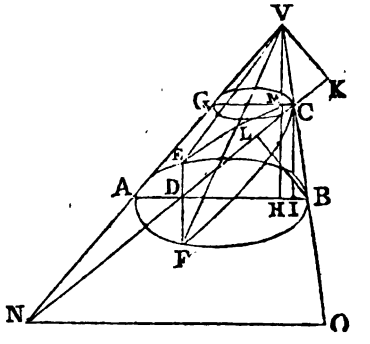
$= 30592.3718 \times 3 = 91777.1154$ ſolid inches $=$ 325·45076 gallons, the content in this caſe.

PROBLEM XXVI.

*To find the Solidity of the Elliptic Hoofs of the Fruſtum of a Cone, made by a Plane cutting Diagonally the Oppoſite Extremities of the Ends.*

* From the ſquare of the diameter of the baſe of the hoof, ſubtract the product of the diameter of the other

* Let AEBF be the baſe of a cone, or of any other pyramid, right, or oblique; AVB a ſection through the vertex by a plane perpendicular to the baſe; and EVF, ECF two other ſections perpendicular to AVB, the former through the vertex, and the latter through the ſide, at C, between V and B. On AB let fall the perpendiculars VH, CI; and on DC the perpendiculars VK, BL; and draw GC parallel to AB, and meeting AV and VH in G and M. Then



other end of the fruſtum, and a mean proportional between the diameters; divide the difference by the difference of the diameters; multiply the quotient by the height of the hoof, and the product by the diameter of its baſe; ſo ſhall the laſt product multiplied by ·2618 give the content of the hoof.—That is, putting

D = AB the diameter of the greater end,
$d$ = GC the leſs diameter, and
$b$ = CI the perpendicular height.

Then $\frac{D^2 - d\sqrt{Dd}}{D-d} \times 0{\cdot}2618\, D b$ = the greater hoof ABC,

And $\frac{D\sqrt{Dd} - d^2}{D-d} \times 0{\cdot}2618\, d b$ = the leſs hoof AGC.

A plane being conceived to be drawn through C and A.—The proof is in corollary 2.

EX-

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Then it is evident that EFBV is a pyramid whoſe baſe is EFB (A), altitude VH ($a$); and therefore its content is equal to $\frac{1}{3} A a$; and that EFCV is a pyramid whoſe baſe is EFC (B), height VK ($b$), and therefore its content equal to $\frac{1}{3} B b$; and moreover that the difference of theſe pyramids, or $\frac{1}{3} A a - \frac{1}{3} B b$ is the content of the hoof EFBC.

But, by the ſimilar triangles ABV, GVC, it is

AB − CG : CI or HV − VM :: AB : HV, or $a = \frac{AB \times CI}{AB - GC}$;

alſo AB − GC : CI :: AB : HV :: GC : VM $= \frac{GC \times CI}{AB - GC}$;

and DC : BD :: (by the ſimilar triangles ICD, DBL) CI : BL :: (becauſe of the ſimilar triangles BCI and CVM, VKC and CLB) VM $(= \frac{GC \times CI}{AB - GC})$ : VK $(b) = \frac{GC \times CI \times DB}{DC \times (AB - GC)}$.

Wherefore the hoof EFBC will be

$= \frac{\frac{1}{3} CI}{AB - GC} \times (A \times AB - B \times \frac{GC \times DB}{DC}$; which is a general theorem for the hoof of any pyramid.

*Corol.* 1. If the baſe be circular, or the pyramid a cone, and the angle CDB be leſs than the angle VA; or, which is the ſame, if CD and VA, produced, interſect in N; the ſection ECF will

EXAMPLE I.

If a conical veſſel, whoſe bottom diameter is 30 inches, be inclined to the horizon till the liquor in it juſt cover the bottom, and its ſurface, diſpoſing itſelf into the poſition AC, cut the ſide BC of the veſſel in c, at the diſtance of 18 inches from the bottom AB; how many wine gallons of liquor are in it, ſuppoſing the diameter GC of the veſſel at c to be 19·2 inches?

Here D = 30, $d$ = 19·2, and $h$ = 18.

Whence $\frac{D^2 - d\sqrt{Dd}}{D-d} \times \cdot 2618\, D h = \frac{30^2 - 19{\cdot}2\sqrt{30 \times 19{\cdot}2}}{30 - 19{\cdot}2} \times 30 \times 18 \times \cdot 2618 = \frac{439\frac{1}{5} \times 30 \times 18 \times \cdot 2618}{10\frac{4}{5}} = \frac{2196 \times 30 \times 18 \times \cdot 2618}{54} = 2196 \times \cdot 2618 \times 10 = 5749{\cdot}128$ inches. Which being divided by 231, thus,

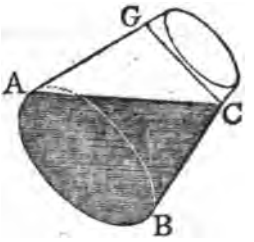
$231 = 3 \times 7 \times 11$

3)5749·128

7)1916·376

11) 273·768

gives 24 888 wine gallons.

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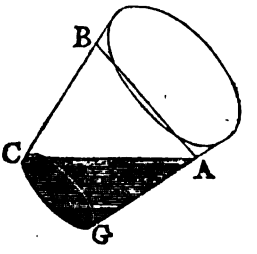
will be a ſegment of an ellipſe whoſe tranſverſe axe is CN, and conjugate $\sqrt{NO \times GC}$, NO being drawn parallel to AB, and meeting VB produced in o. And then the above general theorem will become $\frac{\frac{1}{3}CI}{AB - GC} \times$ (AB × circular ſegment EBF $- \frac{GC \times DB}{DC} \times$ elliptic ſegment ECF) =, by prob. 6 ſect. 3 part 3, $\frac{\frac{1}{3}CI}{AB - GC} \times$ (AB × cir. ſeg. EBF $- \frac{GC \times DB \times \sqrt{NO \times GC}}{DC \times CN}$ × cir. ſeg. whoſe diameter is CN and height CD) =, ſince ſimilar ſegments

EXAMPLE II.

Let there be taken here the ſame dimenſions as in the laſt example, ſuppoſing the veſſel to be narroweſt at the bottom, to find how many ale gallons are in it.




Here $\frac{D\sqrt{Dd-dd}}{D-d} \times \cdot 2618\, db =$

$\frac{30\sqrt{30 \times 19\frac{1}{2} - 19\cdot 2^2}}{30 - 19\frac{1}{2}}$

$\times \cdot 2618 \times 19\frac{1}{2} \times 18 =$

$\frac{351\frac{9}{21} \times 19\frac{1}{2} \times 18 \times \cdot 2618}{10\frac{1}{2}} =$

$\frac{351\frac{9}{21} \times 96 \times 18 \times \cdot 2618}{54} =$

$117\frac{3}{21} \times 96 \times \cdot 2618 = 117\cdot 12 \times 96 \times \cdot 2618 =$ 2943·55 inches. Which being divided by 282, give 10·43, for the number of ale gallons required.

PRO-

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ſegments are as the ſquares of their diameters, $\frac{\frac{1}{2}CI}{AB - GC} \times$ (AE × circular ſegment EBF $- \frac{GC \times DB \times CN \times \sqrt{NO \times GC}}{DC \times AB^2} \times$ ſeg. of the cir. AEBF whoſe height is $\frac{AB \times DC}{CN}$) = the content of the elliptic hoof EFCB.

But, by ſim. triangles, GC — AD : DC :: GC : CN $= \frac{GC \times CD}{GC - AD}$, and GC — AD : DB :: GC : NO $= \frac{GC \times DB}{GC - AD}$; which values of NO and NC being ſubſtituted inſtead of them, in the above expreſſion of the elliptic hoof, will give

$\frac{\frac{1}{2}CI}{AB - GC} \times$ [AB × cir. ſeg. EBF $- \frac{GC^3}{AB^2} \times \left(\frac{DB}{GC - AD}\right)^{\frac{3}{2}} \times$ ſeg. of the cir. AEBF whoſe height is $\frac{AB \times (GC - AD)}{GC}$] $= \frac{\frac{1}{2}b}{D-d} \times$ [D × circ. ſeg. EBF $- \frac{d^2}{D^2} \times \left(\frac{DB}{BD - D + d}\right)^{\frac{3}{2}}$ × ſegment of the circle AB whoſe height is $\frac{D \times (DB - D + d)}{d}$]

=

## PROBLEM XXVII.

*To find the Solidity of the Elliptic Hoofs of the Fruſtum of a Cone made by a Plane cutting off a Part of the Baſe.*

1. From the verſed ſine, or height of the baſe of the hoof, ſubtract the difference of the diameters of the baſe and end of the fruſtum, and divide the remainder by the diameter of the ſaid end; find the tabular ſegment whoſe verſed ſine is equal to the quotient; find alſo the tabular ſegment whoſe verſed ſine is expreſſed by the quotient of the verſed ſine of the baſe of the hoof divided by the diameter of the baſe of the fruſtum; multiply the former ſegment by the cube of the diameter of the end, and by the quotient of the verſed ſine of the baſe of the hoof divided by the difference between the ſaid verſed ſine and the difference of the diameters, and by the root of the ſaid quotient; and multiply the latter



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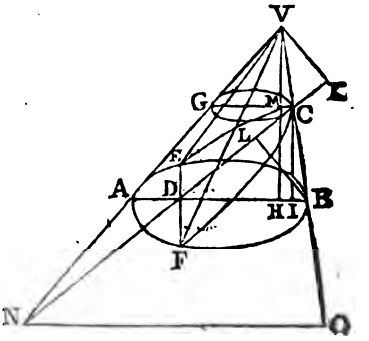
= the elliptic hoof EFCB; putting $b$ for the height of the hoof, and D and $d$ for the diameters of the baſe and end, or top, of the fruſtum, reſpectively.

Or if we would reduce the circular ſegments in the laſt expreſſion to thoſe of a circle whoſe diameter is 1, to be found in the table of circular ſegments at the end of the book, that expreſſion will become $\frac{\frac{1}{4}b}{D-d} \times [D^3 \times$ tab. ſeg. whoſe height is $\frac{BD}{D} - d^3 \times (\frac{BD}{BD-D+d})^{\frac{3}{2}} \times$ tab. ſeg. whoſe height is $\frac{BD-D+d}{d}]$ = the elliptic hoof EFCB. And this is the rule in prob. 27.




And

ſegment by the cube of the greater diameter; multiply the difference of theſe two products by $\frac{1}{3}$ of the height of the hoof, and the product divided by the difference of the diameters will give the content of the hoof required.

2. Subtract

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And if this value of the hoof be taken from $\frac{1}{3}bn \times \frac{D^3 - d^3}{D - d}$ = that of the whole conic fruſtum, the remainder $\frac{\frac{1}{3}b}{D - d} \times : [D^3 - d^3) \times n - D^3 \times$ tab. ſeg. whoſe height is $\frac{BD}{D} + d^3 \times (\frac{BD}{BD - D + d})^{\frac{3}{2}} \times$ tabular ſeg. whoſe height is $\frac{BD - D + d}{d}]$ or $\frac{\frac{1}{3}b}{D - d} \times [-nd^3 + D^3 \times$ tab. ſeg. whoſe height is $\frac{AD}{D} + d^3 \times (\frac{BD}{BD - D + d})^{\frac{3}{2}} \times$ tabular ſeg. whoſe height is $\frac{BD - D + d}{d}]$ will expreſs the meaſure of the complemental elliptic hoof EFCGA.

*Corol.* 2. If the points D and A coincide, the ſection EFC will be a whole ellipſe, and the rules in corollary 1 will become $\frac{1}{3}Dbn \times \frac{DD - d\sqrt{Dd}}{D - d}$ = the elliptic hoof ACB, and $\frac{1}{3}Dbn \times \frac{D\sqrt{Dd} - dd}{D - d}$ = the complemental elliptic hoof ACG. And theſe are the rules in prob. 26.

*Corol.* 3. Since $\frac{b}{D - d} = \frac{H}{D}$, denoting the height of the whole cone by H, the laſt theor. will become $\frac{1}{3}nH \times (D^2 - d\sqrt{Dd}) =$ ACB, and $\frac{1}{3}nH \times (D\sqrt{Dd} - dd) =$ ACG.

*Coro'.* 4. But $\frac{1}{3}nHD^2 =$ the whole cone VAB, and therefore $\frac{1}{3}nHD^2 - \frac{1}{3}nH \times (D^2 - d\sqrt{Dd}) = \frac{1}{3}nHd\sqrt{Dd} =$ the top part VAC. Which top part, conſequently, is to the whole cone VAB, as $\frac{1}{3}nHd\sqrt{Dd}$ to $\frac{1}{3}nHD^2$, or as $d^{\frac{3}{2}}$ to $D^{\frac{3}{2}}$; or the ſquare of the whole cone VAB is to the ſquare of the top part VAC, as $D^3$ to $d^3$.

*Corol.*

2. Subtract the meaſure of the hoof, above found, from that of the whole fruſtum, and the remainder will be the meaſure of the complemental hoof.

That is, if D and $d$ be the extreme diameters of the fruſtum, and $b$ its height; alſo P = the tabular ſegment whoſe verſed ſine is $\frac{BD}{D}$, Q = that whoſe verſed ſine is $\frac{BD-D+d}{d}$, and $n = .785398$ &c.

Then $(P \times D^3 - Q \times d^3 \times \frac{BD}{BD-D+d}\sqrt{\frac{BD}{ED-D+d}}) \times \frac{\frac{1}{3}b}{D-d}$ = the content of the elliptic hoof EFCB.

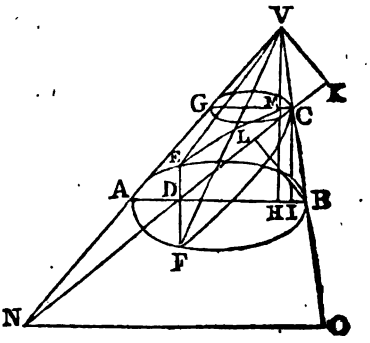
And

$[n(D^3 - d^3) - P \times D^3 + Q \times d^3 \times \frac{BD}{BD-D+d}\sqrt{\frac{BD}{BD-D+d}}] \times \frac{\frac{1}{3}b}{D-d}$ = to that of the complemental hoof EFCGA.

Theſe are proved in corollary 1 to prob. 26.



*Corol.* 5. If the angle CDB be equal to the angle VAB, or if the ſection ECF be parallel to the ſide AG of the cone, it will be a parabola whoſe axe is CD baſe $EF = 2\sqrt{AD \times DB} = 2\sqrt{(D-d) \times d}$, and its area, by prob. 5 ſect 4 part 3, $= \frac{2}{3}DC \times EF = \frac{4}{3}DC\sqrt{(D-d) \times d}$; and therefore the general theorem will become




$\frac{\frac{1}{3}b}{D-d} \times (D \times \text{cir. ſeg. } EBF - \frac{d(D-d)}{DC} \times \frac{4}{3}DC\sqrt{(D-d) \times d})$

$= \frac{1}{3}b \times (\frac{D^3}{D-d} \times \text{tab. ſeg. whoſe height is } \frac{D-d}{D} - \frac{4}{3}d\sqrt{(D-d) \times d})$

= the parabolic hoof EFBC.

Which

EXAMPLE I.

If a veſſel in the form of the fruſtum of a cone, whoſe bottom diameter is 30 inches, be inclined to the horizon till the ſurface of the liquor in it cut the bottom, leaving 10 inches of its diameter dry, and meeting the ſide in c at the diſtance c 1 of 18 inches from the baſe: how many Wincheſter gallons are in it, ſuppoſing the diameter G c of the veſſel at the top of the liquor to be $19\frac{1}{5}$ inches?

Here

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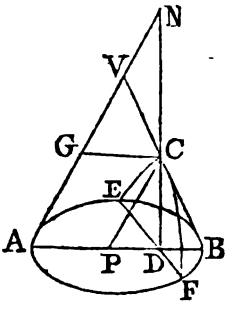
Which being taken from $\frac{1}{3}bn \times \frac{D^3 - d^3}{D - d}$ = the meaſure of the whole fruſtum ABCG, the remainder $\frac{1}{3}b \times [\frac{4}{3}d\sqrt{(D-d)d} - \frac{nd^3}{D-d} + \frac{D^3}{D-d} \times$ tab. ſeg. whoſe height is $\frac{d}{D}$], or $\frac{\frac{1}{3}b}{D-d} \times [(D-d)\frac{4}{3}d\sqrt{d(D-d)} - nd^3 + D^3 \times$ tab. ſegment whoſe height is $\frac{D}{d}$,] will expreſs the meaſure of the complemental parabolic hoof EFCGA.




*Corol.* 6. If the ∠CDB exceed the ∠VAB; or, which is the ſame thing, if DC and AV, produced, interſect above V, as in N; the ſection ECF will be an hyperbola, whoſe tranſverſe axe is CN. But in the hyperbola, as well as in the other conic ſections, the tranſverſe axe $CN = \frac{GC \times CD}{PD}$, CP being drawn parallel to VA, or $= \frac{d \times CD}{D - d - BD}$, and the conjugate of each alſo $= GC\sqrt{\frac{DB}{PD}} = d\sqrt{\frac{DB}{D-d-DB}}$; and the area of the hyperbolic ſection being found by prob. 6 ſect. 5 part 3, and ſubſtituted in the general theorem, will give the ſolidity of the hyperbolic ungula.

*Corol.* 7. If D coincide with 1, or CDB be a right angle, the tranſverſe and conjugate axes of the hyperbolic ſection will become $\frac{2db}{D-d}$ and $d$ reſpectively.

Here $D = 30$, $d = 19\frac{1}{5}$, $b = 18$, and $BD = 30 - 10 = 20$.

Whence $\frac{BD}{D} = \frac{20}{30} = \frac{2}{3} = \cdot 6666\frac{2}{3}$, the tab. verf. the tabular area for which is $\cdot 55622573$; and $\frac{BD-D+d}{d} = \frac{20-30+19\frac{1}{5}}{19\frac{1}{5}} = \frac{46}{96} = \frac{23}{48} = \frac{2\cdot 875}{6} = \cdot 479\frac{2}{3}$, the tabular area anſwering to which is $\cdot 37187178$.

Again, $D^3 = 27000$, which multiplied by $\cdot 55622573$, the former area, produces $15018\cdot 09471$.

And $d^3 \times \frac{BD}{BD-D+d} \times \sqrt{\frac{BD}{BD-D+d}} = (19\frac{1}{5})^3 \times \frac{20}{9\frac{1}{5}} \times \sqrt{\frac{20}{9\frac{1}{5}}} = 19\frac{1}{5} \times (\frac{19\frac{1}{5}}{9\frac{1}{5}})^2 \times 20\sqrt{184} = \frac{19\frac{1}{5} \times 48^2 \times 40}{23 \times 23} \times \sqrt{46} = 22686\cdot 470698$, which multiplied by $\cdot 37187178$, the latter area, gives $8436\cdot 45824$.

Then $6581\cdot 63647$, the difference of theſe two products, being multiplied by $\frac{6}{10\frac{4}{5}} = \frac{30}{54} = \frac{5}{9} = \frac{\frac{1}{3}b}{D-d}$, the quotient of $\frac{1}{3}$ of the height divided by the difference of the diameters, will produce $3656\cdot 4647$ for the ſolidity in inches: which being divided by $268\frac{4}{5}$, the inches in a corn gallon, give $13\cdot 6029$ corn or Wincheſter gallons, for the quantity of liquor in the veſſel.

### EXAMPLE II.

If a veſſel, in form of the fruſtum of a cone, cloſe at both ends, be placed in ſuch a poſition, that the liquor may juſt cover the leſs end, and 10 inches of the diameter of the greater; what number of wine gallons are in it, ſuppoſing the diameters of the two ends to be 30 and $19\frac{1}{5}$ inches, and their diſtance 18 inches?

Here the part filled is the complemental hoof to that in the laſt example. Therefore if from $(D^2 + (D + d)d) \times \frac{1}{3}nb = (30^2 + 19\frac{1}{2} \times 49\frac{1}{2}) \times 6 \times$ ·785398 = 8692·661, the content of the whole fruſtum, be taken 3656·4647, that of the hoof in the laſt example, the remainder 5036·1963 will be the content, in inches, of the complemental hoof; which being divided by 231, give 21·8018 wine gallons for the content required.

### PROBLEM XXVIII.

*To find the Parabolic Hoofs of the Fruſtum of a Cone; that is, of the Hoofs made by a Plane paſſing Parallel to the Side of the Cone.*

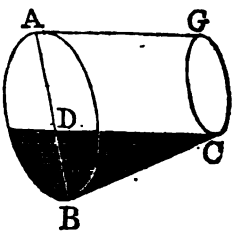
Multiply the baſe of the hoof by the greater diameter of the fruſtum, and divide the product by the difference of the diameters; from the quotient ſubtract $\frac{4}{3}$ of the product ariſing from the multiplication of the leſs diameter by the ſquare root of the product of the leſs diameter and difference of the diameters; then the remainder multiplied by $\frac{1}{3}$ of the height will be the content of the hoof required.

That is, $(\frac{A \times D}{D - d} - \frac{4}{3}d\sqrt{(D - d)d}) \times \frac{1}{3}b$ = the parabolic hoof EFCB; where D and $d$ are the diameters AB and GC, of the fruſtum, $b$ the height CI, and A the baſe EBF of the hoof. [*See the fig. page* 229.]

### EXAMPLE.

If a veſſel, in the form of the fruſtum of a cone, be laid with its upper ſide parallel to the horizon, and if the greateſt diſtance of the ſurface of the liquor from the bottom, whoſe diameter is 30 inches, be 18 inches; how many ale gallons of liquor are in it, ſuppoſing the diameter of the veſſel, at the greateſt

greateſt diſtance of the ſurface of the liquor from the bottom, to be $19\frac{1}{5}$ inches.




Here the dimenſions are the ſame as in the laſt problems; and BD $= D - d = 30 - 19\frac{1}{5} = 10\frac{4}{5} = 10{\cdot}8$ the verſed ſine, or height, of the baſe; which divided by the diameter 30, gives $\cdot 36$ = the tabular verſed ſine; the area anſwering to which is $\cdot 25455056$; which multiplied by 900, the ſquare of the diameter, gives 229·095504 for the circular ſegment, or baſe of the hoof. Then $229{\cdot}0955 \times \frac{30}{10{\cdot}8} = \frac{2290{\cdot}955}{3{\cdot}6} = \frac{3818{\cdot}25834}{6} = 636{\cdot}37639$.

Again $\frac{4}{3}d\sqrt{(D-d)\times d} = \frac{4}{3} \times 19\frac{1}{5}\sqrt{10\frac{4}{5} \times 19\frac{1}{5}}$ $= \frac{4}{3} \times 19{\cdot}2 \times \sqrt{\frac{54}{5} \times \frac{96}{5}} = \frac{4}{3} \times 19{\cdot}2 \times 14{\cdot}4 =$ $4 \times 6{\cdot}4 \times 14{\cdot}4 = 368{\cdot}64$.

Then $(636{\cdot}37639 - 368{\cdot}64) \times \frac{1}{3} = 267{\cdot}73639 \times 6$ $= 1606{\cdot}41834$ inches, the ſolidity; which being divided by 282, give 5·696526 ale gallons, for the quantity of liquor required.

*Note* 1. If from 8692·661, the content of the whole fruſtum, found in the ſecond example of the laſt problem, be taken 1606·418, the content of the hoof, above found, the remainder 7086·243 will be that of the complemental parabolic hoof DAGC.

*Note* 2. If the ſection ECF be an hyperbola, the hoofs may then be found after the manner deſcribed in the ſixth corollary to prob. 27.

BRO-

### PROBLEM XXIX.

*To find the Convex Surface of the Elliptic Hooſs of the Fruſtum of a Cone, made by a Plane cutting the Oppoſite Extremities of the Ends.**

To four times the ſquare of the height of the fruſtum, add the ſquare of the difference of the diameters of the baſe and end; divide the ſquare root of the ſum, by the difference of the ſaid diameters; and call the quotient Q.

Multiply

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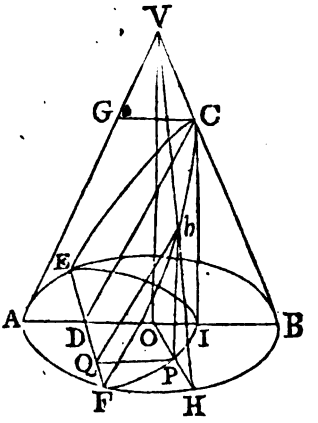
* Let the right-angled triangle VOH be conceived to revolve about the axe VO, the part H*h* of the hypotenuſe deſcribing the ſurface of the ungula, while the part HP of the radius of the baſe of the cone, cut off by the perpendicular *h*P, deſcribes the ſpace FPIEBF.




Then ſince, by the ſimilar triangles, VOH, *h*PH, it is everywhere as HP : H*h* :: HO or OB : HV or BV, the ſpaces generated by theſe lines will be, likewiſe, in the ſame ratio, viz. OB : BV :: the area FIEBF : the ſurface FCEBS $= \frac{BV}{BO} \times$ FIEBF.

And in the ſame manner, by reaſon of the ſimilar triangles DIC, Q P *h*, Q P being drawn parallel to DB, we have DC : DI :: FCEF = B = the ſpace deſcribed by Q *h*, : FIEF the ſpace deſcribed in the ſame time by Q P $= \frac{DI}{DC} \times$ FCEF $= \frac{DI}{DC} \times$ B.

But the ſpace FIEBF = the circular ſeg. FBEF or A − FIEF $= A - \frac{DI}{DC} \times B$. And conſequently

$S = \frac{VB}{OB} \times$ FIEBF $= \frac{VE}{OB} \times (A - \frac{DI}{DC} \times B) = \frac{CB}{IB} \times (A - \frac{DI}{DC} \times B)$ = the convex ſurface of the ungula EFBC.

And

Multiply half the ſum of the ſaid diameters by the ſquare root of their product; and call the product P. Then

1. Q multiplied by P and by ·785398, will give the convex ſurface of the oblique cone ACV.

That is, $\frac{n\sqrt{4b^2+(D-d)^2}}{D-d} \times \frac{D+d}{2}\sqrt{Dd} =$ the curve ſurface of ACV.

2. Q multiplied by the difference between P and the ſquare of the diameter of the baſe, and by ·785398, will give that of the hoof, ABC, including the baſe.

That is, $\frac{n\sqrt{4b^2+(D-d)^2}}{D-d} \times (DD - \frac{D+d}{2}\sqrt{Dd}) =$ the curve ſurface of ABC.

3. Q

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And the method of proceeding will be the ſame for any other kind of pyramid.

*Corol.* 1. If from $\frac{VB}{OB} \times$ the circle AE, the convex ſurface of the whole cone, be taken that of the ungula, above found, the remainder $\frac{VB}{OB} \times$ (FAEF + $\frac{DI}{DC} \times$ FCEF), will expreſs the convex ſurface of the remaining part EFCVA of the cone.

*Corol.* 2. And if from the value of the part laſt found be taken $\frac{VB}{OB} \times$ circle GC $= \frac{VC}{OI} \times$ circle GC, the convex ſurface of the top cone GVC, the remainder $\frac{VB}{OB} \times$ (FAEF + $\frac{DI}{DC} \times$ FCEF − circle GC), will expreſs that of the complemental ungula EFAGC.

*Corol.* 3. If the ∠CDB be leſs than the ∠GAB, or the ſect. FCE be an ellipſe, the ſurface of the ungula will be $\frac{VB}{OB} \times$ (cir. ſeg. FBE − $\frac{DI}{DC} \times$ ellipſe ſeg. FCE) = by proceeding

as

3. Q multiplied by the difference between P and the ſquare of the top diameter, and by ·785398, will give the ſurface of the hoof ACG including the top.

That is, $\frac{n\sqrt{4b^2+(D-d)^2}}{D-d} \times (\frac{D+d}{2}\sqrt{Dd-dd} =$ the curve ſurface of ACG.

Where the letters denote the ſame Quantities as in the foregoing problems.—All theſe are proved in corollary 4.

EX-

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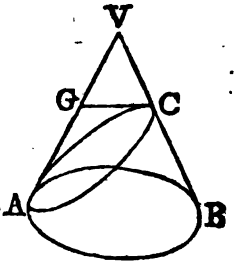
as in cor. 1 prob. 26, $\frac{VB}{OB} \times (EBF - \frac{GC^2 \times DI}{AB^2 \times (GC - AD)}\sqrt{\frac{DB}{GC-AD}}$ $\times$ ſeg. of the circle AB whoſe height is $AB \times \frac{GC-AD}{GC}) =$ $\frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times (EBF - \frac{dd}{DD} \times \frac{DB-\frac{1}{2}(D-d)}{DB-(D-d)}\sqrt{\frac{DB}{DB-(D-d)}}$ $\times$ ſeg. of the circle AB whoſe height is $D \times \frac{DB-(D-d)}{d}) =$ $\frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times (D^2 \times$ tabular ſegment, whoſe height is $\frac{DB}{D} - d^2 \times \frac{DB-\frac{1}{2}(D-d)}{BD-(D-d)}\sqrt{\frac{DB}{DB-(D-d)}} \times$ tab. ſeg. whoſe height is $\frac{DB-(D-d)}{d}$), where $b$ is the height, and D and $d$ the diameters of the baſe and top of the fruſtum.

And the value of the ſurface of the remaining part EFCVA, in corollary 1, will become $\frac{VB}{OB} \times$ (cir. ſeg. FAE $+ \frac{DI}{DC} \times$ ellip. ſeg. FCE) $= \frac{VB}{OB} \times (FAE + \frac{GC^2 \times DI}{AB^2\,(GC-AD)}\sqrt{\frac{DB}{GC-AD}} \times$ ſeg. of the circle AB whoſe height is $AB \times \frac{GC-AD}{GC}) = \frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times$ $(FAE + \frac{dd}{DD} \times \frac{DB-\frac{1}{2}(D-d)}{DB-(D-d)}\sqrt{\frac{DB}{DB-(D-d)}} \times$ ſeg. of the cir.

EXAMPLE.

Required the convex surfaces of the two hoofs of a conic frustum, whose base and top diameters are 30 and $19\frac{1}{5}$, and height 18 inches; the section being made through the contrary extremities of the diameters: together with that of the oblique cone ACV.




Here $D = 30$, $d = 19\frac{1}{5}$, and $b = 18$.

Then $Q = \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} = \frac{\sqrt{4 \times 18^2 + 10{\cdot}8^2}}{10{\cdot}8} = \frac{\sqrt{4^2 + 1{\cdot}2^2}}{1{\cdot}2} = \frac{1}{3}\sqrt{10^2 + 3^2} = \frac{1}{3}\sqrt{109} = 3{\cdot}48010217$.

$P = \frac{D+d}{2}\sqrt{Dd} = 24\frac{3}{5}\sqrt{30 \times 19\frac{1}{5}} = 24\frac{3}{5} \times 24 = 590\frac{2}{5}$.——And then



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cir. AB whose height is $D \times \frac{DB - (D-d)}{d}) = \frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times$ ($D^2 \times$ tab. seg. whose height is $\frac{AD}{D} + d^2 \times \frac{DB - \frac{1}{2}(D-d)}{DB-(D-d)}\sqrt{\frac{DB}{DB-(D-d)}} \times$ tab. seg. whose height is $\frac{DB-(D-d)}{d}$). Or it is $= \frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times (FAE + \frac{dd}{DD} \times \frac{\frac{1}{2}(D+d) - AD}{d-AD}\sqrt{\frac{D-AD}{d-AD}} \times$ segment of the circle AB whose height is $D \times \frac{d-AD}{d}) =$ $\frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times$ ($D^2 \times$ tabular segment whose height is $\frac{AD}{D} + d^2 \times \frac{\frac{1}{2}(D+d) - AD}{d-AD}\sqrt{\frac{D-AD}{d-AD}} \times$ tab. seg. whose height is $\frac{d-AD}{d}$).

Also

1. $Q \times P \times .785398 = 2.73326528 \times 590.4 = 1613.7198213$ inches $= 11.2063876$ feet, the convex ſurface of the oblique cone ACV.

2. $Q \times .785398 \times (D^2 - P) = .785398 \times 3.48010217 \times (900 - 590.4) = 2.73326528 \times 309.6 = 846.2189307$ inches $= 5.87652$ feet, that of the hoof ABC.

3. $Q \times .785398 \times (P - d^2) = .785398 \times 3.48010217 \times (509.4 - 368.64) = 2.73326528 \times 221.76 = 606.1289085$ inches $= 4.2092285$ feet, that of the complemental hoof ACG.

PRO-

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Alſo that of the complemental hoof in cor. 2, will become $\frac{VB}{OB}$ (cir. ſeg. FAE $+ \frac{DI}{DC} \times$ ellip. ſeg. FCE $-$ the cir. CG) $=$ $\frac{VB}{OB} \times (-n \times GC^2 + FAE + \frac{GC^2 \times DI}{AB^2 \times (GC - AD)}) \sqrt{\frac{DB}{GC - AD}} \times$ ſegment of the circle AB whoſe height is AB $\times \frac{GC - AD}{GC}) =$ $\frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times (-ndd + FAE + \frac{dd}{DD} \times \frac{\frac{1}{2}(D+d) - AD}{d - AD}$ $\sqrt{\frac{D - AD}{d - AD}} \times$ ſeg. of the circle AB whoſe height is $D \times \frac{d - AD}{d}) =$ $\frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times (-nd^2 + D^2 \times$ tab. ſeg. whoſe height is $\frac{AD}{D} + d^2 \times \frac{\frac{1}{2}(D+d) - AD}{d - AD} \sqrt{\frac{D - AD}{d - AD}} \times$ tab. ſeg. whoſe height is $\frac{d - AD}{d})$.

*Corol.* 4. If D coincide with A, the rules in the laſt corollary will become $\frac{VB}{BO} \times (AB^2 \times n - n \times AB \times \sqrt{AI \times GC} =$ $\frac{n\sqrt{4b^2 + (D-d)^2}}{D-d} \times (D^2 - \frac{D+d}{2}\sqrt{Dd})$ for the convex ſurface of the ungula ABC.

And

PROBLEM XXX.

*To find the Convex Surfaces of the Elliptic Ungulas of a Cone, made by a Plane cutting off a Part of the Base.*

Multiply continually together the four following quantities, viz. the quotient arising from the division of

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And $\frac{VB}{OB} \times AI\sqrt{AB \times GC} \times n = 2n \times VB \times AI\sqrt{\frac{GC}{AB}} = \frac{n\sqrt{4b^2 + (D-d)^2}}{D-d} \times \frac{D+d}{2}\sqrt{Dd}$ for that of the oblique cone ACV.

Also $\frac{VB \times n}{OB} \times (AI\sqrt{AB \times GC - GC^2}) = \frac{n\sqrt{4b^2 + (D-d)^2}}{D-d} \times (\frac{D+d}{2}\sqrt{Dd-d^2})$ for that of the complemental elliptic hoof ACG.

*Corol.* 5. If DC be parallel to AV, or the section a parabola; since its area B is $\frac{4}{3}DC \times DF = \frac{4}{3}DC\sqrt{AD \times DB}$, the general theor. for the ungula will become

$\frac{VB}{OB} \times (\text{seg. FBE} - \frac{4}{3}DI\sqrt{AD \times DB}) = \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ [seg. FBE to height $DB - \frac{2}{5}(D-d)\sqrt{d(D-d)}$] for the convex surface of the parabolic ungula FEBC.

And the rule in cor. 1 will be

$\frac{VB}{OB} \times (\text{seg. FAE} + \frac{4}{3}DI\sqrt{AD \times DB}) = \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ [(seg. FAE to height $AE + \frac{2}{5}(D-d)\sqrt{d(D-d)}$] for that of the part AEFCV.

Also that in cor. 2 will be $\frac{VB}{OB} \times (\text{seg. FAE} + \frac{4}{3}DI\sqrt{AD \times DB - AD^2} \times n) = \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ (segment FAE to height $AD + \frac{2}{5}(D-d)\sqrt{d(D-d)} - ndd$), for that of the complemental parabolic ungula FAECG.

*Corol.*

of the ſquare of the leſs diameter by that of the greater; the quotient ariſing from the diviſion of the difference between the part of the diameter of the baſe not cut off and half the ſum of the diameters, by the difference between the ſaid part of the diameter of the baſe and the leſs diameter; the ſquare root of the quotient ariſing from the diviſion of the part cut off the diameter of the baſe, by the difference between the part not cut off and the leſs diameter; and ſuch a ſegment of the baſe of the fruſtum, whoſe height is equal to the product ariſing from the multiplication

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*Corol.* 6. If the angle CDB be greater than the angle VAB, or the ſection be an hyperbola, its area being found, and ſubſtituted for B in the general rules, will give the ſurfaces of the hyperbolic ungulas.

*Corol.* 7. If the hyperbolic ſection be perpendicular to the baſe, DI will vaniſh, and the general theorems will become $\frac{VB}{OB} \times$ ſeg. FBE $= \frac{CB}{IB} \times$ ſeg. FBE its height being IB $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ ſeg. of the circle AB, whoſe height is $\frac{D-d}{2}$, for the curve ſurface of the perpendicular ungula CIB.

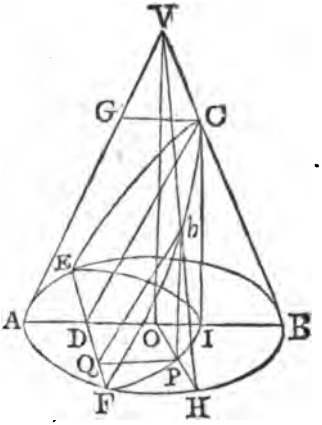
Alſo $\frac{VB}{OB} \times$ ſegment FAE $= \frac{CB}{IB} \times$ ſegment whoſe height is AI $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ ſeg. of the cir. AB whoſe height is $\frac{D+d}{2}$, for that of the remaining part AICV.

And $\frac{VB}{OB} \times$ ſeg. FAE $-$ cir. CG $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ (ſeg. of the circle AB whoſe height is $\frac{D+d}{2} - nd^2$), for that of the complemental perpendicular ungula AICG.

tion of the quotient of the greater diameter divided by the leſs, into the difference between the leſs diameter and the part of the baſe diameter not cut off: and call the laſt product R.—Then

1. The difference between R and the baſe of the hoof, multiplied by the quantity Q, in the laſt problem, will give the convex ſurface of the ungula required.

That is, $\frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times$ (ſeg. FBE $-\frac{dd}{DD} \times \frac{\frac{1}{2}(D+d)-AD}{d-AD}\sqrt{\frac{DB}{d-AD}}$ × ſeg. of the circle AB whoſe height is $D \times \frac{d-AD}{d}$) = the convex ſurface of the hoof FEBC. Uſing ſtill the ſame letters as in the former problems.




2. The ſum of R and the remaining ſegment of the baſe, not included by the hoof above, multiplied by Q, will give the ſurface of the remaining part of the whole cone.

That is, $\frac{\sqrt{4b^2+(D-d)^2}}{D-d} \times$ (ſeg. FAE $+ \frac{dd}{DD} \times \frac{\frac{1}{2}(D+d)-AD}{d-AD}\sqrt{\frac{DB}{d-AD}}$ × ſegment of the circle AB whoſe height is $D \times \frac{d-AD}{d}$) = the convex ſurface of the part FEAVC.

3. From the ſurface of the part in the laſt article, ſubtract that of the little cone at the top, viz. the product of Q by the area of the top of the fruſtum; and the remainder will be the ſurface of the complemental ungula.

That is, $\frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times (-ndd + \text{ſeg. FAE} + \frac{dd}{DD} \times \frac{\frac{1}{2}(D+d) - AD}{d - AD}\sqrt{\frac{DB}{d-AD}} \times$ ſeg. of the circle AB whoſe height is $D \times \frac{d-AD}{d}$) = the convex ſurface of the complemental hoof FEAGC.—All theſe are proved in corollary 3 problem 29.

EXAMPLE.

Required the convex ſurfaces of the parts into which a cone is cut by a plane which enters the ſide at the diſtance of 18 inches from the baſe, and cuts off 20 inches of the diameter of the baſe; that diameter being 30, and the diameter at the top of the ſection $19\frac{1}{3}$ inches.

Here $D = 30$, $d = 19\frac{1}{3}$, $b = 18$, $BD = 20$, and $AD = 10$.

Whence $Q = \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} = \frac{1}{3}\sqrt{109} = 3{\cdot}48010217$.

The ſeg. FBE $= 30^2 \times {\cdot}55622573 = 500{\cdot}603157$.

The ſeg. FAE $= 30^2 \times {\cdot}22917243 = 206{\cdot}255187$.

$R = (\frac{19\frac{1}{3}}{30})^2 \times \frac{14\frac{2}{3}}{9\frac{1}{3}}\sqrt{\frac{20}{9\frac{1}{3}}} \times$ ſeg. whoſe diameter is 30 and height $\frac{30 \times 9\frac{1}{3}}{19\frac{1}{3}}$

$= \frac{8^2 \times 146\sqrt{46}}{5^3 \times 23^2} \times 30^2 \times$ tabular ſeg. whoſe height is $\frac{23}{48}$

$= \frac{6^2 \times 8^2 \times 146\sqrt{46}}{5 \times 23^2} \times {\cdot}37187178 = 320{\cdot}761173$.

And $ndd = {\cdot}785398 \times (19\frac{1}{3})^2 = 289{\cdot}529179$.

Therefore

1. $Q \times (\text{FBE} - R) = 3{\cdot}48010217 \times 179{\cdot}841984 = 625{\cdot}868479$ = the convex ſurface of the hoof FEBC.

2. Q

2. $\mathfrak{Q} \times (\text{FAE} + \text{R}) = 3{\cdot}48010217 \times 527{\cdot}01636 = 1834{\cdot}070779$ = that of the part FAECV.

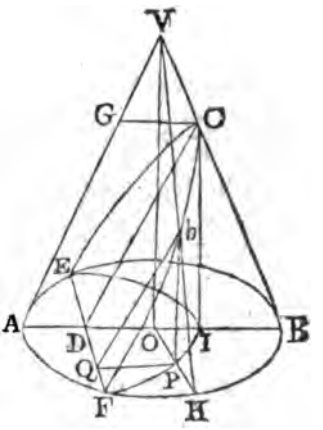
3. $\mathfrak{Q} \times (\text{FAE} + \text{R} - ndd) = 3{\cdot}48010217 \times 237{\cdot}487181 = 826{\cdot}479654$ = that of the complemental hoof FAECG.

## PROBLEM XXXI.

*To find the Convex Surfaces of the Parabolic Hoofs made by a Plane cutting a Cone Parallel to one Side.*

Multiply $\frac{2}{3}$ of the difference of the diameters of the baſe and top of the fruſtum, by the ſquare root of the product of the ſaid difference and the leſs diameter; and call the product s.—Then

1. The difference between s and the baſe of the hoof, multiplied by $\mathfrak{Q}$, in the 29th problem, will give the ſurface of the hoof.




That is, $\frac{\sqrt{4b^2 + (\text{D} - d)^2}}{\text{D} - d} \times$ [ſegment FBE $- \frac{2}{3}(\text{D} - d)\sqrt{d(\text{D} - d)}$] = the convex ſurface of the ungula FEBC. Uſing ſtill the ſame letters.

2. The ſum of s and the complemental baſe, multiplied by $\mathfrak{Q}$, will give the convex ſurface of the remaining part of the cone.

That is, $\frac{\sqrt{4b^2 + (\text{D} - d)^2}}{\text{D} - d} \times$ [ſegment FAE $+ \frac{2}{3}(\text{D} - d)\sqrt{d(\text{D} - d)}$] = the ſurface of the part FAECV.

3. The difference between the circle CG, and the ſum of s and the complemental baſe, multiplied by Q, will give the ſurface of the complemental hoof.

That is, $\frac{\sqrt{4h^2+(D-d)^2}}{D-d} \times$ [ſegment FAE $+\frac{2}{3}(D-d)\sqrt{d(D-d)} - ndd$] = the convex ſurface of the complemental hoof FAECG.

EXAMPLE.

Required the convex ſurfaces of the parabolic hoofs of the fruſtum of a cone, whoſe height is 18, and its diameters 30 and $19\frac{1}{2}$ inches.

Here again $Q = 3{\cdot}48010217$.

$s = \frac{2}{3}(D-d)\sqrt{d(D-d)} = \frac{2}{3} \times 10\frac{1}{2}\sqrt{19\frac{1}{2} \times 10\frac{1}{2}} = 103{\cdot}68$.

The ſeg. FBE $= 30^2 \times$ tab. ſeg. whoſe verſed ſine is $\frac{10\frac{1}{2}}{30}$ or $\frac{3{\cdot}6}{10}$ or $\cdot 36 = 900 \times {\cdot}25455055 = 229{\cdot}095495$.

The ſeg. FAE $= 30^2 \times$ tab. ſeg. whoſe height is $\frac{19\frac{1}{2}}{30}$ or $\frac{6{\cdot}4}{10}$ or $\cdot 64 = 900 \times {\cdot}53084761 = 477{\cdot}762849$.

And $ndd$, as before, $= 289{\cdot}529179$.—Whence

1. $Q \times$ (FBE $- s$) $= 3{\cdot}48010217 \times 125{\cdot}415495 = 436{\cdot}458737$ = the convex ſurface of the hoof FEBC.

2. $Q \times$ (FAE $+ s$) $= 3{\cdot}48010217 \times 581{\cdot}442849 = 2023{\cdot}480521$ = that of the part FAECV.

3. $Q \times$ (FAE $+ s - ndd$) $= 3{\cdot}48010217 \times 291{\cdot}91367 = 1015{\cdot}889396$ = that of the complemental hoof FAECG.

PRO-

## PROBLEM XXXII.

*To find the Curve Surfaces of the Hyperbolic Hoofs of the Fruſtum of a Cone, made by a Plane cutting it Perpendicular to the Baſe.*

1. Multiply the quantity $q$, found as in the preceding problems, by the baſe of the hoof; and the product will be its ſurface.

That is, $\frac{VB}{OB} \times$ ſeg. FBE $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ ſeg. of the circle AB whoſe height is $\frac{D-d}{2}$ = the curve ſurface of the perpendicular hoof FEBC. [See the fig. at page 243.]

2. Multiply $q$ by the complemental baſe, and the product will be the ſurface of the complemental part of the whole cone.

That is, $\frac{VB}{OB} \times$ ſeg. FAE $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times$ ſeg. of the circle AB whoſe height is $\frac{D+d}{2}$ or AD or AI = the curve ſurface of the part FAECV.

3. Multiply $q$ by the difference between the complemental ſegment and the circle GC, and the product will be the ſurface of the complemental hoof.

That is, $\frac{VB}{OB} \times$ (ſeg. FAE $-$ cir. CG) $= \frac{\sqrt{4b^2 + (D-d)^2}}{D-d} \times (-ndd +$ ſeg. of the circle AB whoſe height is $\frac{D+d}{2}$) = the curve ſurface of the complemental hoof FAECG. Uſing ſtill the ſame letters as in the former problems.

*Note.* The two firſt articles are the ſame rule, and ſerve for the whole cone as well as for any perpendicular part.

EXAMPLE.

Required the curve ſurfaces of the perpendicular hoofs of a cone, whoſe baſe diameter is 30 inches; the height of the ſection being 18, and the diameter of the cone at the top of the ſection, $19\frac{1}{2}$ inches.

Here again $\begin{cases} \text{Q} = 3{\cdot}48010217, \\ ndd = 289{\cdot}529179. \end{cases}$

The ſegment FBE $= 30^2 \times$ tab. ſeg. whoſe height is $\frac{30-19\frac{1}{2}}{2\times 30}$ or $\cdot 18 = 900 \times {\cdot}09613453 = 86{\cdot}521077$.

The ſegment FAE $= 30^2 \times$ tab. ſeg. whoſe height is $\frac{30+19\frac{1}{2}}{2\times 30}$ or $\cdot 82 = 900 \times {\cdot}68926363 = 620{\cdot}337267$.

And FAE $- ndd = 330{\cdot}808088$.

Whence, multiplying theſe three each by Q or 3·48010217, we have

1. Q $\times$ FBE $=$ 301·102188 the ſurf. of FEBC.

2. Q $\times$ FAE $=$ 2158·837068 the ſurf. of FEACV.

3. Q $\times$ (FAE $- ndd$) $=$ 1151·245945 the ſurf. of FEACG.

SECT.

## SECTION II.

### OF THE REGULAR BODIES.

#### DEFINITIONS.

1. A Regular ſolid or body, is a ſolid contained under ſome number of like, equal, and regular plane figures.

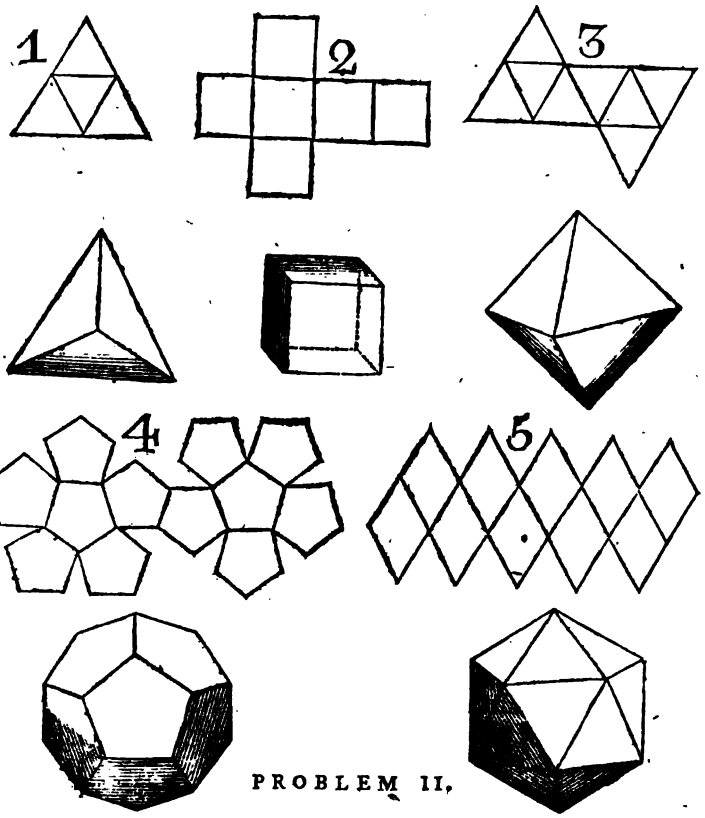
2. The plane figures, under which the ſolid is contained, are the faces of the ſolid. And the ſides of the plane figures, are the edges or linear ſides of the ſolid.

3. There are only five ſorts of regular ſolids, viz.
The tetraedron, or regular triangular pyramid, having four triangular faces;
The hexaedron, or cube, having ſix ſquare faces;
The octaedron, having eight triangular faces;
The dodecaedron, having twelve pentagonal faces;
And the icoſaedron, having twenty triangular faces.

#### PROBLEM I.

*To Conſtruct or Form the Regular Solids.*

Having deſcribed figures, as below, on paſteboard, or ſome other pliable matter, cut them out by the extreme ſides; and cut the other lines half through; then fold them at theſe lines, ſo cut, till the ſides meet; which being faſtened together with glue, or otherwiſe, you will have the form of the bodies. Namely, figure 1 will form the tetraedron; figure 2, the hexaedron; figure 3, the octaedron; figure 4, the dodecaedron; and figure 5, the icoſaedron.




## PROBLEM II.

*To find the Surface of a Tetraedron.*

Multiply the ſquare of the linear edge by the root of 3, and the product will be the whole ſurface, or the ſum of the four faces.

That is, $A^2\sqrt{3}$ is the whole ſurface; A being the linear edge, or a ſide of one of the faces.*

EX-

* DEMONSTRATION.

For, by rule 2 prob. 4 ſect. 1 part 2, one of the faces will be $= \frac{1}{2}\sqrt{3} \times A^2$; which being multiplied by 4, gives $A^2\sqrt{3}$ for all the four faces, *Q. E. D.*

EXAMPLE.

If each ſide of one face of a tetraedron be 1, required the whole ſurface.

Here A = 1; and therefore $A^2\sqrt{3} = \sqrt{3} =$ 1·7320508 = the ſurface required.

PROBLEM III.

*To find the Solidity of a Tetraedron.*

Multiply $\frac{1}{12}$ of the cube of the linear ſide by the root of 2, and the product will be the ſolidity of the tetraedron.

That is, $\frac{1}{12} A^3\sqrt{2}$ is the ſolidity.*

EX-

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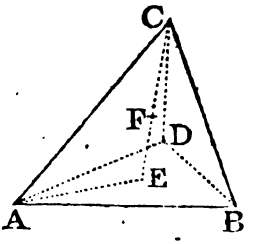
* DEMONSTRATION.




From one angle C of the tetraedron let fall a perpendicular CE upon the oppoſite ſide ADB, and draw EA.

Then $AC^2 = AE^2 + EC^2$; but, from the tables in page 81 and 84, it appears that $\frac{1}{3} AC^2 = \frac{1}{3} AB^2$ is $= AE^2$; therefore, by ſubtracting this equation from the former, $\frac{2}{3} AC^2$ will be $= CE^2$, and hence $CE = AC\sqrt{\frac{2}{3}}$. But ABD is $= \frac{1}{4} AB^2\sqrt{3} = \frac{1}{4} AC^2\sqrt{3}$. Then $\frac{1}{3} CE \times ABD = \frac{1}{3} AC\sqrt{\frac{2}{3}} \times \frac{1}{4} AC^2\sqrt{3} = \frac{1}{12} AC^3\sqrt{2}$ is the ſolidity. *Q. E. D.*

*Corol.* 1. A cube is to a tetraedron of the ſame linear ſide, as 1 is to $\frac{1}{12}\sqrt{2}$, or as 12 is to $\sqrt{2}$.

*Corol.* 2. Put $r$ for the radius EF of the inſcribed ſphere, A for the linear ſide AC, B for the whole ſurface, and C for the ſolidity; and we ſhall have

$$r = \frac{3C}{B} = \frac{\frac{1}{4}A^3\sqrt{2}}{A^2\sqrt{3}} = \frac{1}{4} A\sqrt{\frac{2}{3}} = \frac{1}{12} A\sqrt{6}.$$

*Corol.* 3. Hence alſo $r$ or FE is $= \frac{1}{4}$ EC; for, by the demonſtration, CE is $= A\sqrt{\frac{2}{3}}$.

*Corol.* 4. And hence the radius of the circumſcribed ſphere, is equal to 3 times that of the inſcribed; that is, R = FC = CE − EF = 4FE − EF = 3FE = $3r = \frac{1}{4} A\sqrt{6}$.

SCHO-

EXAMPLE.

If the linear ſide of a tetraedron be 1; required the ſolidity.

Here $A = 1$, and $\frac{1}{12}A^3\sqrt{2} = \frac{1}{12}\sqrt{2} = .11785113$ = the ſolidity required.

PROBLEM IV.

*To find the Surface or Solidity of a Hexaedron or Cube.*

It is evident that 6 times the ſquare of the linear ſide, will be equal to the whole ſurface; and the cube of the linear ſide, equal to the ſolidity of the hexaedron, or cube.*

PRO-

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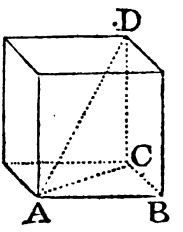
SCHOLIUM.

From this problem and its corollaries, together with the former problem, are deduced theſe equations.

1. $A = \sqrt{\frac{1}{3}B\sqrt{3}} = \sqrt[3]{6C\sqrt{2}} = \frac{1}{3}R\sqrt{6} = 2r\sqrt{6}$.
2. $B = A^2\sqrt{3} = 6\sqrt[3]{C^2\sqrt{3}} = \frac{8}{3}R^2\sqrt{3} = 24r^2\sqrt{3}$.
3. $C = \frac{1}{12}A^3\sqrt{2} = \frac{1}{36}B\sqrt{2B\sqrt{3}} = \frac{8}{27}R^3\sqrt{3} = 8r^3\sqrt{3}$.
4. $R = \frac{1}{4}A\sqrt{6} = \frac{1}{4}\sqrt{2B\sqrt{3}} = \frac{3}{2}\sqrt[3]{\frac{1}{3}C\sqrt{3}} = 3r$.
5. $r = \frac{1}{12}A\sqrt{6} = \frac{1}{12}\sqrt{2B\sqrt{3}} = \frac{1}{2}\sqrt[3]{\frac{1}{3}C\sqrt{3}} = \frac{1}{3}R$.

* SCHOLIUM.

Since, as is evident, the diameter of the inſcribed ſphere is equal to the linear ſide of the hexaedron, and the diameter of the circumſcribed ſphere is the diagonal $AD = \sqrt{AC^2 + CD^2} = \sqrt{AB^2 + BC^2 + CD^2} = \sqrt{3AB^2} = AB\sqrt{3}$; if A, B, and C be put to denote the linear ſide, the ſurface, and the ſolidity of a hexaedron, and $r$, R for the radius of the inſcribed and circumſcribed ſphere; then in the hexaedron, or cube, we ſhall have theſe equations.



### PROBLEM V.

*To find the Surface of an Octaedron.*

Multiply the ſquare of the linear ſide by the root of 3, and double the product will be the ſurface. That is, $2A^2\sqrt{3}$ is the ſurface.*

### EXAMPLE.

The linear ſide of an octaedron being 1, it is required to find the ſuperficies.

Here $A = 1$, and $2A^2\sqrt{3} = 2\sqrt{3} = 3.4641016$ = the ſurface required.

### PROBLEM VI.

*To find the Solidity of an Octaedron.*

Multiply the cube of the linear ſide by the root of 2, and $\frac{1}{3}$ of the product will be the content of the octaedron.

That

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1. $A = \sqrt{\frac{1}{2}}B = \sqrt[3]{C} = \frac{2}{3}R\sqrt{3} = 2r.$
2. $B = 6A^2 = 6\sqrt[3]{C^2} = 8R^2 = 24r^2.$
3. $C = A^3 = \frac{1}{6}B\sqrt{\frac{1}{6}}B = \frac{8}{9}R^3\sqrt{3} = 8r^3.$
4. $R = \frac{1}{2}A\sqrt{3} = \frac{1}{2}\sqrt{\frac{1}{2}}B = \frac{1}{2}\sqrt{3} \times \sqrt[3]{C} = r\sqrt{3}.$
5. $r = \frac{1}{2}A = \frac{1}{2}\sqrt{\frac{1}{6}}B = \frac{1}{2}\sqrt[3]{C} = \frac{1}{3}R\sqrt{3}.$

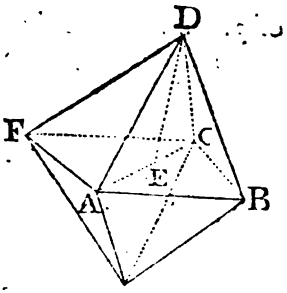
* DEMONSTRATION.

By rule 2 prob. 4 ſect. 1 part 2, one face is $= \frac{1}{4}A^2\sqrt{3}$; conſequently 8 faces, or the whole ſurface, will be $= 2A^2\sqrt{3}$. *Q. E. D.*

That is, $\frac{1}{3}A^3\sqrt{2}$ is the ſolidity.*

EX-

* DEMONSTRATION.




Let E be the middle of the diagonal AC of the ſolid, or the center of the ſolid; and draw DE, which will be equal to AE.

Then, ſince the ſolid is evidently compoſed of two equal ſquare pyramids, each baſe of which, ABCF, is equal to the ſquare of the linear ſide of the ſolid, and the altitude of each equal to DE or AE, half the diagonal of that ſquare; we ſhall have $ABCF \times \frac{1}{2}AE = AB^2 \times \frac{1}{2}AC = \frac{1}{2}AB^2\sqrt{AB^2 + BC^2} = \frac{1}{3}AB^2\sqrt{2AB^2} = \frac{1}{3}AB^3\sqrt{2}$ for the ſolidity. *Q. E. D.*

*Corol.* 1. The radius AE of the circumſcribed ſphere is $= \frac{1}{2}AC = \frac{1}{2}\sqrt{2AB^2} = \frac{1}{2}AB\sqrt{2} = AB\sqrt{\frac{1}{2}}$.

*Corol.* 2. The radius of the inſcribed ſphere, is equal to the quotient ariſing from the diviſion of 3 times the ſolidity by the whole ſurface, equal (from this problem and the laſt) $AB^3\sqrt{2} \div 2AB^2\sqrt{3} = \frac{1}{2}AB\sqrt{\frac{1}{3}} = \frac{1}{6}AB\sqrt{6} = AB\sqrt{\frac{1}{6}}$.

SCHOLIUM.

From the two laſt problems, and the corollaries, are deduced the following equations; in which A denotes the linear ſide of an octaedron, B the whole ſurface, C the ſolidity, $r$ the radius of the inſcribed ſphere, and R the radius of the circumſcribed ſphere.

1. $A = \sqrt{\frac{B\sqrt{3}}{6}} = \sqrt[3]{\frac{3C\sqrt{2}}{2}} = R\sqrt{2} = r\sqrt{6}$.

2. $B = 2A^2\sqrt{3} = 6\sqrt[3]{\frac{C^2\sqrt{3}}{2}} = 4R^2\sqrt{3} = 12r^2\sqrt{3}$.

3. $C = \frac{1}{3}A^3\sqrt{2} = \frac{B\sqrt{B\sqrt{3}}}{18} = \frac{4}{3}R^3 = 4r^3\sqrt{3}$.

4. $R = \frac{1}{2}A\sqrt{2} = \sqrt{\frac{B\sqrt{3}}{12}} = \sqrt[3]{\frac{3}{4}C} = r\sqrt{3}$.

5. $r = \frac{1}{6}A\sqrt{6} = \frac{1}{6}\sqrt{B\sqrt{3}} = \sqrt[3]{\frac{C\sqrt{3}}{12}} = \frac{1}{3}R\sqrt{3}$.

EXAMPLE.

If the linear ſide of an octaedron be 1, what is its ſolidity?

Here $A = 1$, and therefore $\frac{1}{3}A^3\sqrt{2} = \frac{1}{3}\sqrt{2} = 0.47140452079$ is the content required.

PROBLEM VII.

*To find the Surface of a Dodecaedron.*

To 1 add $\frac{2}{5}$ or $\frac{4}{10}$ of the root of 5; multiply the root of the ſum by 15 times the ſquare of the linear ſide; and the product will be the ſurface of the dodecaedron.

That is, $15A^2\sqrt{1 + \frac{2}{5}\sqrt{5}}$ is the ſurface.*

EXAMPLE.

If the linear ſide be 1, required the ſurface.

Here $A = 1$, and therefore $15A^2\sqrt{1 + \frac{2}{5}\sqrt{5}} = 15\sqrt{1 + \frac{2}{5}\sqrt{5}} = 20.6457788075$ is the ſurface required.

PRO-

* DEMONSTRATION.

By rule 2 prob. 4 ſect. 1 part 2, the area of one face is $\frac{5}{4}A^2\sqrt{1 + \frac{2}{5}\sqrt{5}}$, and therefore 12 faces or the whole ſurface will be $12 \times \frac{5}{4}A^2\sqrt{1 + \frac{2}{5}\sqrt{5}} = 15A^2\sqrt{1 + \frac{2}{5}\sqrt{5}}$. *Q. E. D.*

## PROBLEM VIII.

### *To find the Solidity of a Dodecaedron.*

* To 21 times the root of 5 add 47, and divide the ſum by 40; multiply the root of the quotient by 5 times

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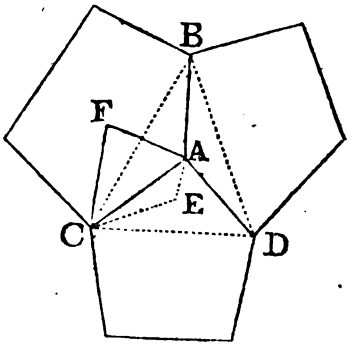
* DEMONSTRATION.




Let A be a ſolid angle of the dodecaedron; and let the extremities of the ſides AB, AC, AD, of the faces which form the angle, be connected by the right lines BC, CD, DB, forming the equilateral triangle BCD within the ſolid; upon the center E of which triangle let fall the perpendicular AE, and to the center F of one of the faces draw the lines AF, CF.

The angle CAD contains 108 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$, to the radius 1; and the angle ADC contains 36 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10-2\sqrt{5}}$; therefore as $\sqrt{10-2\sqrt{5}} : \sqrt{10+2\sqrt{5}} ::$ $CA : CD = AC\sqrt{\frac{10+2\sqrt{5}}{10-2\sqrt{5}}} = AC\sqrt{\frac{5+\sqrt{5}}{5-\sqrt{5}}} = AC\sqrt{\frac{(5+\sqrt{5})^2}{25-5}}$ $= \frac{5+\sqrt{5}}{2\sqrt{5}}\,AC = \frac{1+\sqrt{5}}{2}\,AC.$ And much after the ſame manner, we find $CE = \frac{1}{3}CD\sqrt{3} = CD\sqrt{\frac{1}{3}} = \frac{1+\sqrt{5}}{2\sqrt{3}}\,AC.$

Hence $EA = \sqrt{AC^2 - CE^2} = \sqrt{AC^2 - (\frac{1+\sqrt{5}}{2\sqrt{3}})^2 AC^2} =$ $AC\sqrt{1-\frac{3+\sqrt{5}}{6}} = AC\sqrt{\frac{3-\sqrt{5}}{6}}.$ Then becauſe the chord of an arc is a mean proportional between its verſed ſine and the diameter, and AE is the verſed ſine of the arc whoſe chord is AC, and its diameter equal to that of the circumſcribed ſphere;

5 times the cube of the linear ſide; and the product will be the ſolidity.

That is, $5\,A^3\sqrt{\frac{47+21\sqrt{5}}{40}}$ is the ſolidity.

EX-

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ſphere; we ſhall have $AC^2 \div 2AE = AC^2 \div 2AC\sqrt{\frac{3-\sqrt{5}}{6}}$

$= \frac{1}{2}AC\sqrt{\frac{6}{3-\sqrt{5}}} = \frac{1}{2}AC\sqrt{6 \times \frac{3+\sqrt{5}}{9-5}} = \frac{1}{2}AC\sqrt{3 \times \frac{6+2\sqrt{5}}{4}}$

$= AC \times \frac{1+\sqrt{5}}{4}\sqrt{3} = \frac{\sqrt{3}+\sqrt{15}}{4}AC = R$ the radius of the circumſcribed ſphere.

Again, the angle AFC contains 72 degrees, whoſe ſine is $\frac{1}{4}\sqrt{10+2\sqrt{5}}$; and the angle ACF 54 degrees, whoſe ſine is $\frac{1+\sqrt{5}}{4}$; therefore as $\sqrt{10+2\sqrt{5}} : 1+\sqrt{5} :: AC : AF =$

$\frac{1+\sqrt{5}}{\sqrt{(10+2\sqrt{5})}}AC = AC\sqrt{\frac{5+\sqrt{5}}{10}}$.

But, ſince the radius of the circumſcribed ſphere is the hypotenuſe of a right-angled triangle, whoſe two legs are AF and the radius of the inſcribed ſphere, we ſhall have $\sqrt{R^2 - AF^2} =$

$\sqrt{(\frac{\sqrt{3}+\sqrt{15}}{4}AC)^2 - \frac{5+\sqrt{5}}{10}AC^2} = AC\sqrt{\frac{18+6\sqrt{5}}{16} - \frac{5+\sqrt{5}}{10}}$

$= AC\sqrt{\frac{25+11\sqrt{5}}{40}} = r$ the radius of the inſcribed ſphere.

Then, becauſe the ſolidity of any regular ſolid is equal to the ſurface drawn into $\frac{1}{3}$ of the radius of the inſcribed ſphere, and the ſurface B equal to $15A^2\sqrt{\frac{5+2\sqrt{5}}{5}}$ by the laſt problem, we ſhall have $B \times \frac{1}{3}r = 15A^2\sqrt{\frac{5+2\sqrt{5}}{5}} \times \frac{1}{3}A\sqrt{\frac{25+11\sqrt{5}}{40}}$

$= 5A^3\sqrt{\frac{47+21\sqrt{5}}{40}} = C$ the ſolidity. *Q. E. D.*

*Corollary.*

EXAMPLE.

If the linear ſide be 1, required the ſolidity of the dodecaedron.

Here $A = 1$, and therefore $5A^3\sqrt{\frac{47+21\sqrt{5}}{40}} = 5\sqrt{\frac{47+21\sqrt{5}}{40}} = 7{\cdot}663118960\,5$ is the content.

PRO-

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*Corollary.* From the demonſtration it appears that the radius of the circumſcribed ſphere R is $= \frac{\sqrt{3}+\sqrt{15}}{4}A$, and that of the inſcribed ſphere $r$ is $= A\sqrt{\frac{25+11\sqrt{5}}{40}}$.

SCHOLIUM.

Putting, as before, A, B, C for the linear ſide, ſurface, and ſolidity, and R, $r$ for the radius of the circumſcribed and inſcribed ſphere reſpectively; we ſhall have, in the dodecaedron,

1. $A = \sqrt{\frac{B\sqrt{5-2\sqrt{5}}}{15}} = \sqrt[3]{\frac{C\sqrt{470-210\sqrt{5}}}{5}} = \frac{\sqrt{15}-\sqrt{3}}{3}R = r\sqrt{50-22\sqrt{5}}$.

2. $B = 15A^2\sqrt{\frac{5+2\sqrt{5}}{5}} = 3\sqrt[3]{10C^2\sqrt{130-58\sqrt{5}}} = 10R^2\sqrt{\frac{10-2\sqrt{5}}{5}} = 30r^2\sqrt{130-58\sqrt{5}}$.

3. $C = 5A^3\sqrt{\frac{47+21\sqrt{5}}{40}} = \frac{B}{30}\sqrt{\frac{B\sqrt{650+290\sqrt{5}}}{6}} = \frac{20R^3}{3}\sqrt{\frac{3+\sqrt{5}}{30}} = 10r^3\sqrt{130-58\sqrt{5}}$.

4. $R = \frac{\sqrt{15}+\sqrt{3}}{4}A = \sqrt{\frac{B\sqrt{10+2\sqrt{5}}}{40}} = \sqrt[3]{\frac{3C\sqrt{90-30\sqrt{5}}}{40}} = r\sqrt{15-6\sqrt{5}}$.

5. $r = \frac{A\sqrt{250+110\sqrt{5}}}{20} = \frac{1}{20}\sqrt{\frac{2B\sqrt{650+290\sqrt{5}}}{3}} = \sqrt[3]{\frac{C\sqrt{650+290\sqrt{5}}}{200}} = R\sqrt{\frac{5+2\sqrt{5}}{15}}$.

PROBLEM IX.

*To find the Surface of an Icosaedron.*

Multiply 5 times the square of the linear side by the root of 3, and the product will be the surface.

That is, $5A^2\sqrt{3}$ is the surface.*

EXAMPLE.

The linear side being 1; required the surface.

Here $5A^2\sqrt{3} = 5\sqrt{3} = 8{\cdot}66025403$.

PROBLEM X.

*To find the Solidity of an Icosaedron.*

To 7 add 3 times the root of 5, divide the sum by 2, multiply the root of the quotient by five-sixths of the cube of the linear side, and the product will be the solidity of the icosaedron.

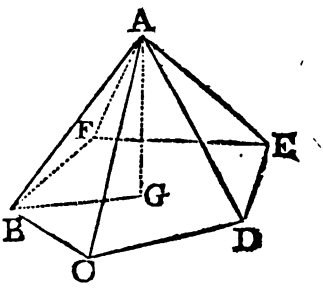
That is, $\frac{5}{6}A^3\sqrt{\frac{7+3\sqrt{5}}{2}}$ is the content.†



* DEMONSTRATION.

For one face is $= \frac{1}{4}A^2\sqrt{3}$, and therefore twenty faces, or the whole surface, will be $20 \times \frac{1}{4}A^2\sqrt{3} = 5A^2\sqrt{3}$. *Q. E. D.*

† DEMONSTRATION.




Let A be a solid angle, of the icosaedron, formed by five faces or triangles whose bases form the pentagon BCDEF, upon whose center G let fall the perpendicular AG, and draw BG.

In the demonstration of problem 8 it was found that BG is $= AB\sqrt{\frac{5+\sqrt{5}}{10}}$, and that the radius $\rho$ of the circle circumscr. one face ABC is $= AB\sqrt{\frac{1}{3}}$. But the radius of the circumscr. sphere is

EXAMPLE.

The linear ſide of an icoſaedron being 1, required the ſolidity.

Here $A = 1$, and therefore $\frac{5}{6}A^3\sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{5}{6}\sqrt{\frac{7+3\sqrt{5}}{2}} = 2{\cdot}1816949905$ is the content.

PROBLEM XI.

*To find the Superficies or Solidity of any Regular Body.*

RULES.

1. Multiply the proper tabular area (taken from the following table) by the ſquare of the linear edge of the ſolid, for the ſuperficies.

2. Multiply

---

is $R = \frac{AB^2}{2AG} = \frac{AB^2}{2\sqrt{AB^2 - BG^2}} = \frac{AB^2}{2\sqrt{AB^2 - \frac{5+\sqrt{5}}{10}AB^2}}$

$= \frac{AB}{2\sqrt{1 - \frac{5+\sqrt{5}}{10}}} = \frac{AB}{2\sqrt{\frac{5-\sqrt{5}}{10}}} = \frac{1}{2}AB\sqrt{\frac{10}{5-\sqrt{5}}} =$

$\frac{1}{2}AB\sqrt{\frac{10}{5-\sqrt{5}} \times \frac{5+\sqrt{5}}{5+\sqrt{5}}} = \frac{1}{2}AB\sqrt{\frac{10\times(5+\sqrt{5})}{25-5}} = AB\sqrt{\frac{5+\sqrt{5}}{8}}.$

And R is the hypotenuſe of a right-angled triangle, of which the one leg is the radius $\rho$ of the circle circumſcribing the face, and the other the radius $r$ of the inſcribed ſphere; therefore $r$ is $= \sqrt{R^2 - \rho^2} = \sqrt{\frac{5+\sqrt{5}}{8}AB^2 - \frac{1}{3}AB^2} =$

$AB\sqrt{\frac{15+3\sqrt{5}-8}{8\times 3}} = AB\sqrt{\frac{7+3\sqrt{5}}{24}}.$

Conſequently the ſolidity $C = \frac{1}{3}rB$, B being the whole ſurface, will be $\frac{1}{3}AB\sqrt{\frac{7+3\sqrt{5}}{24}} \times 5AB^2\sqrt{3} = \frac{5}{8}AB^3\sqrt{\frac{7+3\sqrt{5}}{8}} =$

$\frac{5}{6}AB^3\sqrt{\frac{7+3\sqrt{5}}{2}}$. *Q. E. D.*

SCHO-

2. Multiply the tabular ſolidity by the cube of the linear edge, for the ſolid content.*

| Surfaces and Solidities of Regular Bodies. | | | |
|---|---|---|---|
| N° of Sides | Name | Surface | Solidity |
| 4 | Tetraedron | 1·7320508 | 0·1178513 |
| 6 | Hexaedron | 6·0000000 | 1·0000000 |
| 8 | Octaedron | 3·4641016 | 0·4714045 |
| 12 | Dodecaedron | 20·6457788 | 7·6631189 |
| 20 | Icoſaedron | 8·6602540 | 2·1816950 |



SCHOLIUM.

From this problem and the laſt may be deduced the following equations in the icoſaedron, the letters being as before.

1. $A = \sqrt{\frac{B\sqrt{3}}{15}} = \sqrt[3]{\frac{6}{5}C\sqrt{\frac{7-3\sqrt{5}}{2}}} = R\sqrt{\frac{10-2\sqrt{5}}{5}}$
$= r\sqrt{42-18\sqrt{5}}.$

2. $B = 5A^2\sqrt{3} = 3\sqrt[3]{70\sqrt{3}-30\sqrt{15}} = 2R^2\times(5\sqrt{3}-\sqrt{15})$
$= 3r^2\times(7\sqrt{3}-3\sqrt{15}).$

3. $C = \frac{5}{6}A^3\sqrt{\frac{7+3\sqrt{5}}{2}} = \frac{B}{18}\sqrt{\frac{7\sqrt{3}+3\sqrt{15}}{10}}B$
$= \frac{2}{3}R^3\sqrt{10+2\sqrt{5}} = 10r^3\times(7\sqrt{3}-3\sqrt{15}).$

4. $R = \frac{1}{2}A\sqrt{\frac{5+\sqrt{5}}{2}} = \frac{1}{2}\sqrt{\frac{5\sqrt{3}+\sqrt{15}}{30}}B = \sqrt[3]{\frac{3}{4}C\sqrt{\frac{5-\sqrt{5}}{10}}}$
$= r\sqrt{15-6\sqrt{5}}.$

5. $r = \frac{1}{2}A\sqrt{\frac{7+3\sqrt{5}}{6}} = \frac{1}{6}\sqrt{\frac{7\sqrt{3}+3\sqrt{15}}{10}}B$
$= \frac{1}{2}\sqrt[3]{\frac{7\sqrt{3}+3\sqrt{15}}{30}}C = R\sqrt{\frac{5+2\sqrt{5}}{15}}.$

* DEMONSTRATION.

The numbers in the above table denote the ſurface and ſolidity of each body, when its edge is 1; and becauſe, in ſimilar bodies, the ſurfaces are as the ſquares of the linear edges, and the ſolidities as the cubes of the ſame; therefore the truth of the rules is manifeſt.

# SECTION III.

## OF SOLID RINGS.

### DEFINITION.

BY a ring, in general, is meant a ſolid returning into itſelf; of which every ſection perpendicular to the axe, or line paſſing through the middle of the ſolid, is every-where the ſame figure, and of the ſame magnitude.

### PROBLEM I.

*To find the Surface of a Solid Ring.*

Multiply the axe by the perimeter of a ſection perpendicular to it, and the product will be the ſurface.

### EXAMPLE.

A workman having made for a jeweller a circular ring, or a ring whoſe axe forms the circumference of a circle; it is required to find the expence of the gilding at a penny the ſquare inch, the thickneſs of the ring, or the diameter of a ſection of it, being 2 inches, and the inner diameter, acroſs from ſide to ſide, 18 inches.

Here $18 + 2 = 20 =$ the diameter of the circle formed by the axe; conſequently $20 \times 3{\cdot}14159 =$ the length of the axe. But $2 \times 3{\cdot}14159 =$ the circumference of a ſection of it. Therefore $20 \times 3{\cdot}14159 \times 2 \times 3{\cdot}14159 = 40 \times 3{\cdot}14159^2 = 394{\cdot}785$ ſquare inches, nearly, $= 394{\cdot}785$ pence $=$ £ 1 12 $10\frac{3}{4}$ nearly, the expence required.

PRO-

### PROBLEM II.

*To find the Solidity of a Ring.*

Multiply the axe by a ſection perpendicular to t, and the product will be the ſolidity.

### EXAMPLE.

Required the price of a ring of iron, whoſe dimenſions are the ſame with that in the example to the laſt problem, at 4 pence a pound; a cubic inch of iron weighing 4·423 ounces averdupois.

Here the area of a ſection being $2^2 \times .785398 = 3.14159$, which is the ſame number as that expreſſing half the circumference, and the axe being the ſame as before, it is evident that the ſolidity will be expreſſed by half the ſurface in the laſt example, viz. the ſolidity = 197·3925 cubic inches; which multiplied by 4·423 give 873·065 ounces = 54·56657 pounds; which, at 4 pence each, come to £0 18 $2\frac{1}{4}$, the price required.

### SCHOLIUM.

I omit any more examples, as the manner of operation is the ſame in all forms, with thoſe for priſms, both with regard to the ſurfaces and ſolidities; for, ſince it is evident that any ring is equal to a priſm whoſe altitude and end are reſpectively equal to the axe and ſection of the ring, both in ſurface and ſolidity, the rules for them both muſt be the ſame; and for this reaſon alſo any demonſtration of the rules for rings was quite unneceſſary in this place.

# SECTION IV.

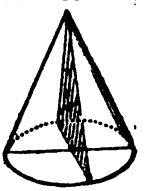
## OF CONIC SECTIONS, AND OF THE FIGURES ARISING FROM, OR DEPENDING ON, THEM.

### DEFINITIONS.

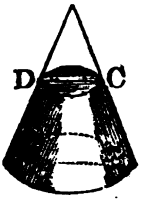
1. CONIC Sections are the plane figures formed by cutting a cone.

According to the different positions of the cutting plane there will arise five different figures or sections.

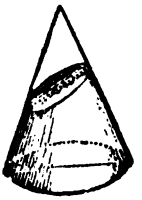
2. If the cutting plane pass through the vertex, and any part of the base, the section will be a triangle.



3. If the cone be cut parallel to the base, the section will be a circle.

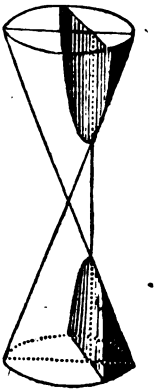



4. The section is called an ellipsis, when the cone is cut obliquely through both sides.



5. The

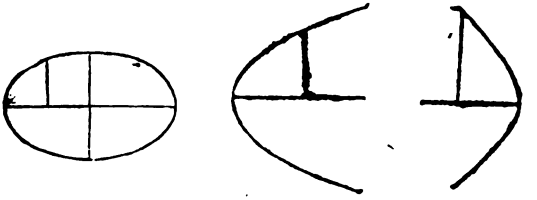
5. The ſection is a parabola, when the cone is cut parallel to one of its ſides.



6. The ſection is an hyperbola, when the cutting plane meets the oppoſite cone continued above the vertex, where it will make another ſection or hyperbola.

7. The vertices of any ſection, are the points where the cutting plane meets the oppoſite ſides of the cone.

8. The tranſverſe axis is the line between the two vertices. And the middle point of the tranſverſe, is the center of the conic ſection.



9. The conjugate axis, is a line drawn through the center, and perpendicular to the tranſverſe.

10. A diameter is any right line drawn through the center, and terminated on each ſide by the curve; the interſections of the diameter and curve being the

vertices of the diameter.—Hence every diameter of the ellipſe and hyperbola have two vertices; but of the parabola, only one; unleſs we conſider the other as infinitely diſtant. And hence, alſo, all the diameters of a parabola are parallel to each other, and infinite.

11. If a tangent to the curve be drawn through the vertex of any diameter, and another diameter be drawn parallel to the tangent, thoſe diameters are ſaid to be conjugates the one to the other.—The axes, or principal conjugate diameters, are perpendicular to each other.

12. An abſciſs is any part of a diameter, terminated at the vertex.

13. An ordinate to any diameter, is a line contained between the diameter and the curve, and is parallel to the conjugate diameter, or to the tangent at the vertex.—The ordinates to the axe are perpendicular to it. And in the ellipſe and hyperbola, every ordinate hath two abſciſſes, in the parabola only one.

14. The parameter of any diameter, is a third proportional to that diameter and its conjugate.

15. The focus is the point of interſection of the axe and an ordinate, to it, which is equal to half the parameter of the axe.—The ellipſe and hyperbola have each two foci, the parabola only one.

16. A ſpheroid, or ellipſoid, is a ſolid generated by the revolution of an ellipſe about one of its axes. It is a prolate one, when the revolution is made about the tranſverſe axis; and oblate, when about the conjugate.

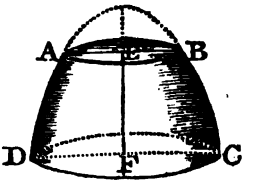
17. A

17. A conoid is a ſolid formed by the revolution of a parabola, or hyperbola, about the axis. And is accordingly called parabolic, or hyperbolic.—The parabolic conoid is alſo called a paraboloid; and the hyperbolic conoid, an hyperboloid.

18. A ſpindle is formed by any of the three ſections revolving about a double ordinate, like the circular ſpindle.

19. A ſegment, of any of theſe figures, is a part cut off at the top, by a plane parallel to the baſe.

20. And a fruſtum is the part left next the baſe, after the ſegment is cut off.



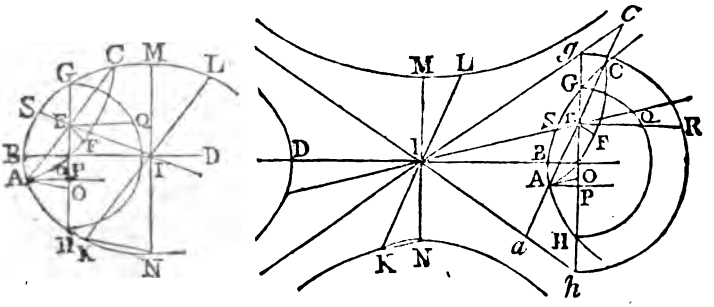

PROPOSITION I.

If any ſolid, formed by the revolution of a conic ſection about its axe, i. e. a ſpheroid, paraboloid, or hyperboloid, be cut by a plane in any poſition; the ſection will be ſome conic ſection, and all the parallel ſections will be like and ſimilar figures.

DEMONSTRATION.

Let ABC be the generating ſection, or a ſection of the given ſolid through its axe BD, and perpendicular to the propoſed ſection AFC, their common interſection

interſection being AC; let GH be any other line meeting the generating ſection in G and H, and cutting AC in E; and erect EF perpendicular to the plane ABC, and meeting the propoſed plane in F.




Then, if AC and GH be conceived to be moved continually parallel to themſelves, will the rectangle AE × EC be to the rectangle GE × EH, always in a conſtant ratio; but if GH be perpendicular to BD, the points G, F, H will be in the circumference of a circle whoſe diameter is GH, ſo that GE × EH will be = EF²; therefore AE × EC will be to EF², always in a conſtant ratio; conſequently AFC is a conic ſection, and every ſection parallel to AFC will be of the ſame kind with, and ſimilar to, it. *Q. E. D.*

*Corol.* 1. The above conſtant ratio, in which AE × EC is to EF², is that of KI² to IN², the ſquares of the diameters of the generating ſection reſpectively parallel to AC, GH; that is, the ratio of the ſquare of the diameter parallel to the ſection, to the ſquare of the revolving axe of the generating plane.

This will appear by conceiving AC and GH to be moved into the poſitions KL, MN, interſecting in I, the center of the generating ſection.

*Corol.*

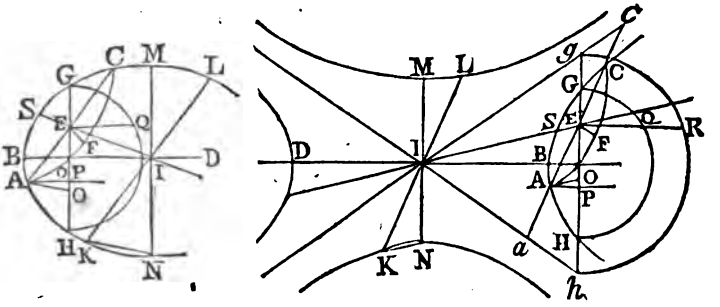
*Corol.* 2. And hence it appears that the axes AC and 2EF of the ſection, ſuppoſing E now to be the middle of AC, will be to each other, as the diameter KL is to the diameter MN of the generating ſection.

*Corol.* 3. If the ſection of the ſolid be made ſo as to return into itſelf, it will evidently be an ellipſe. Which always happens in the ſpheroid, except when it is perpendicular to the axe; which poſition is alſo to be excepted in the other ſolids, the ſection being always then a circle: in the paraboloid the ſection is always an ellipſe, excepting when it is parallel to the axe: and in the hyperboloid the ſection is always an ellipſe, when its axe makes with the axe of the ſolid, an angle greater than that made by the ſaid axe of the ſolid and the aſymptote of the generating hyperbola; the ſection being an hyperbola in all other caſes, but when thoſe angles are equal, and then it is a parabola.

*Corol.* 4. But if the ſection be parallel to the fixed axe BD, it will be of the ſame kind with, and ſimilar to, the generating plane ABC; that is, the ſection parallel to the axe, in a ſpheroid, is an ellipſe ſimilar to the generating ellipſe; in the paraboloid, the ſection is a parabola ſimilar to the generating parabola; and in an hyperboloid, it is an hyperbola ſimilar to the generating hyperbola of the ſolid.

*Corol.* 5. In the ſpheroid, the ſection through the axe is the greateſt of the parallel ſections; but in the hyperboloid, it is the leaſt; and in the paraboloid, all the ſections parallel to the axe, are equal to one another.

For the axe is the greateſt parallel chord line in the ellipſe, but the leaſt in the oppoſite hyperbolas, and all the diameters are equal in a parabola.




*Corol.* 6. If the extremities of the diameters KL, MN, be joined by the line KN, and AO be drawn parallel to KN, and meeting GEH in O, E being the middle of AC, or AE the ſemi-axe, and GH parallel to MN. Then EO will be equal to EF, the other ſemi-axe of the ſection.

For, by ſimilar triangles, KI : IN :: AE : EO.

Or upon GH as a diameter deſcribe a circle meeting EQ, perpendicular to GH, in Q; and it is evident that EQ will be equal to the ſemi-diameter EF.

*Corol.* 7. Draw AP parallel to the axe BD of the ſolid, and meeting the perpendicular GH in P. And it will be evident that, in the ſpheroid, the ſemi-axe EF = EO will be greater than EP; but in the hyperboloid, the ſemi-axe EF = EO, of the elliptic ſection, will be leſs than EP; and in the paraboloid, EF = EO is always equal to EP.

## SCHOLIUM.

The analogy of the ſections of an hyperboloid to thoſe of the cone, are very remarkable, all the three conic ſections being formed by cutting an hyperboloid in the ſame poſitions as the cone is cut.

Thus,

Thus, let an hyperbola and its aſymptote be revolved together about the tranſverſe axe, the former deſcribing an hyperboloid, and the latter a cone circumſcribing it; then let them be ſuppoſed to be both cut by a plane in any poſition, and the two ſections will be like, ſimilar, and concentric figures: that is, if the plane cut both the ſides of each, the ſections will be concentric, ſimilar ellipſes; if the cutting plane be parallel to the aſymptote, or to the ſide of the cone, the ſections will be parabolas; and in all other poſitions, the ſections will be ſimilar and concentric hyperbolas.

That the ſections are like figures, appears from the foregoing corollaries. That they are concentric, will be evident when we conſider that $Cc$ is $= Aa$, producing AC both ways to meet the aſymptotes in $a$ and $c$. And that they are ſimilar, or have their tranſverſe and conjugate axes proportional to each other, will appear thus: Produce GH both ways to meet the aſymptotes in $g$ and $b$; and on the diameters GH, $gb$, deſcribe the ſemi-circles GQH, $gRb$, meeting EQR, drawn perpendicular to GH, in Q and R; EQ and ER being then evidently the ſemi-conjugate axes, and EC, $Ec$, the ſemi-tranſverſe axes of the ſections. Now if GH and AC be conceived to be moved parallel to themſelves, $AE \times EC$ or $CE^2$ will be to $GE \times EH$ or $EQ^2$, in a conſtant ratio, or CE to EQ will be a conſtant ratio; and ſince $cE$ is as $Eg$, and $aE$ as $Eb$, $aE \times Ec$ or $cE^2$ will be to $gE \times Eb$ or $ER^2$, in a conſtant ratio, or $cE$ to ER will be a conſtant ratio; but at an infinite diſtance from the vertex, C and $c$ coincide, or $EC = Ec$, as alſo $EG = Eg$, conſequently $EQ = ER$, and then CE to EQ will be $= cE$ to ER; but as theſe ratios are conſtant, if they be equal to each other in one place, they muſt be always ſo; and conſequently $CE : Ec :: QE : ER$.

And

And this analogy of the ſections will not ſeem ſtrange, when we conſider that a cone is a ſpecies of the hyperboloid; or a triangle a ſpecies of the hyperbola, whoſe axes are infinitely little.

### PROPOSITION II.

If SI be the ſemi-diameter belonging to the double ordinate AEC of the generating plane, AEC being the diameter of the ſection AFC, conceived to be moved continually parallel to itſelf; and if $x$ denote any part of the diameter SI, intercepted by E the middle of AC, and any given fixed point taken in SI; then will the ſection AFC be always as $a + bx + cxx$; $a$, $b$, $c$, being conſtant quantities; $b$ in ſome caſes affirmative, and in others negative; $c$ being affirmative in the hyperbola, negative in the ellipſe, and nothing in the parabola; and $a$ may always be ſuppoſed to denote the diſtance of the given fixed point from the vertex S.

### DEMONSTRATION.

In any conic ſection, $AC^2$ is as $a + bx + cxx$; but all the parallel ſections are like and ſimilar figures, and ſimilar plane figures are as the ſquares of their like dimenſions; therefore the ſection AFC is as $AC^2$, that is, as $a + bx + cxx$. *Q. E. D*

*Corollary.* If the given fixed point, where $x$ begins, coincide with the vertex S, then will $a$ be equal to nothing, and the ſection will be as $bx \pm cxx$, or as $x \pm dxx$, in the hyperbola and ellipſe, and as $bx$, or as $x$, in the parabola.

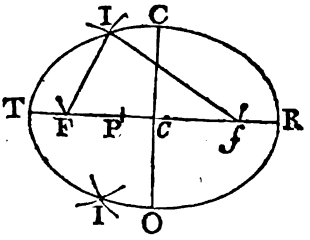
SEC-

## SECTION V.

### OF THE ELLIPSE, AND THE FIGURES GENERATED FROM IT.

#### PROBLEM I.

*To Deſcribe an Ellipſe.**




DRAW the tranſverſe TR, and its conjugate CO, biſecting each other perpendicularly in the center *c*.

With the radius T*c* and center C, deſcribe an arc cutting TR in the points F, *f*; which will be the two foci of the ellipſe.

Take any point P in the tranſverſe; then with the radii TP, PR, and centers F, *f*, deſcribe two arcs interſecting in I; ſo will the point I be in the curve or circumference of the ellipſe.

And thus, by aſſuming ſeveral points P in the tranſverſe, there will be found as many points in the

* The truth of this conſtruction will appear, by obſerving that the tranſverſe axe is equal to the ſum of two lines drawn from the foci to meet in any point in the curve.

the curve as we pleaſe. Through all which let the curve line be drawn.

*Otherwiſe.*

Having found the foci F, *f*, take a thread of the length of the tranſverſe, and faſten its ends with pins in the foci; then ſtretch the thread F I *f* and it will reach to I in the curve: and by moving a pencil round within the thread, keeping it always ſtretched, it will trace out the curve.

## PROBLEM II.

*In an Ellipſe, to find any two Conjugate Diameters, an Ordinate to one of them, and its Abſciſs, one from another; viz. having any Three of them given, to find the Fourth.*

### CASE I.

* *To find the Ordinate.*

As any diameter,
Is to its conjugate;
So is the mean proportional between the abſciſſes,
To the ordinate.

That is, $d : c :: \sqrt{x(d-x)} : \frac{c}{d}\sqrt{x(d-x)} = y$; putting $d$ for the diameter, $c$ its conjugate, and $x$ an abſciſs to the ordinate $y$.

EX-

* The values of the ſeveral quantities in the four caſes of this problem, are found from the general equation expreſſing the property of the curve, viz. $dd : cc :: x(d-x) : yy$.

EXAMPLE.

If the tranſverſe be 35, the conjugate 25, and the abſciſs 7; what is the ordinate.

Here $\frac{c}{d}\sqrt{x \times (d-x)} = \frac{25}{35}\sqrt{7 \times 28} = \frac{5}{7}\sqrt{7^2 \times 2^2}$ $= \frac{5 \times 7 \times 2}{7} = 10$ the ordinate.

CASE II.

*To find the Abſciſſes.*

As the conjugate,
Is to the diameter;
So is the ſquare root of the difference of the ſquares of the ordinate and ſemi-conjugate,
To the diſtance between the ordinate and center.

Then that diſtance being added to, and ſubtracted from, the ſemi-diameter, will give the two abſciſſes.

That is, $x = \frac{1}{2}d \pm \frac{d}{c}\sqrt{\frac{1}{4}cc - yy}$.

EXAMPLE.

What are the two abſciſſes of the ordinate 10, the diameters being 35 and 25?

Here $\frac{1}{2}d \pm \frac{d}{c}\sqrt{\frac{1}{4}cc - yy} = \frac{35}{2} \pm \frac{35}{25}\sqrt{(\frac{25}{2})^2 - 10^2} =$ $\frac{35}{2} \pm \frac{7}{2}\sqrt{5^2 - 4^2} = \frac{35 \pm 21}{2} = 28$ and 7, the two abſciſſes.

CASE III.

*To find the Conjugate.*

As the mean proportional between the abſciſſes,
Is to the ordinate;
So is the diameter,
To its conjugate.

That is, as $\sqrt{x(d-x)} : y :: d : c = \frac{dy}{\sqrt{dx-xx}}$.

EXAMPLE.

What is the conjugate to the diameter 35; the abſciſs to an ordinate of 10 being 7?

Here $\frac{dy}{\sqrt{(d-x)\times x}} = \frac{35\times 10}{\sqrt{28\times 7}} = \frac{350}{14} = 25$ the conjugate.

CASE IV.

*To find the Diameter.*

To or from the ſemi-conjugate, according as the leſs or greater abſciſs is uſed, add or ſubtract the root of the difference of the ſquares of the ordinate and ſemi-conjugate: Then, as the ſquare of the ordinate, is to the product of the conjugate and abſciſs; ſo is the ſum or difference above found, to the diameter.

That is, $d = \frac{cx}{yy} \times (\frac{1}{2}c \mp \sqrt{\frac{1}{4}cc - yy})$.

EXAMPLE.

If an ordinate and its leſs abſciſs be 10 and 7; what is the diameter, ſuppoſing the conjugate to be 25?

Here $d = \frac{cx}{yy} \times (\frac{1}{2}c + \sqrt{\frac{1}{4}cc - yy}) = \frac{25\times 7}{10\times 10} \times (\frac{25}{2} + \sqrt{(\frac{25}{2})^2 - 10^2}) = \frac{7}{4} \times \frac{25+15}{2} = 7\times 5 = 35$ the tranſverſe.

PRO-

## PROBLEM III.

### *To find the Circumference of an Ellipſe.*

### * RULE I.

Multiply the circumference of the circumſcribing circle by the ſum of the ſeries

$$1 - \frac{d}{2.2} - \frac{3dd}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6} - \frac{3.3.5.5.7d^4}{2.2.4.4.6.6.8.8} \text{ \&c,}$$

and the product will be the periphery of the ellipſe ſought; where $d$ is the difference between an unit and the ſquare of the quotient of the leſs axe divided by the greater.



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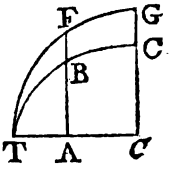
### * DEMONSTRATION.

Let $a$ be the ſemi-tranſverſe axe TC,
$c$ the ſemi-conjugate CC,
$y$ the ordinate AB,
$x$ its diſtance CA from the center,
$z$ the arc BC.




Then, ſince by the nature of the curve, $y$ is $= \frac{c}{a}\sqrt{aa - xx}$, $\dot{y}$ will be $= \frac{-c\dot{x}x}{a\sqrt{aa-xx}}$; and conſequently

$$\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{\dot{x}\sqrt{a^2 - \frac{a^2 - c^2}{aa} \times x^2}}{\sqrt{aa - xx}} = \frac{\dot{x}\sqrt{aa - dxx}}{\sqrt{aa - xx}}, \text{ by}$$

writing $d$ for $\frac{aa - cc}{aa}$ or $1 - \frac{cc}{aa}$, $= \frac{\dot{a}x\sqrt{1 - \frac{dxx}{aa}}}{\sqrt{aa - xx}} = \frac{\dot{a}x}{\sqrt{aa-xx}}$

$\times \left(1 - \frac{dx^2}{2a^2} - \frac{d^2x^4}{2.4a^4} - \frac{3d^3x^6}{2.4.6a^6} \text{ \&c}\right)$ by throwing the numerator into a ſeries.

But

EXAMPLE.

Required the periphery of an ellipſe whoſe axes are 24 and 18.

Here $1 - (\frac{18}{24})^2 = 1 - (\frac{3}{4})^2 = \frac{7}{16} = .4375 = d$. Then

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But the fluent of $\frac{\dot{a}\dot{x}}{\sqrt{aa - xx}}$ is $=$ the correſponding circular arc FG deſcribed with the center $c$ and radius $c$T, which call A: then by page 66 *Coteſii Harmonia Menſurarum*, the fluent comes out $A - \frac{d}{2a^2}B - \frac{d^2}{2.4a^4}C - \frac{3d^3}{2.4.6a^6}D$ &c for the length of the arc BC; where $B = \frac{aaA - x\sqrt{aa - xx}}{2}$, $C = \frac{3aaB - x^3\sqrt{aa - xx}}{4}$, $D = \frac{5aaC - x^5\sqrt{aa - xx}}{6}$, &c.

But when A arrives at T, $x$ is $= a$, and $\sqrt{aa - xx} = 0$; hence the values of the quantities B, C, D, &c, become barely $B = \frac{aa}{2}A$, $C = \frac{3aa}{4}B = \frac{3a^4}{2.4}A$, $D = \frac{5aa}{6}C = \frac{3.5a^6}{2.4.6}A$, &c. And conſequently the ſeries above becomes $A \times (1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \frac{3.3.5d^3}{2.2.4.4.6.6}$ &c) for $\frac{1}{4}$ of the elliptic periphery; where A is $\frac{1}{4}$ of the periphery of the circle, or $=$ the whole circumference of the ellipſe when A is that of the circle. *Q. E. D.*

*Corol.* 1. Hence the peripheries of ſimilar ellipſes are to each other as thoſe of their circumſcribed circles, or as any two ſimilar diameters of thoſe ellipſes.

*Corol.* 2. If $a$ be the leſs ſemi-axe, and $c$ the greater, the fluxion of the arc will be $\dot{z} = \dot{x}\sqrt{\frac{aa + dxx}{aa - xx}}$; and if $a = 1$, and $c = \sqrt{2}$, the ſame fluxion $\dot{z}$ will be

$$= \dot{x}\sqrt{\frac{1 + xx}{1 - xx}} = \frac{1 + xx}{\sqrt{1 - x^4}}\dot{x} = \frac{\dot{x}}{\sqrt{1 - x^4}} + \frac{x^2\dot{x}}{\sqrt{1 - x^4}}.$$

And if we write $v$ for $xx$, we ſhall have $\dot{z} = \frac{\dot{v}}{2\sqrt{v}}\sqrt{\frac{1 + v}{1 - v}}$, as Mr. *Landen* found in page 142 of his *Math. Lucub.*

3 Or,

Then the 2d term $A = \frac{d}{4} = \cdot 10938$

the 3d $B = \frac{3d}{4 \cdot 4} A = \cdot 00897$

the 4th $C = \frac{3 \cdot 5d}{6 \cdot 6} B = \cdot 00164$

the 5th $D = \frac{5 \cdot 7d}{8 \cdot 8} C = \cdot 00039$

the 6th $E = \frac{7 \cdot 9d}{10 \cdot 10} D = \cdot 00011$

the 7th $F = \frac{9 \cdot 11d}{12 \cdot 12} E = \cdot 00003$

the 8th $G = \frac{11 \cdot 13d}{14 \cdot 14} F = \cdot 00001$

their sum is ·12053,
which taken from the first term 1·00000,
leaves ·87947

for the value of the series; and being drawn into 3·1416 × 24, the periphery of the circumscribing circle, gives 66·31056 for the curve of the ellipse required.

* RULE II.

Multiply the sum of the two axes by 1·5706, or the half of 3·1416, and the product will be the circumference nearly.

T 3 That

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Or, by writing $w^n$ for $x$, we shall have universally $\dot{z} = n w^{n-1} \dot{w} \sqrt{\frac{1 + w^{2n}}{1 - w^{2n}}}$. Which theorems are of use in determining some fluents.

* DEMONSTRATION.

It will be evident that this rule is very near the truth, if it be considered that this arithmetical mean between the axes, exceeds

That is, $(t + c) \times \frac{1}{4}p$ = the circumference nearly; putting $t$ for the tranſverſe, $c$ for its conjugate, and $p$ for 3·1416.

EXAMPLE.

Let there be here taken the ſame example as before, in which the axes are 24 and 18.

Then $(24 + 18) \times 1{\cdot}5706 = 42 \times 1{\cdot}5706 = 65{\cdot}9736$ is the circumference nearly.

RULE

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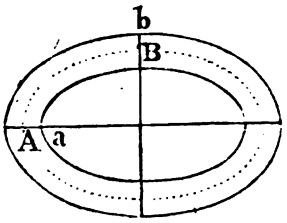
exceeds their geometrical mean; and that the geometrical mean (as will be proved hereafter) is the diameter of a circle equal in area to the ellipſe, which circle is of a leſs ambit than the ellipſe, or any other figure, of the ſame area.

*Otherwiſe.*




Let the dotted curve line repreſent the elliptic curve whoſe length is required. And let curve lines be deſcribed at ſmall and equal diſtances, within it, and without it, as in the annexed figure. Then theſe two curves will be very nearly ellipſes alſo; and the difference of their areas, computed as ellipſes, will be nearly the area of the elliptic ring, or ſpace contained between them, through the middle of which runs the ellipſe whoſe length is ſought; which area being alſo equal to the ſaid length of the ellipſe drawn into the breadth of the ſpace, or diſtance between the other two curves, it follows that the length of the elliptic curve will be nearly equal to the ſaid difference of areas divided by the ſaid breadth or diſtance. Therefore let $t$ = the tranſverſe of the ellipſe,

$c$ = its conjugate,

$a = 3{\cdot}1416$, and $d$ the ſmall ſemi-diſtance Aa or Bb.

Then $(t + 2d) \times (c + 2d) \times \frac{1}{4}a$ = area of the greateſt curve.

And $(t - 2d) \times (c - 2d) \times \frac{1}{4}a$ = area of the leaſt.

The dif. $(t + c) \times ad$ = the annular ſpace.

Therefore $(t + c) \times \frac{1}{4}a$ = the length, nearly.

* RULE III.

Multiply the ſquare root of $\frac{1}{2}$ the ſum of the ſquares of the two axes by 3·1416, and the product will be the circumference nearly.

That is, $p\sqrt{\frac{tt+cc}{2}}$ = the circumference nearly.

EXAMPLE.

Taking again the ſame example,

We have $\sqrt{\frac{24^2+18^2}{2}} \times 3{\cdot}1416 = 15\sqrt{2} \times 3{\cdot}1416$ $= 21{\cdot}2132 \times 3{\cdot}1416 = 66{\cdot}6433$ = the circumference nearly.



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* DEMONSTRATION.

Call the infinite ſeries in rule 1, s;

viz. $s = 1 - \frac{d}{2.2} - \frac{3d^2}{2.2.4.4} - \&c.$

Then ſince $\sqrt{1-\frac{1}{2}d}$ is $= 1 - \frac{d}{2.2} - \frac{d^2}{2^3.4} - \frac{3d^3}{2^4.4.6}$ &c.

we ſhall have $s - \sqrt{1-\frac{1}{2}d} = - \frac{d^2}{64} - \frac{3d^3}{256}$ &c.

Therefore, rejecting this laſt remaining ſeries, on account of its ſmallneſs, we have $s = \sqrt{1-\frac{1}{2}d}$ nearly; and conſequently the circumference of the ellipſe $c = c\sqrt{1-\frac{1}{2}d}$ nearly, $c$ being the circumference of the circumſcribed circle.

Or $c = pt\sqrt{1-\frac{1}{2}d} = pt\sqrt{1-\frac{1}{2}+\frac{cc}{2tt}} = pt\sqrt{\frac{1}{2}+\frac{cc}{2tt}} =$ $p\sqrt{\frac{tt+cc}{2}}$, which is the 3d rule, $p$ being = 3·14159 &c.

* RULE IV.

Find the ſquare root of $\frac{1}{2}$ the ſum of the ſquares of the axes as in the laſt rule, and call it A.

To 3 times the greater axe add its parameter, divide the ſum by 4, and call the quotient B.

Then multiply the difference between B and 3 times A, by 1·5708, and the product will be the periphery nearer than by the 3d rule.

That is, $\frac{1}{2}p \times (3\sqrt{\frac{tt+cc}{2}} - \frac{3t+p}{4}) =$ the periphery, $p$ being the parameter.

EXAMPLE.

Taking the ſame example as in the former rules,

We have $A = \sqrt{\frac{24^2+18^2}{2}} = 15\sqrt{2} = 21{\cdot}2132.$

As 24 : 18 :: 18 : $13\frac{1}{2}$ = the parameter, by the definition.

And

---

* Again, $s - 1 + \frac{1}{4}d = -\frac{3d^2}{64} - \frac{5d^3}{256}$ &c;

but $3s - 3\sqrt{1-\frac{1}{2}d} = -\frac{3d^2}{64} - \frac{9d^3}{256}$ &c;

theref. $2s - 3\sqrt{1-\frac{1}{2}d} + 1 - \frac{1}{4}d = -\frac{d^3}{64}$ &c.

hence $s = \frac{3\sqrt{1-\frac{1}{2}d} - 1 + \frac{1}{4}d}{2}$ nearly,

and ſo $c = \frac{1}{2}pt \times (3\sqrt{1-\frac{1}{2}d} - \frac{4-d}{4}) =$

$\frac{1}{2}p \times (3\sqrt{\frac{tt+cc}{2}} - \frac{3tt+cc}{4t}) = \frac{1}{2}p \times (3\sqrt{\frac{tt+cc}{2}} - \frac{3t+p}{4})$

nearly, where $p$ is the parameter of the axe. And this is the 4th rule.

And $\frac{3 \times 24 + 13\frac{1}{2}}{4} = 21\frac{3}{8} = 21{\cdot}375 =$ B.

Then $(3\text{A} - \text{B}) \times 1{\cdot}5708 = 42{\cdot}2646 \times 1{\cdot}5708 = 66{\cdot}3892$ the circumference nearly.

RULE V.*

Find A and B as in the laſt rule.

Divide the difference of the ſquares of the axes by the ſquare of the greater axe, multiply the ſquare of the quotient by $\frac{1}{16}$ of the greater axe, and call the reſult C.

Then from 5 times A take 3 times B, to the remainder add C, multiply the ſum by 1·5708, and the product will give the circumference of the ellipſe ſtill nearer.

*Or,*

From 5 times A, ſubtract $\frac{1}{16}$ of the ſum of 35 times the greater axe and 14 times its parameter, to the remainder add $\frac{1}{16}$ of the product ariſing from the multiplication of the ſaid parameter, by the quotient of the ſquare of the leſs axe divided by that of the greater, and multiply the ſum by 1·5708.

That

---

* Farther, $s - 1 + \frac{d}{4} + \frac{3d^2}{64} = -\frac{5d^3}{256}$ &c,

or $\frac{4}{5}s - \frac{4}{5} + \frac{d}{5} + \frac{3d^2}{80} = -\frac{d^3}{64}$,

which taken from the laſt approximation, and reduced, gives

$$s = \tfrac{1}{2} \times 5\sqrt{1 - \tfrac{1}{2}d} - 3 \times \frac{4-d}{4} + \frac{dd}{16};$$

and conſequently $c = \frac{1}{2}pt \times (5\sqrt{1 - \frac{1}{2}d} - 3 \times \frac{4-d}{4} + \frac{dd}{16})$

$= \frac{1}{2}p \times (5\sqrt{\frac{tt+cc}{2}} - 3 \times \frac{3t+p}{4} + (\frac{tt-cc}{tt})^2 \times \frac{t}{16})$

$= \frac{1}{2}p \times (5\sqrt{\frac{tt+cc}{2}} - \frac{35t+14p}{16} + \frac{pcc}{16tt})$. Which is the 5th rule.

And thus we may proceed to any degree of accuracy whatever,

That is, $\frac{1}{4}p \times [5\sqrt{\frac{tt+cc}{2}} - 3 \times \frac{3t+p}{4} + (\frac{tt-cc}{tt})^2 \times \frac{t}{16}]$, or $\frac{1}{4}p \times [5\sqrt{\frac{tt+cc}{2}} - \frac{35t+14p}{16} + \frac{pcc}{16tt}]$ = the periphery very nearly.

EXAMPLE.

Taking again the same example as before, we have

A = 21·2132,
B = 21·375,
C = ·2871.

Then $(5A - 3B + C) \times 1{\cdot}5708 = 42{\cdot}2281 \times 1{\cdot}5708$ = 66·3319 = the circumference very near.

Or, to use the latter part of the rule,

$(5A - \frac{35 \times 24 - 14 \times 13\frac{1}{2}}{16} + \frac{13\frac{1}{2} \times 18^2}{16 \times 24^2}) \times 1{\cdot}5708 =$ 42·2281 × 1·5708 = 66·3319, the same as before.

RULE VI.

Take $\frac{1}{2}$ the sum of the quantities in the second and third rules, and it will be the circumference very nearly.*

That is, $\frac{1}{4}p \times (\frac{t+c}{2} + \sqrt{\frac{tt+cc}{2}})$ = the periphery.

EXAMPLE.

Taking, still, the same example,

The number found by the 2d rule is 65·9736,
And by the 3d rule is 66·6433,

their sum is 132·6169,
the half of which is 66·3084,

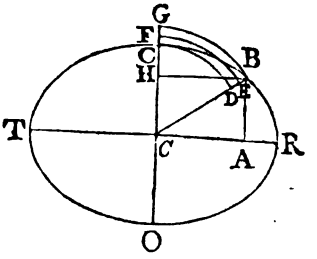
for the circumference, and is nearer to the true number found by rule 1 than any of the other approximations.

PRO-

* For the one being nearly as much too great as the other is too little, $\frac{1}{2}$ their sum must be near the truth.

PROBLEM IV.

*To find the Length of any Arc* BC *of an Ellipſe.*




* RULE I.

If $a$ denote the ſemi-tranſverſe $Tc$,
$c$ the ſemi-conjugate $cc$,
and $x$ the diſtance $cA$ of the ordinate AB from the center,
then will the arc CB be $= x \times$
$(1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6$ &c).

And the ſeries will be the ſame if $x$ be conſidered as an ordinate to either axe $2c$, the other axe being $2a$, the arc beginning at the vertex of the axe $2c$, and terminated by the ordinate $x$.

EX-

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* DEMONSTRATION.

By proceeding as in the inveſtigation of problem 3, we have the fluxion of the curve $\dot{z} = \dot{x}\sqrt{\frac{aa - dxx}{aa - xx}} = \dot{x} \times$
$(1 + \frac{c^2}{2a^4}x^2 + \frac{4a^2c^2 - c^4}{8a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{16a^{12}}x^6$ &c),
reſtoring $c$ and exterminating $d$; and the fluent is
$z = x \times (1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6$ &c).

*Corol.*

EXAMPLE.

If the two axes be 24 and 18, and the diſtance $cA$ 3; required the length of the arc BC.

Here $a = 12$, $c = 9$, and $x = 3$.

| | | |
|---|---|---|
| Then the 1ſt term $=$ | — — | 1·00000 |
| 2d term $A = \frac{c^2}{6a^4}x^2 =$ | — — | 0·00586 |
| 3d term $B = 3 \times \frac{4a^2 - c^2}{20a^4}x^2 A =$ | | 0·00019 |
| 4th term $C = \frac{5}{14a^4} \times \frac{8a^4 - 4a^2c^2 + c^4}{4a^2 - c^2}x^2 B =$ | | 0·00001 |
| | the ſum | 1·00606 |
| | mult. by | 3 |
| | gives the arc BC | 3·01818 |

* RULE II.

Find the length of the circular arc FE intercepted by $cC$, $cB$, and whoſe radius is half the ſum of the lines

---

*Corol.* 1. When $x$ is $= a$, the rule becomes

$$a \times (1 + \frac{c^2}{6a^2} + \frac{4a^2c^2 - c^4}{40a^4} + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^6} \text{ \&c})$$

for one fourth of the periphery of the ellipſe. *Q. E. D.*

*Corol.* 2. By ſuppoſing $a = c$, the ellipſe will become a circle, and the general ſeries for the elliptic arc will become for the circular arc

$$x \times (1 + \frac{x^2}{6a^2} + \frac{3x^4}{40a^4} + \frac{5x^6}{112a^6} \text{ \&c}), =$$

$$x \times (1 + \frac{x^2}{2 \cdot 3a^2} + \frac{3x^4}{2 \cdot 4 \cdot 5a^4} + \frac{3 \cdot 5x^6}{2 \cdot 4 \cdot 6 \cdot 7a^6} \text{ \&c}),$$

the ſame as was determined at page 88.

* For the circular arc FE is equal to the arithmetical mean between the circular arcs BG, CD, deſcribed with the radii $cB$, $cC$;

lines $cc$, $cB$; and it will be equal to the elliptic arc BC nearly.

EXAMPLE.

Taking here the same example,

We have AR $= 9$, and AT $= 15$.

Hence $Tc^2 : cc^2 :: TA \times AR : AB^2 = \frac{9^2 \times 9 \times 15}{12 \times 12} = \frac{9 \times 9 \times 15}{16}$,

and $cB = \sqrt{cA^2 + AB^2} = \sqrt{9 + \frac{9 \times 9 \times 15}{16}} = \frac{3}{4}\sqrt{151}$
$= 9{\cdot}21616$.

Then $\frac{1}{2}cc + \frac{1}{2}cB = 9{\cdot}10808 =$ the radius of the circular arc EF.

But $HB \div cB = 3 \div 9{\cdot}21616 = {\cdot}325515$ is the sine of the angle $ccB$ or arc FE to the radius 1, answering to $18{\cdot}9968$ degrees.

Therefore, by rule 1 prob. 6 sect. 1 part 2,

${\cdot}01745 \times 18{\cdot}9968 \times 9{\cdot}10808 = 3{\cdot}0192$ is the circular arc EF, or elliptic arc BC, nearly.

* RULE III.

Divide the difference of the squares of the semiaxes, by the square of the greater, and call the quotient $q$.

Then

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$cc$; to which it is evident the elliptic arc must be nearly equal; and so much the nearer, the less the arc itself is.

This rule will also be the more exact, the nearer the axes of the ellipse approach towards an equality.

Rule 2 of the last problem is a particular case of this general rule.

* DEMONSTRATION.

By the first rule the arc A is equal to

$$x \times \left(1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6 \text{ \&c}\right).$$

But

Then divide the difference between the ſquare of the greater ſemi-axe, and the product of $\frac{1}{3}$ of the ſquare of the diſtance $cA$ of the ordinate from the center multiplied by $q$, by the difference between the ſquare of the ſaid greater ſemi-axe and $\frac{1}{3}$ of the ſquare of the ſaid diſtance; and the root of the quotient multiplied by the ſaid diſtance, will be the arc nearly.

That is, $x\sqrt{\frac{aa-\frac{1}{3}qxx}{aa-\frac{1}{3}xx}}$ = the arc;

where $a$ = $Tc$ the greater ſemi-axe,

$x$ = $cA$, and $q = \frac{aa-cc}{aa} = 1-\frac{cc}{aa}$,

$c$ being the leſs ſemi-axe $cC$.

EXAMPLE.

Taking again the ſame example,

We have $1-\frac{cc}{aa} = 1-(\frac{9}{12})^2 = 1-(\frac{3}{4})^2 = 1-\frac{9}{16} = \frac{7}{16} = q$.

And $x\sqrt{\frac{aa-\frac{1}{3}qxx}{aa-\frac{1}{3}xx}} = 3\sqrt{\frac{12^2-\frac{1}{3}\times\frac{7}{16}\times 3^2}{12^2-\frac{1}{3}\times 3^2}} = 3{\cdot}017898$ = the arc required nearly.

RULE

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But in the inveſtigation of that rule it appears that $\sqrt{\frac{aa-dxx}{aa-xx}}$ is $= 1+\frac{c^2}{2a^4}x^2+\frac{4a^2c^2-c^4}{8a^8}x^4+\frac{8a^4c^2-4a^2c^4+c^6}{16a^{12}}x^6$ &c; hence it is evident that $x\sqrt{\frac{aa-\frac{1}{3}dxx}{aa-\frac{1}{3}xx}}$ is $= x\times(1+\frac{c^2}{6a^4}x^2+\frac{4a^2c^2-c^4}{72a^8}x^4+\frac{8a^4c^2-4a^2c^4+c^6}{432a^{12}}x^6$ &c).

And conſequently $A = x\sqrt{\frac{aa-\frac{1}{3}dxx}{aa-\frac{1}{3}xx}}$ nearly. *Q. E. D.*

*Corollary.* When the two axes are equal to each other, the ellipſe will become a circle, $d$ will be $= 0$, and the rule will become $x\sqrt{\frac{aa}{aa-\frac{1}{3}xx}}$ for the circular arc.

* RULE IV.

Call the quantity found by the laſt rule B.

Multiply the diſtance $cA$ by the leſs ſemi-axe, divide the product by the ſquare of the greater, to $\frac{1}{6}$ of the ſquare of the quotient add 1, multiply the ſum by the diſtance $cA$, and call the product C.

Then from 9 times B take 4 times C, and $\frac{1}{5}$ of the remainder will be the arc very near.

That is,

$$A = \frac{9B - 4C}{5} = \tfrac{1}{5} \times \left[9x\sqrt{\frac{aa - \frac{1}{3}qxx}{aa - \frac{1}{3}xx}} - \left(1 + \frac{c^2x^2}{6a^4}\right) \times 4x\right].$$

EXAMPLE.

Taking ſtill the ſame example,

The Quantity found by the laſt rule is $3{\cdot}017898 = B$.

And

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* DEMONSTRATION.

For ſince $A = x \times \left(1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{40a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{112a^{12}}x^6 \&c\right)$,

and $B = x \times \left(1 + \frac{c^2}{6a^4}x^2 + \frac{4a^2c^2 - c^4}{72a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{432a^{12}}x^6 \&c\right)$;

We have $A - B = x \times \left(\frac{4a^2c^2 - c^4}{90a^8}x^4 + \frac{40a^4c^2 - 20a^2c^4 + 5c^6}{756a^{12}} \&c\right)$.

But

$$\tfrac{4}{5}\left[B - \left(1 + \frac{c^2}{6a^4}x^2\right)x\right] = x \times \left(\frac{4a^2c^2 - c^4}{90a^8}x^4 + \frac{8a^4c^2 - 4a^2c^4 + c^6}{540a^{12}}x^6 \&c\right).$$

Therefore $A - B - \frac{4}{5}\left[B - \left(1 + \frac{c^2x^2}{6a^4}\right)x\right] = x \times \left(\frac{8a^4c^2 - 4a^2c^4 + c^6}{210a^{12}}x^6 \&c\right)$.

And $A = \frac{1}{5}\left[9B - \left(1 + \frac{c^2x^2}{6a^4}\right) \times 4x\right] =$

$$\tfrac{1}{5}x \times \left[9\sqrt{\frac{aa - \frac{1}{3}dxx}{aa - \frac{1}{3}xx}} - \left(1 + \frac{c^2x^2}{6a^4}\right)\right] \text{ nearly.} \quad Q. E. D.$$

*Corollary.* When $a = c$, the rule becomes

$$\tfrac{1}{5}x \times \left(9\sqrt{\frac{aa}{aa - \frac{1}{3}xx}} - 4 \times \frac{6aa + xx}{6aa}\right)$$

for the circular arc.

And $x \times (1 + \frac{c^2x^2}{6a^4}) = 3 \times (1 + \frac{9^2 \times 3^2}{6 \times 12^4}) = 3 \times (1 + \frac{3}{2 \times 4^4}) = 3{\cdot}017578 = c.$

Then $\frac{9B - 4C}{5} = \frac{27{\cdot}161082 - 12{\cdot}070212}{5} = \frac{15{\cdot}09077}{5} = 3{\cdot}018154 =$ the arc very nearly.

RULE V.

As the ſum of 15 times the parameter of the axe, at whoſe vertex the arc begins, and a third proportional to the ſaid axe, the abſciſs, and the difference between 9 times the ſaid axe and 21 times its parameter; is to the ſum of 15 times the parameter, and a third proportional to the axe, the abſciſs, and the difference between 19 times the axe and 21 times its parameter; ſo is the ordinate, to the length of the arc, very nearly.*

That

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* DEMONSTRATION.

Since $y$ is $= c - \frac{c\sqrt{aa - xx}}{a} = \frac{cx^2}{2a^2} + \frac{cx^4}{8a^4}$ &c, it is evident that a fraction of this form $x \times \frac{A + (B+1)y}{A + By}$, when expanded in a ſeries, will produce a ſeries of the ſame form with that expreſſing the length of the arc in the firſt rule, and which being put equal to it, will afford equations for determining the values of A and B.

Thus $x \times \frac{A + (B+1)y}{A + By}$ is $= x \times \frac{A + (B+1) \times (\frac{cx^2}{2a^2} + \frac{cx^4}{8a^4} \&c)}{A + B \times (\frac{cx^2}{2a^2} + \frac{cx^4}{8a^4} \&c)}$

$= x \times (1 + \frac{cx^2}{2Aa^2} + \frac{Ac - 2Bc^2}{8A^2a^4}x^4$ &c), which put

$= x \times (1 + \frac{c^2x^2}{6a^4} + \frac{4a^2c^2 - c^4}{8a^8}x^4$ &c).

Then

That is, $\dfrac{15p + \frac{19c - 21p}{c} \times y}{15p + \frac{9c - 21p}{c} \times y} \times x =$ the arc CB,

where c = the whole axe CO where the arc begins,
$p$ = its parameter = $TR^2 \div CO$,
$y$ = the abscis CH,
$x$ = the ordinate HB.

EXAMPLE.

Taking, still, the same example,

In which $c = 9$, $a = 12$, c = 18, $p = 12 \times 24 \div 9 = 32$, and $x = 3$.

U Hence

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Then, by equating the corresponding terms, we obtain

$\frac{c}{2Aa^2} = \frac{c^2}{6a^4}$, or $A = \frac{3a^2}{c} = \frac{3}{2}p$; and $\frac{Ac - 2Bc^2}{8A^2a^4} = \frac{4a^2c^2 - c^4}{8a^8}$,

hence $B = \dfrac{9c^2 - 21a^2}{10cc} = \dfrac{9c - \frac{21aa}{c}}{10c} = \dfrac{9c - 21p}{10c}$,

putting c = $2c$, and $p$ = its parameter $\frac{2aa}{c}$.

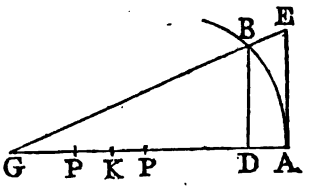
Consequently

$$x \times \frac{A + (B+1)y}{A + By} \text{ is} = x \times \frac{\frac{3}{2}p + \frac{19c - 21p}{10c}y}{\frac{3}{2}p + \frac{9c - 21p}{10c}y} = x \times \frac{15p + \frac{19c - 21p}{c}y}{15p + \frac{9c - 21p}{c}y}$$

*Q. E. D.*

*Corol.* 1. Hence in the ellipse AB, whose vertex is A, either axe AK, and latus rectum AP; if there be drawn the ordinate BD, and GP be taken =

$\frac{1}{2}PA + \dfrac{19AK - 21AP}{10AK} \times AD$,




and there be drawn the right line GBE meeting the tangent AE in E; then will AE be = the arc AB, very nearly.

For,

Hence $y = c - \frac{c\sqrt{aa-xx}}{a} = 9 - \frac{9\sqrt{12^2-3^2}}{12}$

$= 9 - \frac{9\sqrt{15}}{4} = \cdot 2857874715.$

Then

$$x \times \frac{15p + \frac{19c-21p}{c}y}{15p + \frac{9c-21p}{c}y} = 3 \times \frac{15\times 32 + \frac{19\times 18 - 21\times 32}{18}y}{15\times 32 + \frac{9\times 18 - 21\times 32}{18}y}$$

$= 3 \times 1\cdot 0060 56 = 3\cdot 018168 =$ the length of the arc very nearly.

PRO-

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For, by similar triangles, GD : DB :: GA : AE $= \frac{AG \times BD}{DG} =$

$$\frac{AP+PG}{AP+PG-AD} \times DB = \text{(by the const.)} \frac{\frac{3}{2}AP + \frac{19AK-21AP}{10AK} \times AD}{\frac{3}{2}AP + \frac{9AK-21AP}{10AK} \times AD} \times DB;$$

which is the arc, nearly, by the above demonstration.

And this proves the truth of the construction given by *Sir I. Newton*, in his letter of June 13, 1676, to Mr. *Oldenburg*. But Mr. *Jones*, in publishing this letter, has printed AP instead of AD.

Or the same rule may be otherwise expressed, like rule 3 for the hyperbolic arc.

*Corol.* 2. When the ellipse becomes a circle, then AK = AP, and the above rule will become $\frac{15AP - 2AD}{15AP - 12AD} \times DB$ for the length of the circular arc, whose diameter is AP, versed sine AD, and right sine DB. And this is nearer the truth than any of the other approximations, before given for the circular arc, except that in rule 6 for that purpose; for the example in that rule, calculated by this approximation, gives 6·1170598 for the length of the arc. And thus, by making AP = AK, all the rules for the ellipse will be adapted to the circle.

*Corol.* 3. The same rules for the ellipse may also be easily adapted to the other two conic sections, the hyperbola and parabola; namely, by only changing the sign of the axe c or AK for the hyperbola, and by making that axe infinite for the parabola.

## PROBLEM V.

### *To find the Area of an Ellipſe.**

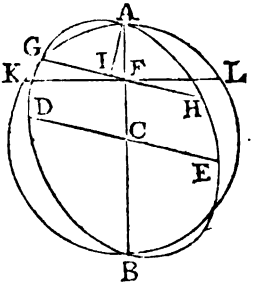
RULE I.

Multiply continually together the two axes and the number ·7854, for the area of the ellipſe.

That is, $ntc$ = the area; putting $t$ = the tranſverſe, $c$ = the conjugate, and $n$ = ·7854. By corollary 4.



* GENERAL INVESTIGATION.




Let AE, DE, be any two conjugate diameters of the ellipſe ADBEA, GH a double ordinate to the diameter AB, AKEL a circle whoſe diameter is AB, KL a double ordinate of the circle to the diameter AB, and AI perpendicular to GH.

Now whilſt KL, by flowing, generates the circular ſegment KAL, GH will deſcribe the elliptic ſegment GAH; but the velocity of KL is to that of GH, as AF to AI; and by the property of all conjugate diameters, AC : CD :: ($\sqrt{AF \times FE}$ =, by the nature of the circle,) KF : FG; therefore as the circular ſegment KAL is to the elliptic ſegment GAH, ſo is AC × AF to CD × AI, and ſo is AC × radius to CD × ſine ∠AFI, and ſo is AC to CD × ſine ∠C; putting 1 for the radius.

*Cor. l.* 1. The whole ellipſe and circle, deſcribed upon any diameter of it, are to each other as the correſponding ſegments cut off them by their particular double ordinates GH, KL, paſſing through the ſame point of that diameter.

*Corol.* 2. As radius is to the ſine of the angle made by any two conjugate diameters, ſo is a mean proportional between the two circles deſcribed upon thoſe diameters, to the ellipſe,

For $\left\{\begin{matrix} r \times d : s \times c :: \odot d : \bigcirc \\ r \times c : s \times d :: \odot c : \bigcirc \end{matrix}\right\}$ by cor. 1.

Hence $r^2 cd : s^2 cd :: r^2 : s^2 :: \odot d \times \odot c : \bigcirc^2$,

And $r : s :: \sqrt{\odot d \times \odot c} : \bigcirc$.

Where $d$ and $c$ are the diameters, $s$ the ſine of the angle made by them to the radius $r$, $\odot$ denotes a circle, and $\bigcirc$ an ellipſe.

*Corol.*

EXAMPLE.

If the axes of an ellipſe be 35 and 25, what is the area?

$.7854 \times 35 \times 25 = 687.225 =$ the area required.

RULE II.

Multiply continually together any two conjugate diameters, the natural ſine of their included angle, and the number .7854.

That is, $dcns =$ the area;
putting $d$ and $c =$ any two conjugate diameters, $s =$ ſine of their included angle, and $n = .7854$.

EX-

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*Corol.* 3. If the conjugate diameters be equal to each other, it will follow that, As radius is to the ſine of the angle made by the equal conjugate diameters; ſo is the circle deſcribed on one of thoſe diameters, to the ellipſe.

*Corol.* 4. The ellipſe is a mean proportional between the two circles deſcribed on the two axes.—For they make with each other a right angle, whoſe ſine is = to the radius.

*Corol.* 5. As radius is to the ſine of the angle made by any two conjugate diameters; ſo is the circle whoſe diameter is a mean proportional between the conjugate diameters, to the ellipſe.—This follows from corollary 2.

*Corol.* 6. The ellipſe is equal to a circle whoſe diameter is a mean proportional between the two axes.——From corollary 4.

*Corol.* 7. As an ellipſe : is to the rectangle of its two axes, or to the rectangle of any two conjugate diameters drawn into the ſine of their included angle, the radius being 1 :: ſo is any circle : to the ſquare of its diameter.——Any two like ſegments or zones of the ellipſe and circle are alſo in the ſame proportion.

*Corol.* 8. Ellipſes, and their like ſegments, are to one another as the rectangles of their axes, or as the rectangles of any conjugate diameters forming the ſame angle in each.

*Corol.* 9. Similar ellipſes are to one another as the ſquares of their like diameters.

*Corol.*

EXAMPLE.

If two conjugate diameters of an ellipſe be 28 and 32, and their included angle 77° $34\frac{1}{4}'$; required its area.

The ſine of 77° $34\frac{1}{4}'$ is ·9765625; therefore ·9765625 × 32 × 28 × ·7854 = 687·225 = the area.

PROBLEM VI.

*To find the Area of the Segment of an Ellipſe cut off by a Double Ordinate to either Axe, that is, by a Line Perpendicular to that Axe.*

RULE I.

Find the correſponding ſegment of the circle deſcribed upon the ſame axe to which the cutting line, or baſe of the ſegment, is perpendicular.



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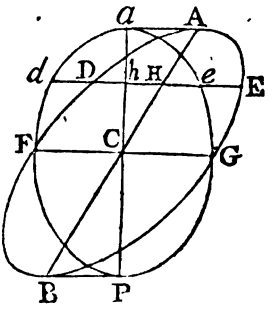
*Corol.* 10. From corollary 7 comes alſo the following conſtruction.




Let ADE be an oblique ſegment of the ellipſe AFBGA, cut off by a double ordinate to the diameter AB, FG being the conjugate. Through the center C draw *a*P perpendicular to FG and meeting A*a* and BP, in *a* and P, both parallel to FG; then about the axes *a*P, FG, deſcribe the ellipſe *a*FPG*a*, meeting the ordinate produced in *e* and *d*. Then will the right elliptic ſegment *dae* be equal to the oblique ſegment DAE, as well as the whole ellipſe *a*FPG*a* equal to the ellipſe AFBGA; moreover their correſponding ordinates *de*, DE, parallel to the common diameter FG, are every where equal, as are the like parts or zones contained between any two of ſuch ordinates. And the ſame may be ſaid of all ellipſes contained between the parallels A*a*, BP, infinitely produced: in which property they reſemble parallelograms of the ſame baſe and between the ſame parallels.

Then as this axe : is to the other axe :: ſo is the circular ſegment : to the elliptic ſegment.

This follows from corollary 7 to the laſt problem.

### RULE II.

Find the tabular circular ſegment whoſe verſed ſine or height is equal to the quotient of the height of the elliptic ſegment divided by its axe. Then multiply continually together this ſegment and the two axes of the ellipſe, for the area of the ſegment required.

This rule follows from the former.

### EXAMPLE.

What is the area of an elliptic ſegment cut off by a line parallel to, and at the diſtance of, $7\frac{1}{2}$ from the leſs axe, the axes being 35 and 25?

Here $17\frac{1}{2} - 7\frac{1}{2} = 10 =$ the height of the ſegment.

And $10 \div 35 = 2 \div 7 = \cdot 2857\frac{1}{7} =$ the tabular verſed ſine; whoſe ſegment is $\cdot 1851669$.

Then $\cdot 1851669 \times 35 \times 25 = 162 \cdot 0210375 =$ the area of the leſs ſegment.

If the greater ſegment had been required,

Then $\cdot 78539816 - \cdot 1851669 = \cdot 60023126$.

And $\cdot 60023126 \times 25 \times 35 = 525 \cdot 2023525 =$ the area of the greater ſegment.

### EXAMPLE II.

What is the area of the elliptic ſegment cut off by a double ordinate perpendicular to the conjugate axe at the diſtance of $7\frac{1}{2}$ from the center, the axes being 35 and 25?

Here $12\frac{1}{2} - 7\frac{1}{2} = 5 =$ the altitude of the ſegment.

And $5 \div 25 = 1 \div 5 = \cdot 2 =$ the tabular verſed ſine; whoſe correſponding ſegment is $\cdot 1118238$.

Hence $\cdot 1118238 \times 25 \times 35 = 97 \cdot 845825 =$ the area of the leſs ſegment.

Again,

Again, ·78539816 — ·1118238 = ·67357436.
And ·67357436 × 25 × 35 = 589·377565 = the greater segment.

### PROBLEM VII.

*To find the Area of an Elliptic Segment cut off by a Double Ordinate to any Diameter; that is, by a Line Oblique to the Axes.*

Divide the abscis AF (fig. to prob. 5) of the double ordinate, by its diameter AB, and find the tabular circular segment whose versed sine is the quotient.—Then multiply continually together the tabular area and the two axes. Or the tabular area, the diameter AB to which the base of the segment is a double ordinate, its conjugate diameter GH, and the sine of their included angle, for the area of the elliptic segment required.*

### EXAMPLE.

The axes of an ellipse being 35 and 25, it is required to find the area of a segment whose base is a double ordinate to a diameter whose length is 33, it being divided by the double ordinate into the two abscisses 7 and 26.

Here FA ÷ AB = 7 ÷ 33 = ·2121$\frac{7}{33}$ = the tabular versed sine; to which corresponds the area ·12162869.



* DEMONSTRATION.

For, by cor. 7 prob. 5, as $AB^2$ : to circular segment LAK :: so is AB × DE × s. ∠C = rectangle of the two axes : to elliptic segment GAH; but circular segment LAK = $AB^2$ × tabular circular segment *s* whose versed sine is FA ÷ AB; therefore $AB^2$ : $AB^2 \times s$ :: 1 : *s* :: rectangle of the axes to elliptic segment. *Q. E. D.*

Hence ·12162869 × 25 × 35 = 106·425104 = the ſegment GAH.

Moreover, ·78539816 — ·12162869 = ·66376947. And ·66376947 × 25 × 35 = 580·79828625 = the greater ſegment GBH.

### PROBLEM VIII.

*To find the Trilineal Area* ABQ, *included by either Axe, a Line drawn from any Point in it, and their Intercepted Arc.*

Draw the ordinate DB meeting the circle deſcribed upon the ſaid axe in C, and draw AC, OC.*

Then

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* DEMONSTRATION.

For, by the inveſtigation and corollaries to prob. 5,

As the axe A : to the axe *a* :: DC : DB
:: circular ſegment DCQ : elliptic ſegment DBQ
:: (becauſe of the common baſe AD) triangle ACD : triangle ABD
:: (by compoſition) the trilineal ACQ : trilineal ABQ. *Q. E. D.*

*Corol.* 1. Since, by rule 2 prob. 6 ſect, 1 part 2, putting OQ = *r*, and DC = *y*,

the circ. arc QC is $= y \times (1 + \frac{y^2}{3 \cdot 2r^2} + \frac{3y^4}{5 \cdot 2 \cdot 4r^4} + \frac{3 \cdot 5y^6}{7 \cdot 2 \cdot 4 \cdot 6r^6}$ &c),

we have the ſect. OCQ $= \frac{1}{2}ry \times (1 + \frac{y^2}{3 \cdot 2r^2} + \frac{3y^4}{5 \cdot 2 \cdot 4r^4} + \frac{3 \cdot 5y^6}{7 \cdot 2 \cdot 4 \cdot 6r^6}$ &c);

which being increaſed or diminiſhed by $\frac{1}{2}$OA × DC = the triangle ACO, we ſhall have

$$\frac{1}{2}y \times (\text{AQ} + \frac{y^2}{3 \cdot 2r} + \frac{3y^4}{5 \cdot 2 \cdot 4r^3} + \frac{3 \cdot 5y^6}{7 \cdot 2 \cdot 4 \cdot 6r^5} \text{ \&c}),$$

for the general value of the trilineal ACQ; and conſequently

$$\frac{cy}{2r} \times (\text{AQ} + \frac{y^2}{3 \cdot 2r} + \frac{3y^4}{5 \cdot 2 \cdot 4r^3} + \frac{3 \cdot 5y^6}{7 \cdot 2 \cdot 4 \cdot 6r^5} \text{ \&c})$$

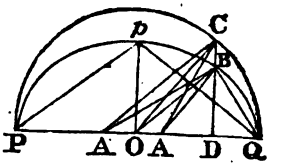
for the elliptic trilineal ABQ; *r* being the radius of the circle, or ſemi-axe OQ of the ellipſe, and *c* the other ſemi-axe.

And ſince *r* : *c* :: *y* : BD = *z*, we ſhall have $y = \frac{rz}{c}$, and conſequently the value of the elliptic trilineal ABQ, expreſſed in terms of its own ordinate *z* or BD, and ſemi-axes *r*, *c*, will be

$$\frac{1}{2}z \times (\text{AQ} + \frac{rz^2}{3 \cdot 2c^2} + \frac{3rz^4}{5 \cdot 2 \cdot 4c^4} + \frac{3 \cdot 5rz^6}{7 \cdot 2 \cdot 4 \cdot 6c^6} \text{ \&c}).$$

*Corol.*

Then, As the ſaid axe :
Is to the other axe : :
So the circ. trilineal ACQ :
To the elliptic trilin. ABQ.




*Note.* It is evident that the triangle ACO added to, or ſubtracted from, the circular ſector OCQ, will give the circular trilineal ACQ, according as AQ is greater or leſs than half the axe.

Or the ſemi-ſegment CDQ increaſed or diminiſhed by the triangle ACD, will give the ſame.

EX-

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*Corol.* 2. If A coincide with O, the triangle AOB will vaniſh, and the trilineal become barely a ſector OBQ, whoſe value will be

$$\tfrac{1}{2}rz \times \left(1 + \frac{z^2}{3 \cdot 2c^2} + \frac{3z^4}{5 \cdot 2 \cdot 4c^4} + \frac{5 \cdot 3z^6}{7 \cdot 2 \cdot 4 \cdot 6c^6} \&c\right).$$

*Corol.* 3. When $z = c$, then the quadrant $Opq$ will be

$$\tfrac{1}{2}rc \times \left(1 + \frac{1}{3 \cdot 2} + \frac{3}{5 \cdot 2 \cdot 4} + \frac{5 \cdot 3}{7 \cdot 2 \cdot 4 \cdot 6} \&c\right).$$

*Corol.* 4. If A coincide with P, then $AQ = QP = 2r$, and the trilineal BPQ will be $= \tfrac{1}{2}rz \times \left(2 + \frac{z^2}{3 \cdot 2c^2} + \frac{3z^4}{5 \cdot 2 \cdot 4c^4} + \frac{5 \cdot 3z^6}{7 \cdot 2 \cdot 4 \cdot 6c^6} \&c\right)$.

And when B coincides with $p$, then will the trilineal $Ppq$ be $= \tfrac{1}{2}rc \times \left(2 + \frac{1}{3 \cdot 2} + \frac{3}{5 \cdot 2 \cdot 4} + \frac{5 \cdot 3}{7 \cdot 2 \cdot 4 \cdot 6} \&c\right)$.

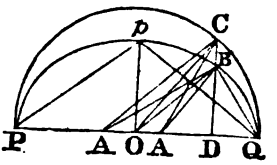
*Corol.* 5. If from the double of the quadrant in corollary 3, be taken the trilineal in corol. 4, there will remain the ſeg. $Ppb = r \times \left(\frac{c - z}{1} + \frac{2c^3 - z^3}{6 \cdot 2c^2} + 3\frac{2c^5 - z^5}{10 \cdot 2 \cdot 4c^4} + 5 \cdot 3\frac{2c^7 - z^7}{14 \cdot 2 \cdot 4 \cdot 6c^6} \&c\right)$.

*Corol.* 6. If from the value of the ſector OBQ in corollary 2, be taken that of the triangle $OBQ = \tfrac{1}{2}rz$, there will remain the ſegment $BQ = \tfrac{1}{2}rz \times \left(\frac{z^2}{3 \cdot 2c^2} + \frac{3z^4}{5 \cdot 2 \cdot 4c^4} + \frac{5 \cdot 3z^6}{7 \cdot 2 \cdot 4 \cdot 6c^6} \&c\right)$.

*Corol.*

EXAMPLE.

Given the axes 35 and 25 of an ellipſe, and an ordinate to the tranſverſe 10; required the area included by the tranſverſe and a line drawn from the top of the ordinate to cut the tranſverſe in an angle of 40 degrees.




Here, as $2p = 25$ : PQ $=$ $35$ :: DB $=$ $10$ : DC $=$ $14$.

As rad. : tang. $\angle$B ($50^\circ$) :: DB : DA $=$ $11{\cdot}917536$.

Hence AD $\times$ $\frac{1}{2}$DC $=$ $83{\cdot}422752$ $=$ the $\triangle$ADC.

As

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*Corol.* 7. When B coincides with $p$, the ſeries in each of the two laſt corollaries becomes, for the value of each of the ſegments $Pp$, $pQ$, $\frac{1}{2}rc \times \left(\frac{1}{3\cdot 2} + \frac{3}{5\cdot 2\cdot 4} + \frac{5\cdot 3}{7\cdot 2\cdot 4\cdot 6}\ \&c\right)$.

*Corol.* 8. If A be the focus of the ellipſe, then

AQ $= r \pm \sqrt{rr - cc}$, and the trilineal

$$ABQ = \tfrac{1}{2}z \times \left(r \pm \sqrt{rr-cc} + \frac{rz^2}{3\cdot 2c^2} + \frac{3rz^4}{5\cdot 2\cdot 4c^4} + \frac{3\cdot 5rz^6}{7\cdot 2\cdot 4\cdot 6c^6}\ \&c\right),$$

$$= \tfrac{1}{2}rz \times \left(1 \pm \sqrt{1 - \frac{cc}{rr}} + \frac{z^2}{2\cdot 3c^2} + \frac{3z^4}{2\cdot 4\cdot 5c^4} + \frac{3\cdot 5z^6}{2\cdot 4\cdot 6\cdot 7c^6}\ \&c\right).$$

*Corol.* 9. Putting $T = ABQ = \frac{1}{2}rz \times \left(a + \frac{z^2}{2\cdot 3c^2} + \frac{3z^4}{2\cdot 4\cdot 5c^4}\ \&c\right)$

$a$ being $= 1 \pm \sqrt{1 - \frac{cc}{rr}}$, and $q = \frac{2T}{r} = z \times \left(a + \frac{z^2}{2\cdot 3c^2} + \frac{3z^4}{2\cdot 4\cdot 5c^4}\ \&c\right)$

we ſhall have, by reverſion,

$$z = \frac{q}{a} - \frac{q^3}{6c^2a^4} + \frac{10-9a}{120c^4a^7}q^5 - \frac{280 - 504a + 225a^2}{5040c^6a^{10}}q^7\ \&c,$$

$$= \frac{2T}{ra} \times \left(1 - \frac{1}{6a^3}p^2 + \frac{10-9a}{120a^6}p^4 - \frac{280 - 504a + 225a^2}{5040a^9}p^6\ \&c\right)$$

where $p = \frac{2T}{rc}$; which ſeries generally converges very faſt, and affords a good direct ſolution of *Kepler*'s problem.

Suppoſe, for example, it were required to find the true anomaly of the earth, and its diſtance from the ſun, anſwering to the

As $25 : 35 :: \sqrt{12{\cdot}5^2 - 10^2} : 10{\cdot}5 =$ OD;

And hence OQ $-$ OD $= 7 =$ DQ. Then, by the table of circular ſegments, the ſemi-ſegment CDQ will come out 68·4920775.

The triangle ADC $+$ ſegment CDQ $= 151{\cdot}9143395$ $=$ the trilineal ACQ.

Theref. as $35 : 25 :: 151{\cdot}9143395 : 108{\cdot}5102425$ $=$ the area of the elliptic trilineal ABQ required.

PRO-

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the mean anomaly of 1 ſign or 30 degrees, the excentricity being ·01681 to the mean diſtance 1.

Here PO $= r = 1$, AO $=$ ·01681, Op $= c = \sqrt{1 - {\cdot}01681^2} =$ ·9998587, the area of a quadrant of the ellipſe $=$ ·785398 &c $\times rc =$ ·7852872; then, as 3 ſigns : 1 ſign :: ·7852872 : ·2617624 $=$ T; hence $\frac{2T}{r} = 2T =$ ·5235248, and $\frac{2T}{rc} = \frac{2T}{c} = \frac{2}{3} \times$ ·78539 &c $=$ ·5235988 $= p$:

Alſo A $= 1{\cdot}01681$, and $\frac{p^2}{a^3} = 2589869$.

Then, 1ſt term A $= \frac{1}{a} =$ — — — $+{\cdot}98346790$

2d — B $= \frac{1}{6}$A $\times \frac{p^2}{a^3} =$ — — $-{\cdot}04245088$

3d — C $= \frac{10 - 9a}{20}$B $\times \frac{p^2}{a^3} =$ — $+{\cdot}00046655$

4th — D $= \frac{280 - 504a + 225a^2}{42}$C $\times \frac{p^2}{a^3} =$ $-{\cdot}00000044$

the ſum $=$ ·94148313

which drawn into $\frac{2T}{r} = 2T =$ ·5235248, gives ·4928897 $=$ DB $= z$,

But

PROBLEM IX.

*To find the Surface of a Spheroid.*

* RULE I.

*For Both Spheroids.*

Let $f$ denote the fixed axe, or the axe about which the ellipse is conceived to revolve, $r$ the revolving axe, $p = 3{\cdot}14159$, and $q = \frac{ff \odot rr}{ff}$: Then

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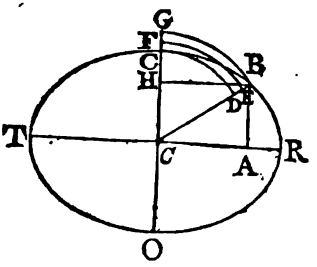
But OD $= \frac{r}{c}\sqrt{cc - zz} = {\cdot}87005233$; hence AD = AO + OD $= {\cdot}88686233$, and as AD : DB :: rad. : tang. $\angle$DAB $= 29{\cdot}063904° = 29°\ 3'\ 50''\ 3'''$ the true anomaly.

Alſo, as rad. : ſec. $\angle$DAB :: AD : AB $= 1{\cdot}0146255$, the diſtance of the earth from the ſun, to the mean diſtance 1.

* DEMONSTRATION.




Put the fixed ſemi-axe TC = CR = $a$, CC = CO = $b$, CA = $x$, AB = $y$, arc CB = $z$, and $3{\cdot}14159 = p$.

Then, $a : b :: \sqrt{aa - xx} : \frac{b}{a}\sqrt{aa - xx} = y$; hence $\dot{y} = -\frac{b\dot{x}x}{a\sqrt{aa - xx}}$, and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{x}^2 + \frac{b^2x^2\dot{x}^2}{a^2(a^2 - x^2)}}$ $= \frac{\dot{x}}{a}\sqrt{\frac{a^4 - x^2(aa - bb)}{aa - xx}}$; and conſequently the fluxion of the ſurface $\dot{s} = 2py\dot{z} = \frac{2pb\dot{x}}{aa}\sqrt{a^4 - x^2(aa - bb)}$

=

Then will the ſurface $s$ of the ſpheroid be expreſſed by either of the following ſeries; uſing the upper or under ſigns, according as it is an oblong or oblate ſpheroid; viz.

$$s = prf \times (1 \mp \frac{1}{2.3}q - \frac{1}{2.4.5}q^2 \mp \frac{3}{2.4.6.7}q^3 - \frac{3.5}{2.4.6.8.9}q^4 \text{ \&c}),$$

or

$$s = prf \times (1 \mp \frac{A}{2.3}q - \frac{3B}{4.5}q \mp \frac{3.5C}{6.7}q - \frac{5.7D}{8.9}q \text{ \&c}).$$

Where A, B, C, &c, are the ſeveral preceding terms.

EX-

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$= \frac{2pbx}{aa}\sqrt{a^4 - c^2x^2}$ (putting $cc = aa - bb$) $= \frac{2pbcx}{aa}\sqrt{\frac{a^4}{cc} - xx}$

$= \frac{2pbcx}{aa} \times (\frac{a^2}{c} - \frac{cx^2}{2a^2} - \frac{c^3x^4}{2.4a^6} - \frac{3c^5x^6}{2.4.6a^{10}}$ &c). And the fluent is

$s = 2pbx \times (1 - \frac{c^2x^2}{2.3a^4} - \frac{c^4x^4}{2.4.5a^8} - \frac{3c^6x^6}{2.4.6.7a^{12}} - \frac{3.5c^8x^8}{2.4.6.8.9a^{16}}$ &c).

And if for $\frac{cc}{aa}$ be written $dd$, the ſeries will become

$s = 2pbx \times (1 - \frac{d^2x^2}{2.3a^2} - \frac{d^4x^4}{2.4.5a^4} - \frac{3d^6x^6}{2.4.6.7a^6} - \frac{3.5d^8x^8}{2.4.6.8.9a^8}$ &c).

and when $x = a$, it is

$= 2pab \times (1 - \frac{d^2}{2.3} - \frac{d^4}{2.4.5} - \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9}$ &c)

$= 2pab \times (1 - \frac{A}{2.3}d^2 - \frac{3B}{4.5}d^2 - \frac{3.5C}{6.7}d^2 - \frac{5.7D}{8.9}d^2$ &c).

where, when $cc$ or $aa - bb$ is affirmative, viz. in the caſe of an oblong ſpheroid, all the ſigns after the firſt will be negative; but for an oblate ſpheroid, $cc$ being negative, the ſigns after the firſt will become alternately *plus* and *minus*. Q. E. D.

*Corol.* 1. The ſeries above will not always converge for an oblate ſpheroid; viz. when $d$ is greater than 1, or when $\frac{bb}{aa}$ is greater than 2, the ſeries will not converge.

*Corol.* 2. Hence the ſurfaces of ſimilar ſpheroids, and alſo of their like parts, are to each other as the rectangles of their axes, or as the ſquares of their like dimenſions.

EXAMPLE.

If the axes be 50 and 40, required the surface of each spheroid.

1. For the Oblong Spheroid.

Here $f = 50, r = 40$, and $\frac{ff-rr}{ff} = \frac{30^2}{50^2} = \frac{9}{25} = .36 = q$.

Hence 1st term A = — — 1.

2d term B $= \frac{A}{2.3}q$ = −0.06

3d term C $= \frac{3B}{4.5}q$ = −0.00324

4th term D $= \frac{3.5C}{6.7}q$ = −0.0004166−

5th term E $= \frac{5.7D}{8.9}q$ = −0.000072900

6th term F $= \frac{7.9E}{10.11}q$ = −0.0000150+

7th term G $= \frac{9.11F}{12.13}q$ = −0.0000035−

8th term H $= \frac{11.13G}{14.15}q$ = −0.0000009−

the sum of the negative terms = −0.0637489, which taken from the 1st term 1, leaves .9362511 for the sum of the series; and this being drawn into $prf = 3.14159265 \times 50 \times 40 = 6283.18531$, will produce 5882.6385 for the surface required.

2. For the Oblate Spheroid.

Here $f = 40$, $r = 50$, and $\frac{rr-ff}{ff} = \frac{30^2}{40^2} = \frac{9}{16} = q$. Then by increasing the values of the terms, found above for the oblong spheroid, in proportion to such power of the ratio of the value of $q$ in this case, to its value in the former, whose exponent is respectively

1 1 less

1 leſs than the number of each term; viz. in the proportion of the 0, 1, 2, 3, &c power of $\frac{9}{25}$ to $\frac{9}{16}$, or of 16 to 25, or of 64 to 100; we ſhall have the

1ſt term $= 1 \times \frac{100^0}{64^0} = 1 \times 1 = +1\cdot$

2d term $= \cdot 06 \times \frac{100}{64} = +0\cdot 09375$

3d term $= \cdot 00324 \times \frac{100^2}{64^2} = -0\cdot 00791$

4th term $= \cdot 0004166 \times \frac{100^3}{64^3} = +0\cdot 00159$

5th term $= \cdot 0000729 \times \frac{100^4}{64^4} = -0\cdot 00044$

6th term $= \cdot 0000150 \times \frac{100^5}{64^5} = +0\cdot 00014$

7th term $= \cdot 0000035 \times \frac{100^6}{64^5} = -0\cdot 00005$

8th term $= \cdot 0000009 \times \frac{100^7}{64^7} = +0\cdot 00002$

ſum of the affirmative terms $+1\cdot 09550$
ſum of the negative terms $-0\cdot 00840$

the difference $1\cdot 0871$;

then $6283\cdot 18531 \times 1\cdot 0871 = 6830\cdot 4507 =$ the ſurface of the oblate ſpheroid required.

## RULE II.

### *For Both Spheroids.*

Divide the revolving axe by the fixed axe, and call the quotient $q$. Find the difference between 1 and the ſquare of $q$, and call the ſquare root of that difference $s$.

For an oblong ſpheroid, multiply $\cdot 01745329$ by the degrees in the arc whoſe ſine is $s$, and call the product P.

For

For an oblate ſpheroid, multiply 2·302585 by the logarithm of the ſum of $s$ and $q$, and call the product P, alſo.

Multiply P by the fixed axe, and divide the product by $s$; to the quotient add the revolving axe; then the ſum multiplied by $\frac{1}{4}$ the circumference of the greateſt circle, viz. by 3·14159 and by $\frac{1}{4}$ the revolving axe, will give the ſurface required.

That is,

putting $f$ = the fixed axe,
$r$ = the revolving axe,
$q$ = $r \div f$,
$s$ = $\sqrt{1 \circ qq}$,
$p$ = 3·14159 &c,
P = ·01745329 × degrees to the ſine $s$, in the oblong ſpheroid,
and P = 2·302585 × log.of $s + q$ in the oblate ſpheroid;

Then $(\frac{Pf}{s} + r) \times \frac{1}{2}pr$ = the ſurf. of the ſpheroid.*

EXAMPLE.

The axes of a ſpheroid being 50 and 30, required the ſurface.

1. For the Oblong Spheroid.

Here $f = 50$, $r = 30$, $q = \frac{30}{50} = ·6$, $s = \sqrt{1 - qq}$ $= \sqrt{1 - ·36} = \sqrt{·64} = ·8$, which is the ſine of 53·130105 degrees.

Then

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* DEMONSTRATION.

For the fluent of $\frac{2phx}{aa}\sqrt{a^4 - c^2x^2}$, the fluxion of the ſurface in the demonſtration of the firſt rule, when $x = a$, is $pb \times (b + \frac{aa}{c}P)$,

where

Then $53{\cdot}130105 \times {\cdot}01745329 = {\cdot}92729513 =$ P, and $(\frac{50 \times {\cdot}92729513}{{\cdot}8} + 30) \times \frac{30 \times 3{\cdot}14159}{2} =$ $4144{\cdot}8263 =$ the surface of the oblong spheroid.

2. For the Oblate Spheroid.

Here $f = 30$, $r = 50$, $q = \frac{30}{36} = \frac{5}{6}$, and $s = \sqrt{1 - \frac{25}{36}} = \sqrt{\frac{16}{36}} = \frac{4}{6}$.

Hence $s + q = \frac{5}{6} + \frac{4}{6} = \frac{9}{6} = 3$, whose log. is ${\cdot}4771213$.

And $P = {\cdot}4771213 \times 2{\cdot}302585 = 1{\cdot}0986124$.

Then $(\frac{30 \times 1{\cdot}0986124}{\frac{4}{6}} + 50) \times \frac{50 \times 3{\cdot}14159}{2} =$ $5868{\cdot}39918 =$ the surface of the oblate spheroid.

RULE III.

*For the Oblong Spheroid.*

Find $q$ and $s$ as in the last rule.

Find also the area of the circular middle zone whose radius is half the quotient of the fixed axe divided by $s$, and the distance of each end from the center is half the fixed axe; and call this zone $z$.

Then multiply continually together $z$, $q$, $s$, and $3{\cdot}4159$ &c, for the surface of the oblong spheroid.

That is, $pqsz =$ the surface of the oblong spheroid ABDP; where $z$ is equal to the middle zone EFGH of the circle whose radius is $\frac{\frac{1}{2}f}{s}$.*



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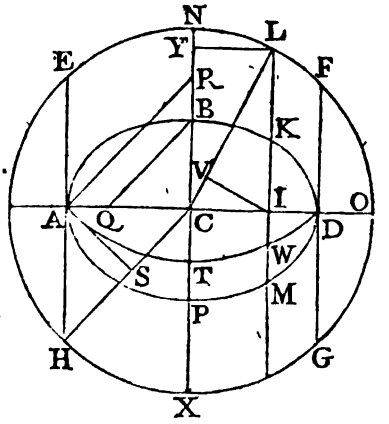
where P $= {\cdot}017453 \times$ deg. in arc to sine $\frac{c}{a}$, when $a$ is greater than $b$; or P $= 2{\cdot}3025$ &c $\times$ log. of $\frac{b+c}{a}$, when $a$ is less than $b$; $c$ being $= \sqrt{aa - bb}$. *Q. E. D.*

* DEMONSTRATION.

For since the fluxion of the surface BKMP is $\frac{2pbcx}{aa}\sqrt{\frac{a^4}{cc} - xx}$, and




EXAMPLE.

Taking again the ſame example,

We have $\frac{5.f}{s} = \frac{25}{\cdot 8} = 31\cdot 25 =$ the radius, or $62\cdot 5$ = the diameter of the circle.

Hence CO — CD $= 31\cdot 25 - 25 = 6\cdot 25 =$ DO.

And DO ÷ 2CO $= 6\cdot 25 \div 62\cdot 5 = \cdot 1 =$ the tabular verſed ſine, to which belongs the area $\cdot 04087528$; and this taken from $\cdot 39269908$, the tabular ſemi-circle, leaves $\cdot 3518238$ for half the tabular zone.

Whence

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and that of the circular area NLIC $= \dot{x}\sqrt{\frac{a^4}{cc} - xx}$, and being to the former in the conſtant ratio of 1 to $\frac{2pbc}{aa}$, their fluents muſt be continually in the ſame ratio; and conſequently the ſurface BKMP $= \frac{2pbc}{aa} \times$ CNLI $= pqs \times$ 2CNLI $= p \times \frac{BC}{CN} \times$ 2CNLI.

And the whole ſurface ABDPA $= pqs \times$ EFGH. *Q. E. D.*

Whence $\cdot 3518238 \times 2 \times 62 \cdot 5 \times 62 \cdot 5 = 2748 \cdot 6234375$ = the middle zone EFGH = Z.

Then $pqsz = 3 \cdot 14159 \times \cdot 6 \times \cdot 8 \times 2748 \cdot 6234375$ = 4144·8265 the ſurface, nearly the ſame as before.

## RULE IV.

### *For the Oblong Spheroid.*

Between the ſemi-conjugate BC and the ſum of the conjugate BP and circular arc EF, find a mean proportional, and it will be the radius of a circle equal to the ſurface of the ſpheroid; the circle being deſcribed as in the laſt rule.

That is, multiply continually together BC, BP + EF, and 3·14159, for the ſurface.*



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* DEMONSTRATION.

Apply BQ = AC, and draw AR parallel to BQ.

Then is AR = the radius CN; for, by the conſtruction,

CN is $= \frac{aa}{\sqrt{aa - bb}} = \frac{AC^2}{\sqrt{QB^2 - BC^2}} = \frac{AC^2}{CQ} = \frac{AC \times BQ}{QC} =$ AR by ſimilar triangles. Draw CH, and AS perpendicular to it.

Then, by the laſt rule, the ſurface BAP is $= p \times \frac{BC}{CN} \times 2CNEA$ or $2CXHA = p \times BC \times \frac{2 \text{ ſector } CXH + 2 \triangle HAC}{CH} = p \times BC \times (NE + AS)$.

But, by ſimilar triangles, $\begin{cases} HC : AC :: AH : AS, \\ AR : QB :: CR : CB. \end{cases}$

And ſince HC = AR, and AC = QB, by the conſtruction; as alſo AH = CR, by reaſon of the equal hypotenuſes HC, AR, and common baſe AC; the fourth terms AS, CB, muſt alſo be equal to each other. And conſequently the ſurface BAP = $p \times$ BC $\times$ (EN + BC) = $p \times$ BC $\times$ (AT + BC); and that of the whole ſpheroid = $p \times$ BC $\times$ (ATD + BP) = $p \times$ BC $\times$ (ENF + BP), the

EXAMPLE.

Taking ſtill the ſame example,

We have the radius CN $= \frac{\frac{1}{4}f}{s} = \frac{25}{\cdot 8} = 31\frac{1}{4}$.

Then as $31\frac{1}{4} : 25 =$ AC the ſine of the arc EN $::$ $1 : \frac{4}{5} = \cdot 8 =$ the ſine of the ſimilar arc to the radius

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the arc ATD being deſcribed with the center R and radius RA; viz. equal to a circle whoſe radius is a mean proportional between BC and ATD + BP. *Q. E. D.*

SCHOLIUM.

This conſtruction, for finding a circle equal to the ſurface of an oblong ſpheroid, was firſt given by Mr. *Huygens* in his *Horologium Oſcillitorium*, *prop.* 9, but without demonſtration.—The conſtruction is here rendered general for any part or zone BKMP of the ſpheroid, viz. that its curve ſurface is equal to a circle whoſe radius is a mean proportional between BC and NL + IV or TW + IV, IV being perpendicular to CL. For in the ſame manner as $p \times$ BC $\times$ (EN + AS) was found for the ſurface BAP, may the ſurface of any zone BKMP be found to be $p \times$ BC $\times$ (NL + IV).

In imitation of the two rules above for expreſſing the ſurface of the oblong ſpheroid, by means of a circular arc and a circular or elliptic area, I ſhall here ſhew the inveſtigation of others for that of the oblate ſpheroid, by means of the parabolic arc, or hyperbolic area.

Thus if $x$ be the abſciſs, and $y$ the ordinate, of a parabola whoſe parameter is $n$; ſince $x = \frac{yy}{n}$, and $\dot{x} = \frac{2y\dot{y}}{n}$, the fluxion of the arc will be

$$\sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\frac{4y^2\dot{y}^2}{nn} + \dot{y}^2} = \frac{\dot{y}}{n}\sqrt{nn + 4yy} = \frac{\dot{y}}{m}\sqrt{mm + yy},$$

putting $m = \frac{1}{2}n = \frac{1}{4}$ the parameter; which being to that of the oblate ſurface, viz.

$$\frac{2pb\dot{x}}{aa}\sqrt{a^4 + c^2x^2} = \frac{2pbc\dot{x}}{aa}\sqrt{\frac{a^4}{cc} + x^2} = \frac{2pb\dot{x}}{m}\sqrt{mm + xx},$$

(putting $m = \frac{a^2}{c}$, and ſuppoſing $x = y$), in the conſtant ratio

of

dius 1; to which belong 53·130105 degrees; hence $53{\cdot}130105 \times {\cdot}01745329 \times 31\frac{1}{4} = 28{\cdot}977972 =$ the arc EN.

Then BP × (BC + EN) × 3·14159 = 4144·8262 = the ſurface as before, nearly.



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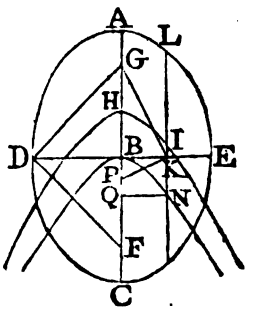
of 1 to $2bp$, their fluents muſt be in the ſame ratio; and conſequently if P be put to denote the parabolic arc, then will $2pbP$ expreſs the value of the oblate ſurface, which therefore is equal to a circle whoſe radius is a mean proportional between $b$ and 2P, or between $2b$ and P.—And hence this conſtruction.—Let DE be the leſs or fixed axe of the ellipſe, AC the greater axe, and F the focus; join DF, and draw DG perpendicular to DF; biſect BG in H; with the vertex H and focus B, deſcribe a parabola HI: Then draw any ordinate LK to the axe DE, meeting the parabola in I; and the ſurface generated by the arc AL will be equal to a circle whoſe radius is a mean proportional between AC and the parabolic arc HI. Or the ſurface generated by AL, will be equal to the curve ſurface of a cylinder whoſe diameter is AC and height HI.




Again, ſince $P = \frac{2y}{m}\sqrt{mm+yy} - \frac{A}{b}$, A being the area of an hyperbola whoſe ſemi-axes are $b$ and $m$, and $y$ the ordinate; ſee cor. 7 rule 1 prob. 6 ſect 5; the ſurface generated by AL will be equal to $2p \times (\frac{2by}{m}\sqrt{mm+yy} - A)$; and hence this conſtruction is evident. Let LK, produced, meet, in N, the hyperbola BN, whoſe vertex is B, center A, and ſemi-conjugate axe BG; draw GK, and KP perpendicular to it, and draw the ordinate NQ: Then will the difference between the rectangle AC × KP and the hyperbolic area BQN, be to the ſurface generated by AL, as the radius of a circle, is to its circumference.

RULE V.

*For Both Spheroids.*

Find $q$ as in the firſt rule; then to or from 1, add or ſubtract $\frac{1}{3}$ of $q$, according as it is the oblate or oblong ſpheroid; and call the ſquare root of the ſum or difference A. Or $A = \sqrt{1 \pm \frac{1}{3}q}$.

Multiply the product of the two axes by 3·14159, and call the product P.

Then the product of A and P will be the ſurface nearly.

That is, $pfr\sqrt{1 \pm \frac{1}{3}q}$.*

EXAMPLE.

Let here be taken here the ſame example as at rule 1, in which the axes are 50 and 40.

1. For the Oblong Spheroid.

Here $q = \frac{9}{25}$, and $\sqrt{1 - \frac{9}{3 \times 25}} = \frac{1}{5}\sqrt{22} = 0.93808 = A$.

But $3.14159 \times 50 \times 40 = 6283.1853 = P$.

Then $A \times P = 5893.6278$ the ſurface nearly.

2. For

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* For ſince, by rule 1, the value of the ſurface is $s = 4abp \times (1 \pm \frac{d^2}{2.3} - \frac{d^4}{2.4.5} \pm \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9}$ &c), and $4abp\sqrt{1 \pm \frac{1}{3}d^2} = 4abp \times (1 \pm \frac{d^2}{2.3} - \frac{d^4}{2.4.9} \pm \frac{3d^6}{2.4.6.27} - \frac{3.5d^8}{2.4.6.8.81}$ &c) we ſhall have, by taking the latter of theſe quantities from the former,

$s - 4abp\sqrt{1 \pm \frac{1}{3}d^2} = -\frac{d^4}{2.5.9} \pm \frac{5d^6}{4.7.27} - \frac{5d^8}{2.8.81}$ &c, and conſequently $s = \frac{1}{4}abp\sqrt{1 \pm \frac{1}{3}d^2}$ nearly, which is rule 5.

2. For the Oblate Spheroid.

Here $q = \frac{9}{16}$, and $\sqrt{1 + \frac{9}{3 \times 16}} = \frac{1}{4}\sqrt{19} = 1{\cdot}089725 = \text{A}$.

Then $\text{A} \times \text{P} = 1{\cdot}0897 \times 6283{\cdot}1853 = 6846{\cdot}7869$ the surface of the oblate spheroid nearly.

RULE VI.

*For Both Spheroids.*

To or from 1, add or subtract, according as it is for the oblate or oblong spheroid, $\frac{1}{6}$ of $q$, and call $\frac{4}{5}$ of the sum or difference B.

Then multiply P by the difference between A and B, and $\frac{9}{5}$ of the product will be the surface nearly.

That is,
$\frac{9}{5}\text{P} \times (\text{A} - \text{B}) = \frac{9}{5} \times pfr \times (\sqrt{1 \pm \frac{1}{3}q} - \frac{4}{5} \times \overline{1 \pm \frac{1}{6}q})$ = the surface nearly.*



* Again, by transposing the two first terms of the series in the value of s, we shall have

$$s - 4abp \times (1 \pm \frac{d^2}{6}) = 4abp \times (-\frac{d^4}{2.4.5} \pm \frac{3d^6}{2.4.6.7} - \frac{3.5d^8}{2.4.6.8.9} \&c),$$

$$\text{or } \frac{4}{5} \times [s - (1 \pm \frac{d^2}{6}) \times 4abp] = 4abp \times (-\frac{d^4}{2.5.9} \pm \frac{d^6}{4.7.9} - \frac{5d^8}{4.8.81} \&c);$$

which taken from

$$s - 4abp\sqrt{1 \pm \tfrac{1}{3}d^2} = -\frac{d^4}{2.5.9} \pm \frac{5d^6}{4.7.27} - \frac{5d^8}{2.8.81} \&c,$$

the remainder above found, it leaves

$$\tfrac{1}{5}s - 4abp \times (\sqrt{1 \pm \tfrac{1}{3}d^2} - \tfrac{4}{5} \times \overline{1 \pm \tfrac{1}{6}d^2}) = \pm\frac{d^6}{2.7.27} - \frac{5d^8}{4.8.81} \&c,$$

and consequently $s = \frac{9}{5} \times 4abp \times [\sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{9} \times (1 \pm \frac{1}{6}dd]$
$= \frac{4}{5}abp \times [9\sqrt{1 \pm \frac{1}{3}dd} - 4(1 \pm \frac{1}{6}dd)]$ more nearly, which is rule 6.

EXAMPLE.

Taking the ſame example, we ſhall have

1. For the Oblong Spheroid.

$B = \frac{4}{9} \times (1 - \frac{1}{6}q) = \frac{4}{9} \times (1 - \frac{9}{6 \times 25}) = \frac{4}{9} \times \frac{47}{50} =$ ·417777 &c.

Then $A - B =$ ·93808 − ·417777 = ·5203.

And $\frac{9}{7} \times$ ·5203 × 6283·1853 = 5884·2029 = the ſurface of the oblong ſpheroid very nearly.

2. For the Oblate Spheroid.

$B = \frac{4}{9} \times (1 + \frac{1}{6}q) = \frac{4}{9} \times (1 + \frac{9}{6 \times 16}) = \frac{4}{9} \times \frac{35}{32} =$ ·48611 &c.

Then $A - B =$ 1·0897 − ·4861 = ·6036.

And $\frac{9}{7} \times$ ·6036 × 6283·1853 = 6826·6806 = the ſurface of the oblate ſpheroid nearly.

RULE VII.

*For Both Spheroids.*

From the ſum or difference of 1 and $\frac{1}{6}$ of $q$, according as it is for the oblate or oblong ſpheroid, take $\frac{1}{4 \cdot 6}$ of the ſquare of $q$, and call $\frac{8}{27}$ of the remainder c.*

Then

* Farther, by tranſpoſing the firſt three terms of the original ſeries, and reducing the reſult, to make the firſt term of it equal to that of the laſt remainder, we ſhall have

$\frac{8}{27} \times [s - 4abp(1 \pm \frac{1}{6}d^2 - \frac{1}{46}d^4)] = 4abp \times (\pm \frac{d^5}{2.7.27} - \frac{5d^9}{6.8.81}$ &c),

which

Then from A ſubtract the ſum of B and C, multiply the remainder by P, and $\frac{2.7}{7}$ of the product will be the ſurface of the ſpheroid, ſtill nearer.

That is, $\frac{2.7}{7}$ P × (A − B − C) = the ſurface nearly; where $c = \frac{8}{27} \times (1 \pm \frac{1}{6}q - \frac{1}{40}qq)$, and the value of the other letters as before.

EXAMPLE.

Taking ſtill the ſame example, in which the axes are 40 and 50;

1. For the Oblong Spheroid.

Here again $q = \frac{9}{25} = .36$, P = 6283·1853, A = ·93808, B = ·417777 &c, and $c = \frac{8}{27} \times (1 - \frac{1}{6}q - \frac{1}{40}qq)$ = ·277558.

Then $\frac{2.7}{7}$ P × (A − B − C) = 6283·1853 × ·93628 = 5882·8206 the ſurface of the oblong ſpheroid very near.

2. For the Oblate Spheroid.

Here $q = \frac{9}{16} = .5625$, P = 6283·1853, A = 1·089725, B = ·486111, and $c = \frac{8}{27} \times (1 + \frac{1}{6}q - \frac{1}{40}qq)$ = ·32173.

Then $\frac{2.7}{7}$ × (A − B − C) × P = 1·0872 × 6283·1853 = 6831·079 = the ſurface of the oblate ſpheroid very nearly.

PRO-

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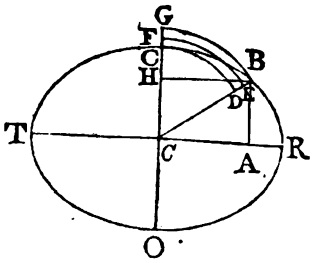
which being taken from the laſt remainder, $\frac{5}{3}s - 4abp \times [\sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{5}(1 \pm \frac{1}{8}dd)] = \pm\frac{d^6}{2.7.27} - \frac{5d^8}{4.8.81}$ &c, leaves $\frac{7}{27}s - 4abp \times [\sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{5}(1 \pm \frac{1}{8}dd) - \frac{8}{27}(1 \pm \frac{1}{8}d^2 - \frac{1}{40}d^4)] = -\frac{5d^8}{8.12.81}$ &c, and conſequently $s = \frac{2.7}{7} \times 4abp \times [\sqrt{1 \pm \frac{1}{3}dd} - \frac{4}{5}(1 \pm \frac{1}{8}dd) - \frac{8}{27}(1 \pm \frac{1}{8}d^2 - \frac{1}{40}d^4)]$ $= \frac{4}{7}abp \times [27\sqrt{1 \pm \frac{1}{3}dd} - 12(1 \pm \frac{1}{8}dd - 8(1 \pm \frac{1}{8}d^2 - \frac{1}{40}d^4)]$, nearer ſtill; and this is rule 7.

## PROBLEM X.

*To find the Curve Surface of the Fruſtum of a Spheroid, contained between two Planes cutting the Spheroid Perpendicular to the Fixed Axe, and one of them paſſing through the Center.*




### RULE I.

*For Both Spheroids.*

Let $f$ denote the fixed, and $r$ the revolving axe; put $p = 3{\cdot}14159$, and $q = \frac{ff\varpi rr}{ff}$, as in the laſt problem; alſo $b =$ the height $cA$ of the fruſtum, and $z = \frac{4qbb}{ff}$; then will the value of the ſurface be expreſſed by

$$prb \times (1 \pm \tfrac{1}{2\cdot3}z - \tfrac{1}{2\cdot4\cdot5}z^2 \pm \tfrac{3}{2\cdot4\cdot6\cdot7}z^3 - \tfrac{3\cdot5}{2\cdot4\cdot6\cdot8\cdot9}z^4 \text{ \&c}),$$

or $prb \times (1 \pm \frac{1}{2\cdot3}\text{A}z - \frac{3}{4\cdot5}\text{B}z \pm \frac{3\cdot5}{6\cdot7}\text{C}z - \frac{5\cdot7}{8\cdot9}\text{D}z$ &c).

Where A, B, C, &c, are the ſeveral preceding terms, and the upper or under ſigns to be uſed, according as it is the oblate or oblong ſpheroid.*

EX-

* See the inveſtigation of rule 1 of the laſt problem.

EXAMPLE I.

What is the ſurface of the fruſtum of a ſpheroid, whoſe height is 15; the diameter of the greater end being 40, and that of the leſs 32?

Here CO = 40 = the revolving axe, 2AB = 32, and HB = cA = 15; hence CH = Cc − AB = 4, and HO = Cc + AB = 36; and conſequently, by caſe 3 prob. 2, we have as

$\sqrt{\text{CH} \times \text{HO}} = 12$ ; HB = 15 :: CO = 40 : TR = 50 the fixed axe; which ſhews that the fruſtum is that of an oblong ſpheroid.

Then, as in the example to rule 1 of the laſt problem, $f = 50$, $r = 40$, $q = \frac{9}{25} = .36$. But $b = 15$, and $z = \frac{4qbb}{ff} = \frac{4 \times 9 \times 15 \times 15}{25 \times 50 \times 50} = .1296$.

Hence the 1ſt term A = — + 1.

2d — B $= \frac{1}{2.3}$ A$z$ = −0.0216

3d — C $= \frac{3}{4.5}$ B$z$ = −0.0004199+

4th — D $= \frac{3.5}{6.7}$ C$z$ = −0.0000195−

5th — E $= \frac{5.7}{8.9}$ D$z$ = −0.0000012+

6th — F $= \frac{7.9}{10.11}$ E$z$ = −0.0000001−

the ſum of the negative terms is −0.0220407,
which taken from the firſt term 1.0000000,
leaves 0.9779593

for the value of the infinite ſeries; which being drawn into $prb = 3.14159 \times 40 \times 15 = 1884.955592$, produces 1843.4098512 for the ſurface required.

EX

EXAMPLE II.

It is required to find the curve ſurface of the fruſtum $2cABC$ of a ſpheroid, whoſe height is 16; the diameter of the greater end being 50, and that of the leſs 30.

Here the revolving axe $TR = 50$, $cH = AB = 16$, and $HB = cA = 15$; hence $AR = cR - cA = 10$, and $TA = Tc + cA = 40$; then, by caſe 3 prob. 2, we ſhall have as $\sqrt{AR \times AT} = 20 : AB = 16 :: TR = 50 : CO = 40 =$ the fixed axe; which indicates the fruſtum to be that of an oblate ſpheroid.

Wherefore $f = 40$, $r = 50$, $q = \frac{9}{16}$, $b = 16$, and $z = \frac{4qbb}{ff} = \frac{4 \times 9 \times 16 \times 16}{16 \times 40 \times 40} = \frac{36}{100} = .36$, which being the ſame as the converging quantity $q$ in the example to rule 1 of the laſt prob. the ſeveral terms of the ſeries muſt be the ſame as there found, viz.

| | | | |
|---|---|---|---|
| the 1ſt term | A | = | + 1 |
| 2d — | B | = | + 0·06 |
| 3d — | C | = | − 0·00324 |
| 4th — | D | = | + 0·0004166 − |
| 5th — | E | = | − 0·000072900 |
| 6th — | F | = | + 0·0000150 + |
| 7th — | G | = | − 0·0000035 − |
| 8th — | H | = | + 0·0000009 − |
| 9th — | I | = | − 0·0000002 + |
| ſum of the affir. terms | | = | + 1·0604325 |
| ſum of the neg. terms | | = | − 0·0033166 |
| their difference | | = | 1·0571159 the value of the ſeries; |

which being drawn into $prb = 3.14159 \times 50 \times 16 = 2513.27412287$, will produce 2656·822036 for the ſurface of the oblate fruſtum required.

RULE

RULE II.

*For Both Spheroids.*

1. Multiply the ſquare root of the difference of the ſquares of the axes, by the height of the fruſtum; divide the product by the ſquare of the fixed axe; and call double the quotient $q$.

That is, $q = \frac{2b\sqrt{ff - rr}}{ff}$, $b$ being the height of the fruſtum, $f$ the fixed, and $r$ the revolving axe.

2. In the oblong ſpheroid, call the ſquare root of the difference between 1 and the ſquare of $q$, A. But in the oblate ſpheroid, let A be the root of the ſum of 1 and the ſquare of $q$.

That is, $\text{A} = \begin{cases} \sqrt{1 - qq} \text{ in the oblong,} \\ \sqrt{1 + qq} \text{ in the oblate.} \end{cases}$

3. In the oblong ſpheroid, let the product of ·01745329 and the degrees whoſe ſine is $q$, be called B. But in the oblate ſpheroid, let B be the product of 2·30258509 and the logarithm of the ſum of $q$ and the root of the ſum of 1 and the ſquare of $q$.

That is, $\text{B} =$
$\begin{cases} \text{·0174 \&c} \times \text{degrees whoſe ſine is } q \text{ in the oblong,} \\ \text{2·302 \&c} \times \text{log. of } q + \sqrt{1 + qq} \text{ in the oblate.} \end{cases}$

4. Divide B by $q$, to the quotient add A, multiply the ſum by the continual product of the height, revolving axe, and the number 3·14159, and half the laſt product will be the ſurface required.

That is, $\frac{1}{2}prb \times (\text{A} + \frac{\text{B}}{q}) =$ the ſurface.*

EX-

* For this is the fluent of the fluxion of the ſurface in the inveſtigation of the rules in the laſt problem.

EXAMPLE I.

Let there be taken here the firſt example to the laſt rule, in which $f$ is $= 50$, $r = 40$, and $b = 15$.

Then $\frac{2b\sqrt{ff-rr}}{ff} = \frac{30^2}{50^2} = \frac{9}{25} = \cdot 36 = q$.

And $A = \sqrt{1-qq} = \sqrt{1-\frac{9^2}{25^2}} = \frac{4\sqrt{34}}{25} = \cdot 9329523$.

Alſo the ſine $q = \cdot 36$ anſwers to $21\cdot 1001965$ degrees, which being drawn into $\cdot 01745329$, will produce $\cdot 3682678 = B$.

Hence $(A + \frac{B}{q}) \times \frac{1}{2}prb = \cdot 9779593 \times 1884\cdot 955592$ $= 1843\cdot 4098512 =$ the ſurface, the ſame as before.

EXAMPLE II.

Let there be taken the oblate fruſtum, in the ſecond example to the preceding rule, in which the axes are 50 and 40, and the height 16.

Here then $f = 40$, $r = 50$, and $b = 16$.

Hence $\frac{2b\sqrt{rr-ff}}{ff} = \frac{32 \times 30}{40 \times 40} = \frac{3}{5} = \cdot 6 = q$.

And $\sqrt{1+qq} = \sqrt{1+\frac{3^2}{5^2}} = \frac{1}{5}\sqrt{34} = 1\cdot 16619038$ $= A$.

Likewiſe $q + \sqrt{1+qq} = 1\cdot 76619038$, whoſe logarithm is $\cdot 24703757$, which being drawn into $2\cdot 30258509$ will produce $\cdot 56882503 = B$.

Then $(A + \frac{B}{q}) \times \frac{1}{2}prb = 1\cdot 057116 \times 2513\cdot 27412287$ $= 2656\cdot 822287 =$ the ſurface, nearly the ſame as before.

RULE III.

*For the Oblong Spheroid.*

1. Divide the ſquare of the fixed axe, by the root of the difference of the ſquares of the axes; and call the quotient $d$.

That is, $d = \frac{ff}{\sqrt{ff - rr}}$.

2. Find the area of the circular zone whoſe diameter is $d$, and its height from the center equal to the height $b$ of the fruſtum of the ſpheroid; and call the area of that zone $z$.

3. Then as the diameter of the circle, is to the circular zone; ſo is the circumference of the greateſt circle of the ſpheroid, to the ſurface of its fruſtum.

That is, $d : z :: pr : \frac{prz}{d}$ = the ſurface of the fruſtum.*

EXAMPLE.

It is required to find the ſurface of the fruſtum of an oblong ſpheroid whoſe height is 15, the axes being 50 and 40.

Here $\frac{ff}{\sqrt{ff - rr}} = \frac{50^2}{30} = \frac{250}{3} = 83\frac{1}{3} = d$ the diam. of the circle.

And, by the table of circular areas, the area of the zone will be found to be $\cdot 17603268 \times \frac{250^2}{3^2}$.

Alſo $pr = 3\cdot 14159 \times 40$ = the greateſt circumference of the ſpheroid.

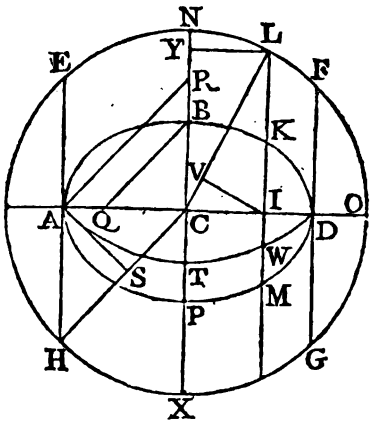
Then,

* This is the ſame with rule 3 to the laſt problem, only expreſſed in other terms.

Then, as $\frac{250}{3}$ : ·17603268 × $\frac{250^2}{3^2}$ :: 3·14159 × 40 : ·17603268 × 3·14159 × 40 × $\frac{250}{3}$ = 1843·4099 the ſurface, nearly the ſame as before.

RULE IV.

*For the Oblong Spheroid.*




Let BKMP be the fruſtum, whoſe ſurface is ſought, NLO a quadrant of the circle mentioned in the laſt problem; draw CL, and IV perpendicular to it.

Then the ſurface of the fruſtum will be equal to the circle whoſe radius is a mean proportional between BC and NL + IV.

Or the ſurface will be equal to the continual product of NL + IV, BC, and 3·14159.*

EXAMPLE.

Let there be taken here the ſame example as before, in which the axes are 50 and 40, and the height 15.

Then

* This is proved at rule 4 to the laſt problem.

Then ſince, by the example to the laſt rule, CL = CN = $\frac{125}{3}$, we ſhall have as CL = $\frac{125}{3}$ : YL = 15 :: 1 = radius : $\frac{9}{25}$ = ·36 = the ſine of 21·1001965 degrees, which are thoſe contained in the arc NL; and therefore ·01745329 × 21·1001965 × $\frac{125}{3}$ = 15·3444958 = NL. And, by ſimilar triangles, ſince IL = $\sqrt{CL^2 - CI^2}$ = $\sqrt{\frac{125^2}{3^2} - 15^2}$ = $\frac{20\sqrt{34}}{3}$ = 38·8730126, as CL = $\frac{125}{3}$ : LI :: CI = 15 : IV = 13·9942845.

And hence (NL + IV) × 3·14159 × BC = 29·3387803 × 3·14159 × 20 = 1843·409933 = the ſurface, nearly the ſame as before.

## RULE V.

### *For Both Spheroids.*

The 5th rule to the laſt problem will ſerve, if inſtead of $q$ we uſe $z$, as found in rule 1 of this problem, and $b$ the height inſtead of $f$ the fixed axe.

That is, $prb\sqrt{1 \pm \frac{1}{3}z}$ = the ſurface nearly.

### EXAMPLE I.

Taking the ſame example as in the laſt rule; we have $r$ = 40, $b$ = 15, and $z$, found in the firſt example to rule 1, = ·1296.

Conſequently $prb\sqrt{1 - \frac{1}{3}z}$ = 1884·955592 × ·97816 = 1843·7879 = the ſurface nearly.

### EXAMPLE II.

If we take the oblate fruſtum, whoſe height is 16, and the axes 40 and 50; we ſhall have $r$ = 50,

$b = 16$, and $z$, found in the ſecond example to rule 1, $= \cdot 36$.

Then $prh\sqrt{1 + \frac{1}{3}z} = 2513 \cdot 27412287 \times 1 \cdot 0583$ $= 2659 \cdot 797 =$ the ſurface nearly.

## RULE VI.

### *For Both Spheroids.*

This rule will be the ſame as the 6th rule to the laſt problem, uſing $z$ inſtead of $q$, and $b$ for $f$, as in the laſt rule. That is,

$\frac{2}{3}P \times (A - B) = \frac{2}{3}prb \times [\sqrt{1 \pm \frac{1}{3}z} - \frac{4}{9}(1 \pm \frac{1}{6}z)]$ $=$ the ſurface nearly.

### EXAMPLE I.

Taking the 1ſt example to the laſt rule; we have $P = 1884 \cdot 955592$, $A = \sqrt{1 - \frac{1}{3}z} = \cdot 97816157$, and $B = \frac{4}{9} \times (1 - \frac{1}{6}z) = \cdot 43484444$.

Then $\frac{2}{3}P \times (A - B) = 1884 \cdot 955592 \times \cdot 97797$ $= 1843 \cdot 4298 =$ the ſurface nearly.

### EXAMPLE II.

Taking here the 2d example to the laſt rule; we have $P = prb = 2513 \cdot 27412287$, $A = \sqrt{1 + \frac{1}{3}z}$ $= 1 \cdot 0583005$, and $B = \frac{4}{9} \times (1 + \frac{1}{6}z) = \cdot 47\frac{1}{9}$.

Then $\frac{2}{3}P \times (A - B) = 2513 \cdot 27412287 \times 1 \cdot 0569409$ $= 2656 \cdot 3821 =$ the ſurface nearly.

RULE

RULE VII.

*For Both Spheroids.*

This rule alſo will be the ſame with the 7th to the laſt problem, uſing $z$ and $b$ inſtead of $q$ and $f$.

That is, $\frac{2}{7}$ P $\times$ (A $-$ B $-$ C) $=$ $\frac{2}{7}prb \times (\sqrt{1 \pm \frac{1}{3}z} - \frac{4}{9}(1 \pm \frac{1}{6}z) - \frac{7}{27}(1 \pm \frac{1}{6}z - \frac{1}{40}zz)$ $=$ the ſurface very nearly.

EXAMPLE I.

Taking, ſtill, the ſame example of the oblong ſpheroid; we ſhall have, as before, P $=$ 1884·955592, A $=$ ·97816157, and B $=$ ·43484444; alſo C $= \frac{8}{27} \times (1 - \frac{1}{6}z - \frac{1}{40}zz) =$ ·28977188.

Then $\frac{2}{7}$ P $\times$ (A$-$B$-$C) $=$ 1884·955592 $\times$ ·97796 $=$ 1843·411 $=$ the ſurface very nearly.

EXAMPLE II.

Let there be taken, again, the oblate fruſtum, in which, as before, P $=$ 2513·27412287, A $=$ 1·0583005, and B $=$ ·47$\frac{1}{9}$; alſo C $= \frac{8}{27} \times (1 + \frac{1}{6}z - \frac{1}{40}zz) =$ ·31311407.

Then $\frac{2}{7}$ P $\times$ (A$-$B$-$C) $=$ 2513·27412287 $\times$ 1·05714 $=$ 2656·8825 $=$ the ſurface very nearly.

SCHOLIUM.

It is evident that the double of the fruſtum will give the middle zone; and that the fruſtum being added to, or taken from, half the ſpheroid, will give the greater or leſs ſegment.

## PROBLEM XI.

### *To find the Solidity of a Spheroid.*

### RULE I.

Multiply continually together the fixed axe, the ſquare of the revolving axe, and the number ·52359877 or $\frac{1}{6}$ of 3·14159, and the laſt product will be the ſolidity.*

That

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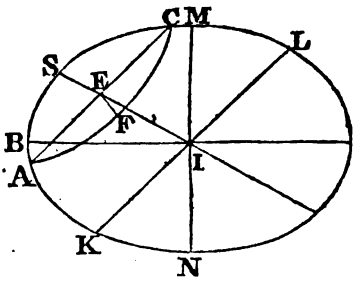
* DEMONSTRATION.




Put $f$ = BI the fixed ſemi-axe, $r$ = IM the revolving ſemi-axe of the ſpheroid, $a$ = SI any ſemi-diameter of the ſection NBM, $b$ = IK its ſemi-conjugate, $y$ = AE an ordinate to the diameter SI, or a ſemi-axe of the elliptic ſection AFC parallel to KL, and $z$ = EF its other ſemi-axe, alſo $x$ = EI, $s$ = the ſine of the angle AES, or of the angle KIS, to the radius 1, and $p$ = 3·14159.

Then, by the property of the ellipſe KSL, $aa : bb :: aa - xx : bb \times \frac{aa - xx}{aa} = yy$; and, by prop. 1 ſect. 2, $b : r :: y : \frac{ry}{b} = z$.

But the fluxion of the ſolid KACL is $psyz\dot{x} = \frac{psryy\dot{x}}{b}$ by writing for $z$ its value $\frac{ry}{b}$, $= pbsr\dot{x} \times \frac{aa - xx}{aa}$ by ſubſtituting for $yy$ its value $bb \times \frac{aa - xx}{aa}$, $= pfrr\dot{x} \times \frac{aa - xx}{aaa}$ by putting for $abs$ its value $rf$; and hence the fluent $pfrrx \times \frac{aa - \frac{1}{3}xx}{aaa}$ or $\frac{1}{3}pfrr \times \frac{3aa - xx}{aaa}$ will be the value of the fruſtum KACL; which, when EI or $x$ becomes SI or $a$, gives $\frac{1}{3}pfrr$ for the value of

That is, $\frac{1}{6}ptt\,c$ = the oblate, and $\frac{1}{6}ptc\,c$ = the oblong ſpheroid, where $p = 3{\cdot}14159$, $t$ = the tranſverſe, and $c$ = the conjugate axe of the generating ellipſe.



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of the ſemi-ſpheroid KSL; or the whole ſpheroid = $\frac{1}{3}p$FRR, putting F and R for the whole fixed and revolving axes. *Q. E. D.*

*Corol.* 1. From the foregoing demonſtration it appears that the value of the general fruſtum KAECL is expreſſed by $\frac{1}{3}pfrrx \times \frac{3aa - xx}{aaa}$.

And if for $fr$ be ſubſtituted its value $abs$, the ſame fruſtum will alſo be expreſſed by $\frac{1}{3}pbrsx \times \frac{3aa - xx}{aa}$.

Alſo, if for $aa$ be put its value $\frac{bbxx}{bb - yy}$, the laſt expreſſion will become $\frac{1}{3}prsx \times \frac{2bb + yy}{b}$ or $\frac{1}{3}psx \times (2br + \frac{ryy}{b})$; which, by writing $z$ inſtead of its value $\frac{ry}{b}$, gives $\frac{1}{3}psx \times (2br + yz)$ for the value of the fruſtum, viz. The ſum of the area of the leſs end and twice that of the greater, drawn into one-third of the altitude or diſtance of the ends.

And out of this laſt expreſſion may be expunged any one of the four quantities $b$, $r$, $y$, $z$, by means of the proportion $b : r :: y : z$.

When the ends of the fruſtum are perpendicular to the fixed axe; then $a$ is $= f$, and the value of the fruſtum becomes $\frac{1}{3}prrx \times \frac{3ff - xx}{ff}$ for the value of the fruſtum whoſe ends are perpendicular to the fixed axe, its altitude being $x$.

And when the ends of the fruſtum are parallel to the fixed axe, $a$ is $= r$, and the expreſſion for ſuch a fruſtum becomes $\frac{1}{3}pfx \times \frac{3rr - xx}{r}$.

*Corol.* 2. If to or from $\frac{2}{3}pfrr$, the value of the ſemi-ſpheroid, be added or ſubtracted $\frac{1}{3}pfrrx \times \frac{3aa - xx}{aaa}$, the value of the general

RULE II.

Multiply the area of the generating ellipſe, by $\frac{2}{3}$ of the revolving axe, and the product will be the content of the ſpheroid.

That is, $\frac{2}{3}t$ A = the oblate, and $\frac{2}{3}c$ A = the oblong ſpheroid; where A is the area of the ellipſe.

And this rule is evidently taken from the former.

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general fruſtum KACL, there will reſult $\frac{1}{3}pfrrbb \times \frac{3a-b}{aaa}$ for the value of a general ſegment, either greater or leſs than the ſemi-ſpheroid, whoſe height, taken upon the diameter paſſing through its vertex and center of its baſe, is $b = a \pm x$.

When $a$ coincides with $f$, the above expreſſion becomes $\frac{1}{3}prrbb \times \frac{3f-b}{ff}$ for the value of a ſegment whoſe baſe is perpendicular to the fixed axe.—And here if we put R for the radius of the ſegment's baſe, and for $rr$ its value $\frac{RRff}{2fb-bb}$, the ſaid ſegment will become $\frac{1}{3}pRRb \times \frac{3f-b}{2f-b}$.

And when $a$ coincides with $r$, the general expreſſion will become $\frac{1}{3}pfbb \times \frac{3r-b}{r}$ for the value of the ſegment whoſe baſe is parallel to the fixed axe.—And if we put F, R, for the two ſemi-axes, of the elliptic baſe of this ſegment, reſpectively corre-ſponding or parallel to $f$, $r$, the ſemi-axes of the generating ellipſe, when parallel to the baſe of the ſegment, and for $\frac{f}{r}$ and $r$ ſubſtitute their values $\frac{F}{R}$ and $\frac{RR+bb}{2b}$, the ſaid fruſtum will be expreſſed by $\frac{1}{3}pFb \times \frac{3RR+bb}{2R}$ in which the dimenſions of itſelf only are concerned.

*Corol.* 3. A ſemi-ſpheroid is equal to $\frac{2}{3}$ of a cylinder, or to double a cone, of the ſame baſe and height; or they are in proportion as the numbers 3, 2, 1. For the cylinder is $= 4nfrr = \frac{12}{3}nfrr$, the ſemi-ſpheroid $= \frac{8}{3}nfrr$, and the cone $= \frac{4}{3}nfrr$.

*Corol.* 4. When $f = r$, the ſpheroid becomes a ſphere, and the expreſſion $\frac{8}{3}fnrr$ for the ſemi-ſpheroid becomes $\frac{8}{3}nr^3$ for the

2 ſemi-

EXAMPLE.

Required the content of an oblate, and of an oblong, ſpheroid; the axes being 50 and 30.

Firſt, $50 \times 30 \times .78539816 = 1178.09724 =$ the area of the ellipſe.

Then $1178.09724 \times \frac{2}{3} \times 30 = 23561.9448 =$ the oblong ſpheroid.

And $1178.09724 \times \frac{2}{3} \times 50 = 39269.908 =$ the oblate one.



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ſemi-ſphere, as in prob. 24 ſect. 1.—And in like manner $f$ and $r$ being ſuppoſed equal to each other in the values of the fruſtums and ſegments of a ſpheroid, in the preceding corollaries, will give the values of the like parts of a ſphere.

*Corol.* 5. All ſpheres and ſpheroids are to each other as the fixed axes drawn into the ſquares of the revolving axes.

*Corol.* 6. Any ſpheroids, and ſpheres, of the ſame revolving axe, as alſo their like or correſponding parts cut off by planes perpendicular to the ſaid common axe, are to one another as their other or fixed axes. This follows from the foregoing corollaries.

*Corol.* 7. But if their fixed axes be equal, and their revolving axes unequal, the ſpheroids and ſpheres, with their like parts terminated by planes perpendicular to the common fixed axe, will be to each other as the ſquares of their revolving axes.

*Corol.* 8. An oblate ſpheroid is to an oblong ſpheroid, generated from the ſame ellipſe, as the longer axe of the ellipſe is to the ſhorter. For, if T be the tranſverſe axe, and C the conjugate; the oblate ſpheroid will be $= \frac{2}{3} \pi T^2 C$, and the oblong $= \frac{2}{3} \pi C^2 T$; and theſe quantities are in the ratio of T to C.

*Corol.* 9. And if about the two axes of an ellipſe, be generated two ſpheres and two ſpheroids, the four ſolids will be continual proportionals, and the common ratio will be that of the two axes of the ellipſe; that is, as the greater ſphere, or the ſphere upon the greater axe, is to the oblate ſpheroid, ſo is the oblate ſpheroid to the oblong ſpheroid, ſo is the oblong ſpheroid to the leſs ſphere, and ſo is the tranſverſe axe to the conjugate. For theſe four bodies will be as $T^3$, $T^2C$, $TC^2$, $C^3$, where each term is to the conſequent one, as T to C.

## PROBLEM XII.

*To find the Content of the Fruſtum of a Spheroid; its Ends being Perpendicular to one of the Axes, and one of them paſſing through the Center.*

### RULE I.

To the area of the leſs end, add twice that of the greater; multiply the ſum by the altitude of the fruſtum, and $\frac{1}{3}$ of the product will be the content.—By corollary 1 to the laſt problem.

That is, $(2D^2 + d^2) \times \frac{1}{3}an =$ the fruſtum whoſe ends are perpendicular to the fixed axe. Where D is the diameter of the greater end, $d$ that of the leſs, $a$ the altitude, and $n = .785398$.

And $(2TC + tc) \times \frac{1}{3}an =$ the fruſtum whoſe ends are parallel to the fixed axe. Where T and C are the tranſverſe and conjugate axes of the greater end, and $t$ and $c$ thoſe of the leſs end.

*Note.* It is evident that the double of the fruſtum will give the content of the zone, or ſpheroidal caſk.

### EXAMPLE I.

There is a caſk in the form of the middle fruſtum, or zone, of an oblong ſpheroid; the bung diameter is 30, the head diameter 18, and the length of the caſk 40 inches; what is the content in ale and wine gallons?

Here $D = 30$, $d = 18$, and $a = 40$.

Therefore $(2D^2 + d^2) \times \frac{1}{3}an = (2 \times 30^2 + 18^2) \times .2618 \times 40 = 2124 \times 10.472 = 22242.528 =$ the content in inches.

Then, ſince the gallon ale meaſure contains 282 cubic inches, and the wine gallon 231, we have $22242.528 \div 282 = 78.874$ the ale gallons.

And $22242.528 \div 231 = 96.288$ the wine gallons.

EX-

EXAMPLE II.

If a veſſel, in the form of the middle fruſtum of an oblate ſpheroid, have the diameter of each end 40, in the middle 50, and its length 18 inches; what is its content in ale and wine gallons?

Here $(2D^2 + d^2) \times \frac{1}{3}an = (2 \times 50^2 + 40^2) \times .2618 \times 18 = 118800 \times .2618 = 31101.84$ cubic inches.

Then $31101.84 \div 282 = 110.29$ ale gallons.

And $31101.84 \div 231 = 134.64$ wine gallons.

EXAMPLE III.

In the fruſtum of an oblong ſpheroid, the greater end is the generating ellipſe, whoſe axes are 50 and 30, the axes of its leſs end 40 and 24, and its height 9 inches; required the content in Wincheſter buſhels.

Here $(2Dc + dc) \times \frac{1}{3}an = (2 \times 50 \times 30 + 40 \times 24) \times .2618 \times 9 = 9330.552$ cubic inches.

But 268.8 inches = a gal. or 2150.4 a corn buſhel.

And therefore $9330.552 \div 2150.4 = 4.339$ buſhels.

EXAMPLE IV.

In the fruſtum of an oblate ſpheroid, the greater end is the generating ellipſe, whoſe axes are 50 and 30, and the height is 20 inches; required the ſolidity.

Here, by the nature of the ellipſe, as $50 : 30 :: \sqrt{(25 + 20) \times (25 - 20)} = \sqrt{45 \times 5} = 15 : 9$ = the ſemi-conjugate axe of the leſs end.

And,

And, by prop. 1 ſect. 2, as 30 : 50 :: 9 : 15 = the ſemi-tranſverſe axe of the leſs end.

Then $(2TC + tc) \times \frac{1}{3}na = (2 \times 50 \times 30 + 30 \times 18) \times \cdot 2618 \times 20 = 118 \times 30 \times 20 \times \cdot 2618 = 70800 \times \cdot 2618 = 18535 \cdot 44$ cubic inches.

RULE II.

From 3 times the ſquare of the ſemi-axe perpendicular to the ends of the fruſtum, ſubtract the ſquare of the height of the fruſtum; then multiply the difference by $\frac{1}{3}$ of the height, and the product by 3·14159 &c; and call the laſt product P.

Then

1. *If the ends be parallel to the fixed axe,*

As the revolving axe is to the fixed axe, ſo will P be to the content of the fruſtum.

2. *When the ends are perpendicular to the fixed axe,*

As the ſquare of the fixed axe is to the ſquare of the revolving axe, ſo is P to the content of the fruſtum.

That is, $\frac{3ff - bb}{3ff} \times pbrr$ will be the fruſtum whoſe ends are perpendicular to the fixed axe. And $\frac{3rr - bb}{3r} \times pbf$ the fruſtum whoſe ends are parallel to the fixed axe: $f$ being the fixed, and $r$ the revolving ſemi-axe, $b$ the height, and $p$ = 3·14159.

EXAMPLE I.

If the axes of an oblong ſpheroid be 50 and 30, required the content of a fruſtum whoſe ends are perpendicular to the fixed axe, and one of them paſſing through the center, the height of the fruſtum being 20 inches.

Here

Here $\frac{3ff-bb}{3ff} \times pbrr = \frac{3\times 25^2-20^2}{3\times 25^2} \times 3{\cdot}14159 \times 20 \times 15^2 = 1475 \times 3{\cdot}14159 \times \frac{1.2}{5} = 11121{\cdot}23799$ = the content required.

EXAMPLE II.

If from the ſame ſpheroid be cut a fruſtum whoſe height is 9, the ends being parallel to the fixed axe, and one of them paſſing through the center, what will be its content?

Here $\frac{3rr-bb}{3r} \times pbf = \frac{3\times 15^2-9^2}{3\times 15} \times 9 \times 25 \times 3{\cdot}14159 = 594 \times 5 \times 3{\cdot}14159 = 9330{\cdot}53018$ = the content required.

EXAMPLE III.

If from an oblate ſpheroid, whoſe axes are 50 and 30, a fruſtum, whoſe height is 9, and its ends perpendicular to the fixed axe, be cut; what will be its ſolidity?

Here $\frac{3ff-bb}{3ff} \times pbrr = \frac{3\times 15^2-9^2}{3\times 15^2} \times 9 \times 25^2 \times 3{\cdot}14159 = 594 \times \frac{2.5}{3} \times 3{\cdot}14159 = 15550{\cdot}88363$ the content required.

EXAMPLE IV.

In the ſame ſpheroid, it is required to find the content of a fruſtum whoſe height is 20, the ends being parallel to the fixed axe.

Here $\frac{3rr-bb}{3r} \times pbf = \frac{3\times 25^2-20^2}{3\times 25} \times 20 \times 15 \times 3{\cdot}14159 = 1475 \times 4 \times 3{\cdot}14159 = 18535{\cdot}39665$ the content required.

PRO-

PROBLEM XIII.

*To find the Content of a Spheroidal Caſk, not full, ſtanding upon its End, the Axe being Perpendicular to the Horizon. That is, of the Fruſtum of an Oblong Spheroid, the Ends being Perpendicular to the Axe, but neither of them paſſing through the Center.*

Multiply the difference of the ſquares of the diameters of the ends, by 4 times the ſquare of the difference between the height of the liquor and half the length of the caſk, and divide the product by the ſquare of the length of the caſk; ſubtract the quotient from 3 times the ſquare of the bung diameter, and multiply the remainder by the aforeſaid difference between the height of the liquor and $\frac{1}{2}$ the length of the caſk; then the product multiplied by ·261799 will give the quantity by which the caſk is more or leſs than half full; and which, therefore, being added to, or taken from, half the content of the caſk, will give the content of the part filled.

That is, $[(2b^2 + h^2)\,\frac{1}{2}l \pm d(3b^2 - \frac{4dd}{ll}(b^2 - h^2))] \times \frac{1}{3}n$ = the content of the part filled, called the ullage; uſing the upper or under ſigns, according as the caſk is more or leſs than half full; where $b$ and $h$ = the bung and head diameters, $l$ = the length of the caſk, $n$ = ·785398, and $d = \frac{1}{2}l \sim w$, $w$ being the wet part of $l$, or the height of the liquor in the caſk.*

EX-

* DEMONSTRATION.

For, if D be the diameter at the ſurface of the liquor, by the property of the ellipſe $b^2 - h^2 : \frac{1}{4}l^2 :: b^2 - D^2 : d^2$; and hence $D^2 = b^2 - \frac{4dd}{ll} \times (b^2 - h^2)$.

But, by the laſt problem, $(2b^2 + h^2) \times \frac{1}{6}ln$ = half the content

EXAMPLE I.

If a ſpheroidal caſk, whoſe head and bung diameters are 18 and 30 inches, and length 40 inches, be filled to the height of 30 inches; how many ale and wine gallons are in it?

Here $b = 30$, $b = 18$, $l = 40$, $w = 30$, and $d = \frac{1}{2}l \sim w = 30 - 20 = 10$.

Then $[3b^2 - \frac{4dd}{ll}(b^2 - b^2)] \times d = (2700 - \frac{400}{1600} \times 48 \times 12) \times 10 = (2700 - 144) \times 10 = 25560$.

And $(2b^2 + b^2) \times \frac{1}{2}l = (1800 + 18 \times 18) \times 20 = 42480$.

Conſequently $(42480 + 25560) \times \frac{1}{231}n = 68040 \times .2617993878 = 17812.8303458$ is the content in inches; and being divided by 282 and 231, gives 63.166065 ale, and 77.111819 wine gallons.

EXAMPLE II.

If the height of the liquor in the ſame caſk be only 10 inches, required the ullage.

Here $d = \frac{1}{2}l \sim w = 20 - 10 = 10$ the ſame as before, and therefore the part which in the laſt example was added, muſt here be ſubtracted; ſo that $(42480 - 25560) \times \frac{1}{231}n = 16920 \times .2617993878 = 4429.6456415$ is the content in inches $= 15.707963$ ale, and 19.175955 wine gallons.

PRO-

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tent of the caſk, and $[3b^2 - \frac{4dd}{ll}(b^2 - b^2)] \times \frac{1}{3}dn =$ the fruſtum by which the part filled is more or leſs than half the caſk.

Conſequently $[(2b^2 + b^2)\frac{1}{2}l \pm (3b^2 - \frac{4dd}{ll} \times (b^2 - b^2)d)] \times \frac{1}{231}n =$ the part filled.

## PROBLEM XIV.

*To find the Solidity of the Segment of a Spheroid whoſe Baſe is Perpendicular to one of the Axes.*

### RULE I.

From 3 times the ſemi-axe perpendicular to the baſe of the ſegment, take the height of the ſegment, multiply the remainder by $\frac{1}{3}$ of the ſquare of the height; then multiply the product by 3·14159, and call the laſt product $\mathfrak{q}$.

Divide the axe which is parallel to the baſe by the other axe, and call the quotient $q$.

Then for the ſegment whoſe baſe is perpendicular to the fixed axe, multiply $\mathfrak{q}$ by the ſquare of $q$; and for the other ſegment, multiply $\mathfrak{q}$ by $q$.

That is, $\mathfrak{q}qq = \frac{3f-b}{3ff} \times prrbb =$ the ſegment whoſe baſe is perpendicular to the fixed axe.

And $\mathfrak{q}q = \frac{3r-b}{3r} \times pfbb =$ the other ſegment, whoſe baſe is parallel to the fixed axe.

Where $b$ is the height of the ſegment, and the other ſymbols as in the 12th problem.

### EXAMPLE. I.

If from an oblong ſpheroid, whoſe axes are 50 and 30, be cut a ſegment whoſe baſe is perpendicular to the fixed axe, its height being 5; required the content of it.

Here $\frac{3f-b}{3ff} \times prrbb = \frac{3\times25-5}{3\times25^2} \times 15^2 \times 5^2 \times 3{\cdot}14159$
$= 70 \times 3 \times 3{\cdot}14159 = 659{\cdot}73445 =$ the content required.

EX-

EXAMPLE II.

If from the ſame ſpheroid be cut a ſegment whoſe height is 6, what will its content be, ſuppoſing its baſe to be parallel to the fixed axe?

Here $\frac{3r-h}{3r} \times pfhh = \frac{3\times 15-6}{3\times 15} \times 25 \times 6^2 \times 3{\cdot}14159$ $= 39 \times 20 \times 3{\cdot}14159 = 2450{\cdot}44226 =$ the content required.

EXAMPLE III.

Required the ſolidity of the ſegment of an oblate ſpheroid, whoſe axes are 50 and 30; the height of the ſegment being 6, and its baſe perpendicular to the fixed axe.

Here $\frac{3f-h}{3ff} \times prrhh = \frac{3\times 15-6}{3\times 15\times 15} \times 25 \times 25 \times 6 \times$ $6 \times 3{\cdot}14159 = 1300 \times 3{\cdot}14159 = 4084{\cdot}07044$ $=$ the content required.

EXAMPLE IV.

To find the content of a ſegment of the ſame ſpheroid, its baſe being parallel to the fixed axe, and its height 5.

Here $\frac{3r-h}{3r} \times pfhh = \frac{3\times 25-5}{3\times 25} \times 15\times 5\times 5\times 3{\cdot}14159$ $= 70 \times 5 \times 3{\cdot}14159 = 1099{\cdot}55742 =$ the content required.

RULE II.

*For the Segment whoſe Baſe is Parallel to the Fixed Axe; having given its Height and the Diameters of its End.*

To the ſquare of the height add 3 times the ſquare of the leſs or greater ſemi-axe of the baſe, according

according as it is the ſegment of an oblong or oblate ſpheroid; divide the ſum by the ſame ſemi-axe, and multiply the quotient by the other ſemi-axe, the product by the height, and this product by ·52359, will give the content of the ſegment.

That is, $\frac{3A^2+b^2}{6A} \times p\,B\,b =$ the content of the ſegment.

Where $b$ is its height, A the leſs or greater ſemi-diameter of the baſe, according as the ſegment is that of an oblong or oblate ſpheroid; B is the other ſemi-diameter, and $p = 3{\cdot}14159$.

EXAMPLE I.

It is required to find the content of the ſegment of an oblong ſpheroid, whoſe baſe is parallel to the fixed axe; its height being 6, and the axes of its elliptic baſe 40 and 24.

Here $\frac{3AA+bb}{6A} \times p\,B\,b = \frac{3\times 12^2+6^2}{6\times 12} \times 20\times 6\times 3{\cdot}14159$ $= 13 \times 60 \times 3{\cdot}14159 = 2450{\cdot}44226$ the content required.

EXAMPLE II.

Required the content of the ſegment of an oblate ſpheroid, whoſe baſe is parallel to the fixed axe; its height being 5, and the diameters of its baſe 18 and 30.

Here $\frac{3A^2+b^2}{6A} \times p\,B\,b = \frac{3\times 15^2+5^2}{6\times 15} \times 9\times 5\times 3{\cdot}14159$ $= 70 \times 5 \times 3{\cdot}14159 = 1099{\cdot}55742$ the content required.

RULE

RULE III.

*For a Segment whoſe Baſe is Perpendicular to the Fixed Axe; having given the Height, the Diameter of its Baſe, and a Diameter in the Middle between its Baſe and Vertex.*

To the ſquare of the diameter of the baſe, add 4 times the ſquare of the diameter in the middle, or the ſquare of twice this diameter; multiply the ſum by the height, and the product again by ·1308996 for the content.

That is, $(D^2 + 4d^2) \times \frac{1}{6}nb =$ the content of the ſegment; D being the diameter of the baſe, $d$ the diameter in the middle, $b$ the height, and $n =$ ·785398.*



* DEMONSTRATION.

By rule 1 the ſegment is $= 4nrrbb \times \frac{3f-b}{3ff}$. But, by the nature of the ellipſe, D is $= \frac{2r\sqrt{2fb-bb}}{f}$, and $d = \frac{2r\sqrt{fb-\frac{1}{4}bb}}{f}$; hence $f = b \times \frac{4dd - DD}{8dd - 4DD}$, and $\frac{r}{f} = \frac{\sqrt{4dd - 2DD}}{2b}$; which values being ſubſtituted in the above value of the ſegment, gives $\frac{1}{2}nb \times (DD + 4dd)$ for the value of the ſegment in terms of D, $d$, and $b$.

SCHOLIUM.

This theorem I have inveſtigated in order to expreſs the value of this ſegment independent of the axes of the ſpheroid. For this purpoſe I was under a neceſſity of introducing another dimenſion of the ſegment, becauſe it is not determinable from its baſe and height alone, as the other ſegment was.

EXAMPLE I.

What is the content of the ſegment of an oblong ſpheroid, whoſe baſe is perpendicular to the fixed axe; its height being 5, the diameter of its baſe 18, and its middle diameter $3\sqrt{19}$?

Here $(DD + 4dd) \times \frac{1}{6}nb = [18^2 + (6\sqrt{19})^2] \times \frac{5}{6} \times$ $\cdot 785398 = (3^2 + 19) \times 30 \times \cdot 785398 = 210 \times$ $3 \cdot 14159 = 659 \cdot 73445$ the content required.

EXAMPLE II.

Required the content of the ſpheroidal ſegment whoſe height is 6, its baſe diameter 40, and diameter in the middle 30; the baſe being perpendicular to the fixed axe of the ſpheroid.

Here $(DD + 4dd) \times \frac{1}{6}bp = (40^2 + 60^2) \times \frac{6}{6} \times \cdot 785398$ $= (2^2 + 3^2) \times 20^2 \times \cdot 785398 = 1300 \times 3 \cdot 14159$ $= 4084 \cdot 07044$ the content required.

PROBLEM XV.

*To find the Content of the Second Segment of a Spheroid.*

As a ſphere is to the ſpheroid, ſo is any part of the ſphere to the like part of the ſpheroid.*

PRO-

* DEMONSTRATION.

For the like parts of any quantities are as the wholes. And that the ſecond ſegments, FGH, *fgh*, are like parts of the ſphere and ſpheroid, is evident from the nature of the figures.

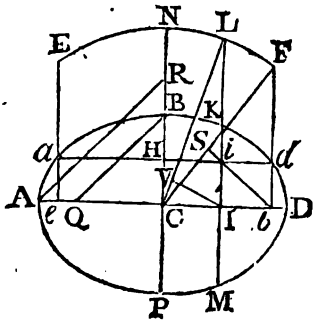
*Corol.*

## PROBLEM XVI.

*To find the Surface of an Elliptic Spindle; or of any Fruſtum or Segment of it.*

### GENERAL RULE.

From the ſpheroidal ſurface, generated by any arc BK of the ellipſe, take the product of the ſame arc BK, and the circumference of the circle whoſe diameter is equal to twice CH, the diſtance of the centers of the ſpheroid and ſpindle; and the remainder will be




 the

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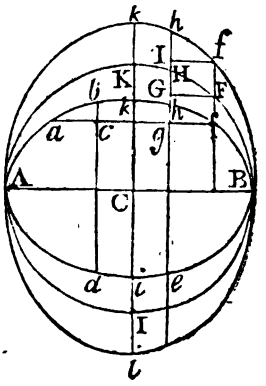
*Corol.* 1. And if the ſphere AKBI and ſpheroid A*k*B*i* have one axe AB common; then the whole ſolids and the ſecond ſegments FGH, *fgh*, will be to each other, as the ſquares of the other axes KI, *ki*, if the common axe AB be the fixed one. Or they will be to each other, as the other axes KI, *ki*, ſimply, if the common axe be the revolving one. For, if A be the axe of the ſphere, F and R the fixed and revolving axes of the ſpheroid; the bodies will be to each other as $A^3$ to $FR^2$; hence if A = F, they will be as $A^2$ to $R^2$; but if A = R, they will be as A to F.




*Corol.* 2. Hence may be found the true quantity of liquor in a ſpheroidal caſk, not full, whoſe axe is parallel to the horizon.

For if from the ſegment *akf* be taken the double of the ſecond ſegment *fgh*, there will remain the part *bhgc*; which taken from the whole caſk *bhed* will leave the part *cdeg*.

And after the ſame manner may be found the quantity of liquor in a ſpheroidal caſk partly filled, and ſtanding a-tilt, with its axe inclined to the horizon.

the ſurface of the part of the ſpindle generated by the arc BK about *ad* parallel to AD the fixed axe of the ſpheroid.*

EXAMPLE I.

Given the axes of an ellipſe 50 and 40, to find the ſurface of the ſpindle generated from an arc of that ellipſe, the length of the ſpindle being 30.

Here AD = 50, BP = 40, $ad = eb = 30$, $Ae = AC - eC = 25 - 15 = 10$, and $eD = CD + Cb = 25$

* DEMONSTRATION.

For the fluxion $\dot{s}$ of the ſpindular ſurface is $= \dot{z}$ the fluxion of the arc BK drawn into $iK \times 2p = 2p\dot{z} \times (1K - 1i)$; and the fluent is equal to the ſpheroidal ſurface BKMP $- 2pz \times 1i$ or CH.

*Corol.* 1. The ſpindular ſurface generated by BK, is equal to the ſpheroidal ſurface BKMP — the ſurface of a ſphere whoſe axe is a mean proportional between BK and 2CH. For $2pz \times$ CH = that ſpheric ſurface.

Or $2pz \times$ CH = a circle whoſe radius is $\sqrt{2CH \times BK}$.

*Corol.* 2. When CH is equal to nothing, the ſpindle becomes barely a ſpheroid: And when H falls below C, the ſurface of the ſphere muſt be added.

What has been hitherto done, anſwers to the ſurface generated by an arc about a line parallel to either axe of the ellipſe.

*Corol.* 3. From B the extremity of the leſs axe of the ellipſe, apply BQ = AC the ſemi-tranſverſe; and parallel thereto draw AR meeting CB, produced in R; with the center C and radius AR deſcribe the arc EF meeting the parallels *a*E, CB, IK, *b*d in F, N, I, F: Draw CL and CF, and perpendicular to them IV and *b*s.

Then the circle whoſe radius is a mean proportional between the ſum and difference of two lines, of which the one is a mean proportional

$25+15=40$. Then as AC : CB :: $\sqrt{Ae \times eD}$ : $ea$ = CH = 16, the diſtance of the centers of the ellipſe and ſpindle.

By example 1 to problem 10, the ſpheroidal ſurface generated by the arc $ad$, is 3686·8197024; and by problem 4, the length of the elliptic arc $ad$ is 30·852.

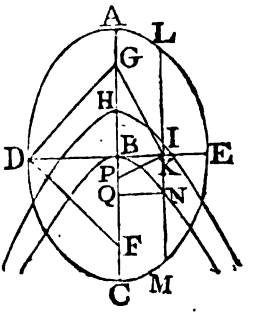
Therefore $3686{\cdot}8197 - 30{\cdot}852 \times 3{\cdot}1416 \times 32 = 3686{\cdot}8197 - 3101{\cdot}5872 = 585{\cdot}2325$ is the ſurface of the ſpindle.



proportional between BC and IV + LN, and the other a mean proportional between BK and 2CH, will be equal to the ſpindular ſurface generated by BK about $ad$ parallel to the tranſverſe axe. For the ſpheroidal ſurface is equal to a circle whoſe radius is a mean proportional between BC and IV + LN, by rule 4 to prob. 9; and $2p \times$ BK $\times$ CH is equal to a circle whoſe radius is a mean proportional between BK and 2CH; but the ſpindular ſurface is equal to the difference of thoſe two quantities; and the difference of two circles is equal to a circle whoſe radius is a mean proportional between the ſum and difference of the radii of the two circles; therefore, &c.

And, in the ſame manner, the ſurface of half the ſpindle, generated by the arc B$d$, is equal to a circle whoſe radius is a mean proportional between the ſum and difference of two lines, the one of which is a mean proportional between BC and $bs$ + NF, and the other a mean proportional between B$d$ and 2CH.

*Corol.* 4. In like manner, for the ſurface of the ſpindle whoſe axe is parallel to the leſs axe of the ellipſe: having conſtructed the annexed figure, as in page 309, the ſpindular ſurface generated by the arc CM, about NQ, will be equal to a circle whoſe radius is a mean proportional between the ſum and difference of two lines, of which the one is a mean proportional between AC and HI, and the other a mean proportional between CM and 2BQ.




This, like the laſt corollary, will be evident by comparing what is done above, with what is done in page 309.

EXAMPLE II.

Required the ſurface of the fruſtum of a ſpindle generated from an arc of the ſame ellipſe as in the laſt example; the height of the fruſtum being 15, and the central diſtance 10.

Here the arc generating the fruſtum, is equal to half the arc in the laſt; and conſequently that arc is = 15·426, and the ſpheroidal fruſtum = 1843·40985.

Then 1843·4098 — 15·426 × 3·1416 × 20 = 1843·4098 — 969·246 = 874·163 the ſurface of the fruſtum required.

PROBLEM XVII.

*To find the Solidity of an Elliptic Spindle.*

RULE I.

1. Divide the ſquare of the perpendicular axe DE by 3 times the ſquare of the parallel axe AB, and multiply the quotient by the cube of FG the axe or length of the ſpindle; and call the product P.

2. Find the area of the elliptic ſegment FDG from which the ſpindle is generated; multiply this area by 4 times CH the central diſtance, and call the product Q.

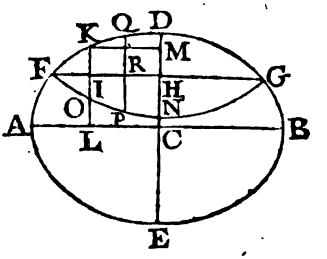
3. Multiply 1·57079 by the difference between P and Q, and the product will be the content of the ſpindle FDGN.

That

That is, $2n \times (\frac{aa}{3bb}l^3 - 4cs) =$ the ſpindle FDGN. Where $n = \cdot 78539$, $a =$ DE, $b =$ AE, $l =$ FG, $c =$ CH, and s = the ſegment FDGF.*



* DEMONSTRATION.




Let FDGN be a ſpindle generated by the arc FDG of the ellipſe ADBE. Put $a =$ FH = half the axe of the ſpindle; $b =$ CH = LI = the central diſtance, or diſtance of the centers of the ellipſe and ſpindle; $c =$ AC = the one ſemi-axe of the ellipſe, and $d =$ CD = the other ſemi-axe; $x =$ HI = CL; and $y =$ IK = HM.

By the property of the ellipſe, $c : d :: \sqrt{cc-xx} : \frac{d}{c}\sqrt{cc-xx} =$ LK, hence $IK = KL - LI = \frac{d\sqrt{cc-xx}}{c} - b = y$, and the fluxion of the ſolid $\dot{s} = py\dot{y}\dot{x} = p\dot{x} \times (dd - \frac{ddxx}{cc} - \frac{2bd\sqrt{cc-xx}}{c} + bb)$ $= p\dot{x} \times (dd - bb - \frac{ddxx}{cc} - 2b \times \frac{d\sqrt{cc-xx}}{c} - 2b^2) = pdd\dot{x} \times \frac{aa-xx}{cc} - 2pby\dot{x}$; and the fluent is $s = pddx \times \frac{3aa-xx}{3cc} - 2bp \times$ area HIKD = the fruſtum DKON.

When $x$ is $= a$, the above theorem will become $\frac{2pdd}{3cc} \times a^3 - 2bp \times$ area FDH = DFN the half of the ſpindle.

And if from the ſemi-ſpindle be taken the fruſtum, there will remain $pddee \times \frac{3a-e}{3cc} - 2bp \times$ area FKI = the ſegment FKO of the ſpindle, $e$ being the height FI of the ſegment.

*Corol.* 1. If $d$ be ſuppoſed $= c$, the ellipſe will become a circle, and accordingly the theorems above given will become the ſame with thoſe before found for the circular ſpindle.

EXAMPLE.

The axes of an ellipſe being 50 and 30, required the ſolidity of a ſpindle generated from an arc of it, about

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*Corol.* 2. If H coincide with C, $b$ will vaniſh, $a$ will be $= c$, and the theorems will become the ſame with thoſe before found for the ſpheroid.

*Corol.* 3. Putting D = DN the greateſt, and $d$ = KO the leaſt diameter of the fruſtum, $b$ = HI its height, C = CH the central diſtance, $s$ = the elliptic ſemi-ſegment KDM, and $n$ = ·78539&c. Then, in the foregoing theorems, $b = x$, $b = c$, $d = c + \frac{1}{2}D$, area HIKD $= s + \frac{1}{2}db$, $c = b \times \frac{c + \frac{1}{2}D}{\sqrt{\frac{1}{4}DD + Dc - \frac{1}{4}dd - dc}}$, and $a = b\sqrt{\frac{\frac{1}{4}DD + Dc}{\frac{1}{4}DD + Dc - \frac{1}{4}dd - dc}}$; which values being ſubſtituted in the theorems above, give $\frac{1}{3}nb \times [2DD + dd - 8c(-D + d + \frac{3s}{b})]$ for the value of the fruſtum, or half a caſk in the form of the middle zone of an elliptic ſpindle; and $\frac{1}{3}nl \times [2DD - 8c(-D + \frac{3s}{l})]$ for half the ſpindle when $s$ = the area DFH, and $l$ = HF.

*Corol.* 4. But in real practice, ſuch as caſk gauging, none of theſe rules can be uſed, becauſe we have not given either the axes of the ellipſe or the central diſtance; and to accommodate rules to that purpoſe, we muſt introduce another dimenſion of a fruſtum, beſides its length and greateſt and leaſt diameters. Thus, putting $m$ = PQ the diameter through R the middle of the length HI, and the other letters as before. Then, by the property of the ellipſe, $(c + \frac{1}{2}D)^2 - (c + \frac{1}{2}d)^2 : 4 :: (c + \frac{1}{2}D)^2 - (c + \frac{1}{2}m)^2 : 1$, hence $4(c + \frac{1}{2}m)^2 - 3(c + \frac{1}{2}D)^2 = (c + \frac{1}{2}d)^2$, and $c = \frac{1}{4} \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m}$. And the laſt two theorems become $\frac{1}{3}nb \times [2D^2 + d^2 - 2 \times \frac{3D^2 + d^2 - 4m^2}{-3D - d + 4m}(-D + d + \frac{3s}{b})]$ for the fruſt. DKON, and $\frac{1}{3}nl \times [2D^2 - 2 \times \frac{3D^2 - 4m^2}{-3D + 4m}(-D + \frac{3s}{l})]$ for the ſemi-ſpindle DFN.

about its chord parallel to, and at the distance of, 9 from the transverse axe.

Here $15 - 9 = 6 =$ DH the height of the generating segment; and, by the property of the ellipse, DC : AC :: $2\sqrt{\text{DH} \times \text{HE}}$ : FG $= 40$, the base of the elliptic segment, or length of the spindle; also, by prob. 6, the area of the segment is 167·7345.

Then $\frac{3\cdot14159}{2} \times (\frac{30\times30}{3\times50\times50} \times 40^3 - 4\times9\times167\cdot7345)$
$= 18 \times 3\cdot14159 \times (213\frac{1}{3} - 167\cdot7345) = 2578\cdot55$
= the content required.

RULE II.

Divide 3 times the generating segment by the length of the spindle; from the quotient subtract the greatest diameter of the spindle; multiply the remainder by 4 times the central distance, and subtract the product from the square of the greatest diameter; then the difference multiplied by the length of the spindle, and the product by ·5236 will give the content of the spindle.

That is, $\frac{2}{3}nl \times [\text{D}^2 - 4c(-\text{D} + \frac{3s}{l})] =$ the spindle.

Where D = DN the greatest diameter, and the rest of the symbols as in the last rule.—By corollary 3.

EXAMPLE.

Required the solidity of an elliptic spindle, whose length is 40, greatest diameter 12, and the central distance 9.

Here $\frac{12}{2} = 6 =$ the height of the generating segment, which is therefore the same as before, the area being 167·7345.

Then

Then

$3{\cdot}14159 \times \frac{40}{6} \times [12^2 - 4 \times 9 \times (-12 + \frac{3 \times 167{\cdot}7345}{40})]$
$= 3{\cdot}14159 \times 80 \times (12 - 1{\cdot}74024) = 2578{\cdot}56$ the content required.

### RULE III.

From three times the ſquare of the greateſt diameter, take 4 times the ſquare of the diameter in the middle between the greateſt diameter and the end; and from 4 times the ſaid middle diameter, take 3 times the ſaid greateſt diameter: Divide the former difference by the latter, and $\frac{1}{4}$ of the quotient will be the central diſtance.

That is, $\frac{1}{4} \times \frac{3DD - 4mm}{4m - 3D} =$ the central diſtance; D being the greateſt diameter, and $m$ the middle diameter.

Then proceed as in the laſt rule.

This is proved in corollary 4.

### EXAMPLE.

If, as in the example to the laſt rule, the greateſt diameter be 12, and the length 40, required the content, ſuppoſing the diameter at $\frac{1}{4}$ of the length to be $6 \times (\sqrt{21} - 3)$ or 9·49546.

Here $\frac{1}{4} \times \frac{3DD - 4mm}{4m - 3D} = \frac{1}{4} \times \frac{3 \times 12^2 - 4 \times 6^2(\sqrt{21} - 3)^2}{4 \times 6(\sqrt{21} - 3) - 3 \times 12}$
$= \frac{12}{4} \times \frac{3 - 21 + 6\sqrt{21} - 9}{2\sqrt{21} - 6 - 3} = 3 \times \frac{6\sqrt{21} - 27}{2\sqrt{21} - 9} = 3 \times 3 = 9$
= the central diſtance, the ſame as in the laſt example; and therefore the content will be 2578·56, as before.

PRO-

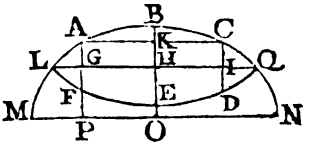
## PROBLEM XVIII.

*To find the Solidity of the Fruſtum of an Elliptic Spindle, or the Content of a Caſk in form of the Middle Zone of ſuch a Spindle.*

### RULE I.

1. Divide the ſquare of the leſs axe of the ellipſe by the ſquare of the greater; multiply the quotient by the length of the fruſtum, or half the length of the caſk; then multiply the product by the difference between the ſquare of the ſaid half length of the caſk, and 3 times the ſquare of half the length of the whole ſpindle; and call $\frac{1}{3}$ of the product P.

That is, $P = ccl \times \frac{3LL - ll}{3tt}$, $t$ being the tranſverſe MN, and $c$ the conjugate axe 2BO of the ellipſe, $l$ = GH half the length of the caſk, and L = LH half the length of the whole ſpindle.




2. Multiply the central diſtance OH by the generating area ABHG; and call double the product Q.

That is, Q = 2OH × ABHG.

3. Then the difference between P and Q multiplied by 3·14159, will give the content of the fruſtum ABEF, or half the caſk ACDF.

That is, $(P - Q) \times p$ = ABEF = $\frac{1}{2}$ACDF.

This follows from the demonſtration of the laſt problem.

*Note.* It is evident that the ſum or difference of two fruſtums, will give the ullage of a caſk ſtanding with its axe perpendicular to the horizon.

EX-

EXAMPLE.

The axes of an ellipſe being 40 and $66\frac{2}{3}$; it is required to find the content of the fruſtum of a ſpindle generated by an arc of the ellipſe; the length of the fruſtum being 20, and the central diſtance 4.

Here, as $40 : 66\frac{2}{3} :: \sqrt{16 \times 24} : 13\frac{1}{3}\sqrt{6} =$ LH half the length of the ſpindle.

And $ccl \times \frac{3ll - ll}{3tt} = 40 \times 40 \times 20 \times \frac{3 \times 6 \times 13\frac{1}{3} \times 13\frac{1}{3} - 20 \times 20}{3 \times 66\frac{2}{3} \times 66\frac{2}{3}} = 24 \times 280 = 6720 =$ P.

But the elliptic ſegment ABK is $= 54\frac{1}{2}$, to which adding the rectangle AH $= 20 \times 12 = 240$, makes $294\frac{1}{2} =$ the generating area ABHG; which being multiplied by 20H or 8, produces $2356 =$ Q.

Then $(P - Q) \times p = (6720 - 2356) \times 3{\cdot}14159 = 13709{\cdot}91033$ the content required.

RULE II.

Divide 3 times the elliptic ſegment whoſe chord is the length of the caſk, by the ſaid length; to the quotient add the leaſt or head diameter; and from the product ſubtract the greateſt or bung diameter; and multiply the remainder by 8 times the central diſtance; then take the product from the ſum of the ſquare of the head diameter, and double the ſquare of the bung diameter; multiply the difference by the length, and the product again by ·2618 for the content of the caſk.

That is, $[2DD + dd - 8c(-D + d + \frac{3s}{l})] \times \frac{1}{3}nl$ = the content of the caſk; where $D =$ BE the bung diameter, $d =$ AF the head diameter, $c =$ HO the central

central diſtance, $l =$ G I the length, $s =$ the ſegment A B C, and $n = \cdot 785398$.

This is proved at corollary 3 to the laſt problem.

EXAMPLE.

The bung and head diameters of a caſk, which is the middle zone of an elliptic ſpindle, being 32 and 24, and its length 40 inches; required the content in ale and wine gallons, ſuppoſing the diſtance of the centers of the ellipſe and ſpindle to be 4 inches.

Here $D = 32$, $d = 24$, $l = 40$, $c = 4$, and the ſegment A B C $= s = 109$.

Then $(2DD + dd - 8c(-D + d + \frac{3s}{l})] \times \frac{1}{3}nl =$ $(2624 - 5 \cdot 6) \times 40 \times \cdot 261799 = 27419 \cdot 8219$ cubic inches.

Which being divided by 282, and 231, we have $97 \cdot 2334$ ale gallons, and $118 \cdot 7005$ wine gallons, for the content required.

RULE III.

From the ſum of the ſquare of the leaſt or head diameter, and 3 times the ſquare of the greateſt or bung diameter, take 4 times the ſquare of the diameter equidiſtant from the two former; and from 4 times the ſaid middle diameter, take the ſum of the ſaid leaſt and 3 times the greateſt diameter; then divide the former difference by the latter, and $\frac{1}{4}$ of the quotient will be the central diſtance; with which proceed as in the 2d rule.

That is, $\frac{1}{4} \times \frac{3DD + dd - 4mm}{-3D - d + 4m}$ is the central diſtance; $m$ being the middle diameter, and D and $d$ the other diameters, as before.

By corollary 4 to the laſt problem.

EXAMPLE.

If the bung and head diameters be 32 and 24, and the length 40 inches, as in the laſt example; required the content, ſuppoſing the middle diameter to be $4\sqrt{91} - 8$.

Here $\frac{3DD+dd-4mm}{-3D-d+4m} \times \frac{1}{4} = \frac{3 \times 22^2 + 24^2 - 64(\sqrt{91}-2)^2}{-3 \times 32 - 24 + 16(\sqrt{91}-2)}$ $\times \frac{1}{4} = \frac{-76+8\sqrt{91}}{-19+2\sqrt{91}} = 4 = c$ the central diſtance, the ſame as in the laſt example; and ſince the other parts are all the ſame, the content muſt likewiſe be the ſame.

PROBLEM XIX.

*To find the Content of the Segment of an Elliptic Spindle.*

1. Multiply the ſquare of the altitude of the ſegment by the ſquare of the leſs axe of the ellipſe, and divide the product by the ſquare of the greater axe; multiply the quotient by the difference between $\frac{1}{2}$ the length of the whole ſpindle, and $\frac{1}{3}$ of the altitude of the ſegment, and call the product P.

That is, $P = aacc \times \frac{\frac{1}{2}l - \frac{1}{3}a}{tt}$; where $c$ is the conjugate, and $t$ the tranſverſe axe of the ellipſe, $a$ the altitude of the ſegment, and $l$ the length of the whole ſpindle.

2. Multiply double the generating area by the diſtance of the centers of the ellipſe and ſpindle, and call the product Q.

That is, $Q = 2CA$, where $C = OH$, and $A =$ the area LAG.

3. Then

3. Then the difference between P and Q drawn into 3·14159, will be the content of the ſegment LAF of the ſpindle.

That is, $(P-Q)\times p = (aacc\times\frac{\frac{1}{2}l-\frac{1}{2}a}{tt}-2AC)\times p$ = the content of the ſegment.

By the demonſtration of problem 17.

EXAMPLE.

The axes of an ellipſe being 50 and 30, required the content of the ſegment of a ſpindle, whoſe height is 10, and the central diſtance 9.

Here $t = 50$, $c = 30$, $a = 10$, and $c = 9$.

Then, by the nature of the ellipſe, as 30 : 50 :: $\sqrt{(15+9)\times(15-9)} = \sqrt{24\times 6} = 12 : 20 = \frac{1}{2}l =$ LH half the length of the ſpindle.

And, by the ſame, as 50 : 30 :: $\sqrt{MP\times PN} = \sqrt{(25+10)\times(25-10)} = \sqrt{35\times 15} = 5\sqrt{21} : 3\sqrt{21} = 13{\cdot}71495534 =$ AP. Hence BO — AP $= 1{\cdot}285 =$ BK.

But $\frac{BK}{2BO} = \frac{1{\cdot}285}{30} = {\cdot}0428\frac{1}{7}$, and $\frac{BH}{BO} = \frac{6}{30} = \frac{1}{5} = {\cdot}2$; the tabular numbers anſwering to which two quotients, in the table of circular ſegments, are ·01166678 and ·1118238; hence $50\times 30\times(\cdot 1118238-\cdot 01166678) = 1500\times\cdot 10015702 = 150{\cdot}23553 =$ the area LBQ; from the half of which taking the rectangle AH = AG × GH = 4·714955 × 10 = 47·149, leaves 27·968765 = A for the generating area LAG.

Then

Then $(aacc \times \frac{\frac{1}{2}l - \frac{1}{3}a}{tt} - 2AC) \times p =$ $(100 \times 900 \times \frac{20 - 3\frac{1}{3}}{2500} - 18 \times 27{\cdot}968765) \times 3{\cdot}14159$ $= (36 \times \frac{5}{7}^{\circ} - 18 \times 27{\cdot}968765) \times 3{\cdot}14159 = 18 \times 5{\cdot}364568 \times 3{\cdot}14159 = 303{\cdot}35917 =$ the content of the ſegment LAF required.

## PROBLEM XX.

*To find the Content of an Univerſal Spheroid, or a Solid conceived to be Generated by the Revolution of a Semi-Ellipſe about its Diameter, whether that Diameter be one of the Axes of the Ellipſe or not.*

### RULE I.

Divide the ſquare of the product of the axes of the ellipſe, by the axe of the ſolid, or the diameter about which the ſemi-ellipſe is conceived to revolve; multiply the quotient by ·5236, and the product will be the content required.

That is, $\frac{T^2 C^2}{d} \times {\cdot}5236 =$ the content; T and C being the tranſverſe and conjugate axes of the ellipſe, and *d* the axe of the ſolid.

### RULE II.

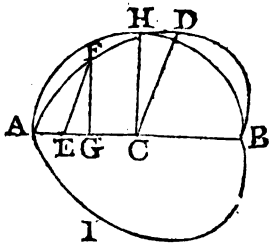
The continual product of ·5236, the diameter about which the revolution is made, the ſquare of its conjugate diameter, and the ſquare of the ſine of the angle made by thoſe diameters, the radius being 1, will be the content.

That

That is, $dccss \times .5236 =$ the content; $c$ being the conjugate diameter to $d$, and $s$ the sine of the angle made by the diameters.*



* DEMONSTRATION.




Let ADBIA be a section of the solid through its axe AB; draw DC the semi-conjugate diameter to AB, as also the ordinate FE parallel to it, and let fall the perpendicular FG.

Put T and C for the transverse and conjugate axes of the ellipse, $d =$ the diameter AB, or axe of the solid, $c =$ 2CD $= \sqrt{T^2 + C^2 - d^2}$ its conjugate diameter; $a = \frac{TC}{dc} =$ sine of the $\angle$DCB or $\angle$FEG, and $b =$ its cosine, $x =$ AE, and $p = 3.14159$.

Then $d : c :: \sqrt{dx - xx} : \frac{c\sqrt{dx - xx}}{d} =$ EF, hence

EF $\times a = \frac{ac\sqrt{dx-xx}}{d} =$ FG, and EF $\times b = \frac{bc\sqrt{dx-xx}}{d} =$ GE;

then AG $=$ AE $+$ EG $= x + \frac{bc\sqrt{dx-xx}}{d}$; conseq. $p \times$ FG$^2 \times$ GA $=$

$\frac{paaccx}{dd} \times (dx - xx + \frac{\frac{1}{2}d - x}{d} \times bc\sqrt{dx - xx}) =$ the fluxion of the solid; the fluent of which is

$\frac{paacc}{dd} \times [\frac{1}{2}dx^2 - \frac{1}{3}x^3 + \frac{bc}{3d} \times (dx - xx)^{\frac{3}{2}}] =$ the measure of the part generated by AFG. And when $x = d$, it becomes $\frac{1}{12}pda^2c^2 = \frac{pT^2C^2}{6d}$ for the value of the whole solid.

*Corol.* 1. If $d =$ T, the rule becomes $\frac{1}{6}pTC^2$ for the oblong spheroid. And if $d =$ C, it will be $\frac{1}{6}pCT^2$ for the oblate spheroid. Also if T, C, and $d$, be all equal, the rule will be $\frac{1}{6}pd^3$ for the sphere. Which are the same with the rules before found for the same bodies.

*Corol.*

EXAMPLE I.

If the axes of an ellipſe be 50 and 30, and it be cut in two by a diameter whoſe length is 40; required the content of the ſolid generated by one of the halves about that diameter.

By rule 1, $\frac{50\times 50\times 30\times 30}{40}\times \cdot 5236 = 225000\times \cdot 1309 = 29452\cdot 5$ is the content required.

EXAMPLE II.

The diameter of an ellipſe about which it revolves being 40, its conjugate diameter $30\sqrt{2}$, and the ſine of the angle made by thoſe diameters $\frac{5}{8}\sqrt{2}$; required the content of the ſolid formed by the revolution of the ellipſe.

By rule 2, we have $40\times 30\times 30\times 2\times \frac{5}{8}\times \frac{5}{8}\times 2\times \cdot 5236 = 225000\times \cdot 1309 = 29452\cdot 5$ the content, the ſame as before.

SEC-

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*Corol.* 2. Draw CH perpendicular, and DH parallel to AB, and about the axes AB and 2CH deſcribe the ſemi-ellipſe AHB; then the ſpheroid generated by the revolution of the ſemi-ellipſe AHB, about AB, will be equal to the ſolid generated by the ſemi-ellipſe AFDB about the ſame axe AB.

For $2CH = ac$, and therefore the ſolid AFBI, or $\frac{1}{6}pda^2c^2$, is $= \frac{1}{6}pd\times (2CH)^2 =$ the ſpheroid whoſe axes are $d$ and 2CH.

And ſo the ſolids generated by all ſemi-ellipſes upon the ſame baſe and between the ſame parallels, are all equal to each other.

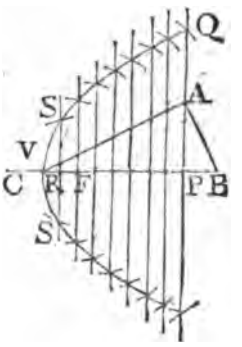
# SECTION VI.

## OF PARABOLIC LINES, AREAS, SURFACES, AND SOLIDITIES.

### PROBLEM I.

*To Conſtruct a Parabola; having given any Ordinate* PQ *to the Axe, and its Abſciſſa* VP.

1. FIND the focus F thus: Biſect PQ in A; draw AV, and AB perpendicular to it; take VF = PB, and F will be the focus.




*Arithmetically.* Divide the ſquare of the ordinate by 4 times the abſciſſa, and the quotient will be the focal diſtance VF.

2. In the axe, produced without the vertex V, take VC = VF; draw ſeveral double ordinates SRS; then with the radii CR, and center F, deſcribe arcs cutting the correſponding ordinates in the points S.

Draw a curve through all the points of interſection, and it will be the parabola required.

PROBLEM II.

*Of any Abſciſs* x, *its Ordinate* y, *and Parameter* p; *having Two given, to find the Third.*

CASE I.

*To find the Parameter.*

Divide the ſquare of the ordinate by its abſciſs, and the quotient will be the parameter.

Or, take a third proportional to the abſciſſa and ordinate, for the parameter.

That is, $p = yy \div x$.

EXAMPLE.

If the abſciſs be 9, and its ordinate 6; required the parameter.

Here $6 \times 6 \div 9 = 36 \div 9 = 4 =$ the parameter.

CASE II.

*To find the Abſciſs.*

Divide the ſquare of the ordinate by the parameter, and the quotient will be the abſciſs.

That is, $x = yy \div p$.

EXAMPLE.

If the ordinate be 6, and the parameter 4; required the abſciſs.

Here $6 \times 6 \div 4 = 36 \div 4 = 9 =$ the abſciſs.

CASE III.

*To find the Ordinate.*

Multiply the parameter by the abſciſs, and the ſquare root of the product will be the ordinate.

That is, $y = \sqrt{px}$.

EX-

EXAMPLE.

The abſciſs being 9, and the parameter 4; required the ordinate.

Here $\sqrt{9} \times 4 = \sqrt{36} = 6 =$ the ordinate.

PROBLEM III.

*Of any Two Abſciſſes* A, B, *taken upon the ſame Diameter, and their Two Ordinates* a, b; *having any Three given, to find the Fourth.*

The abſciſſes are to one another as the ſquares of their ordinates. That is, As any abſciſs is to the ſquare of its ordinate, ſo is any other abſciſs to the ſquare of its ordinate; and the contrary. Or, as the root of an abſciſs is to its ordinate, ſo is the root of another abſciſs to its ordinate.

Hence $\begin{cases} \sqrt{A} : \sqrt{B} :: a : a\sqrt{\frac{B}{A}} = \frac{a\sqrt{AB}}{A} = b. \\ \sqrt{B} : \sqrt{A} :: b : b\sqrt{\frac{A}{B}} = \frac{b\sqrt{AB}}{B} = a. \end{cases}$

And $\begin{cases} aa : bb :: A : \frac{Abb}{aa} = B. \\ bb : aa :: B : \frac{Baa}{bb} = A. \end{cases}$

EXAMPLE I.

If an abſciſs of 9 correſpond to an ordinate of 6, required the ordinate whoſe abſciſs is 16.

Here $\sqrt{9} : \sqrt{16} :: 6 : \frac{6 \times 4}{3} = 8$ the ordinate.

EXAMPLE II.

Required the abſciſs correſponding to the ordinate 6, the ordinate belonging to the abſciſs 16 being 8.

Here $8^2 : 6^2 :: 16 : 9 =$ the abſciſs.

SCHOLIUM.

The demonstration of the three preceding problems are omitted here, as they properly belong to, and are to be found in, all treatises of conic sections.

PROBLEM IV.

*To find the Length of the Curve or Arc of a Parabola, cut off by a Double Ordinate to the Axe.*

RULE I.*

Divide the double ordinate by the parameter, and call the quotient $q$.

Add

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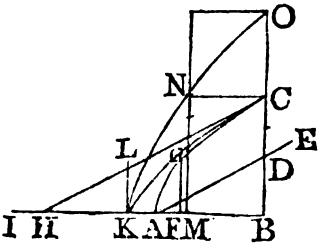
* DEMONSTRATION.

Putting $z$ = any curve beginning at the vertex, $y$ = the ordinate to the axe at the extremity of the curve, $x$ = its abscis, and $a = \frac{1}{2}$ the parameter of the axe, the equation to the curve is $2ax = yy$; hence $2a\dot{x} = 2y\dot{y}$, and $\dot{x}^2 = \frac{y^2\dot{y}^2}{aa}$; consequently $\dot{z} = \sqrt{\dot{y}^2 + \dot{x}\dot{x}} = \sqrt{\dot{y}^2 + \frac{y^2\dot{y}^2}{aa}} = \frac{\dot{y}\sqrt{aa+yy}}{a}$; and the corrected fluents give $z = \frac{y\sqrt{aa+yy}}{2a} + \frac{1}{2}a \times$ hyp. log. of $\frac{y+\sqrt{aa+yy}}{a} = \frac{1}{2}aq\sqrt{1+qq} + \frac{1}{2}a \times$ hyp. log. of $(q + \sqrt{1+qq})$, writing $q$ for $\frac{y}{a}$. And the double of this quantity will give the value of the double curve, as in the rule.

*Corol.* 1. If for $a$ be substituted its value $\frac{yy}{2x}$, there will be obtained $z = \sqrt{xx + \frac{1}{4}yy} + \frac{1}{2}a \times$ hyp. log. of $\frac{x+\sqrt{xx+\frac{1}{4}yy}}{\frac{1}{2}y}$. And this proves the truth of the construction in COTES's *Harmonia Mensurarum*, page 22, viz. that if F be the focus,




and

Add 1 to the ſquare of $q$, and call the root of the ſum $s$.

To the product of $q$ and $s$ add the hyperbolic logarithm of their ſum; then the laſt ſum multiplied by half the parameter, will be the length of the whole curve, on both ſides of the axe.

That is, the curve $c = a \times (qs + \text{hyp. log. of } q + s)$; where $q =$ the quotient of the double ordinate divided by the parameter, $s = \sqrt{1 + qq}$ and $a = \frac{1}{2}$ the parameter.



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and if AD, drawn to biſect BC in D, be produced till DE be = AF × hyp. log. of $\frac{AB + AD}{BD}$; then AE will be equal the curve AC.——For $\frac{1}{2}a =$ AF, and $\sqrt{xx + \frac{1}{4}yy} =$ AD.

*Corol.* 2. From ſect. 5 it will appear that the area of an hyperbola whoſe ſemi-tranſverſe axe is $d$, ſemi-conjugate $a$, and ordinate $y$, is $A = \frac{dy\sqrt{aa + yy}}{2a} - \frac{1}{2}ad \times$ hyp. log. of $\frac{y + \sqrt{aa + yy}}{a}$; but the parabolic curve is $c = \frac{y\sqrt{aa + yy}}{2a} + \frac{1}{2}a \times$ hyp. log. of $\frac{y + \sqrt{aa + yy}}{a}$; hence $\frac{A}{d} + c = \frac{y\sqrt{aa + yy}}{a} = \sqrt{y^2 + \frac{y^4}{aa}} = \sqrt{yy + 4xx} =$ 2AD = HC the tangent to the point C, meeting the axe BA produced in H: Conſequently $c =$ HC $- \frac{A}{d}$. Where the ſemi-tranſverſe $d$ may be taken at pleaſure.

If there be taken $d = y$, we ſhall obtain $c =$ HC $- \frac{A}{y}$; and to find the diſtance of the ordinate $y$ from the center of the hyperbola, we ſhall have $a : y$ or $d :: \sqrt{aa + yy} ; \frac{y\sqrt{aa + yy}}{a} =$ the ſaid diſtance, and which, therefore, is = HC. Whence this conſtruction.—In the axe produced take BI = HC; with the center I, ſemi-tranſverſe IK = the ordinate BC, and ſemi-conju-

*Note* 1. If the common logarithm of any number be multiplied by 2·302585093, the product will be the hyperbolic logarithm of the ſame number.

2. If the value of $s$ run into decimals, it will be much eaſieſt found by a trigonometrical table; for $s$ is the ſecant of the arc whoſe tangent is $q$, the radius being 1.

EXAMPLE.

Required the length of the curve of a parabola cut off by a double ordinate, to the axe, whoſe length is 12, the abſciſs being 2.

Here

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gate KL = the ſemi parameter FG of the parabola, deſcribe the hyperbola KC, which will paſs through C; and let the rectangle CBMN be equal the hyperbolic area KCB: Then will IM be equal to the parabolic curve AC.

When the abſciſs and tranſverſe axe of an hyperbola are given, or conſtant; not only the ordinate, but the area alſo, is as the conjugate axe; and therefore the quotient ariſing from the diviſion of the area by the ordinate, is a conſtant quantity; and conſequently the parabolic curve AC = C = HC − $\frac{B}{z}$, where B is the area, and $z$ the ordinate of any hyperbola, whoſe center is I, vertex K, and abſciſs KE. From hence ariſes the following general conſtruction, given by Mr. HUYGENS in his *Horolog. Oſcillat.* but without demonſtration.—With the center I and vertex K, as before, and any conjugate taken at pleaſure, deſcribe the hyperbola KO meeting BC, produced, if neceſſary, in O; and making the rectangle OM = the hyperbolic area KOE, MI will be equal to the parabolic curve AC, as before.

*Corol.* 3. When $y = a$ = FG, then $z = \frac{1}{2}a \times (\sqrt{2} + \text{hyp. log. of } 1 + \sqrt{2}) = \frac{1}{2}a \times 2{\cdot}2955871 = 2{\cdot}2955871 \times AF = 1{\cdot}1477935 \times FG$ = the curve AG; F being the focus.

*Corol.* 4. The lengths of ſimilar parts of parabolic curves, are as their parameters, or ordinates, or abſciſſes. For $q$ is the ſame in each.

Here $x = 2$, and $y = 6$.

Hence $a = \frac{yy}{2x} = \frac{36}{4} = 9$, and $q = \frac{y}{a} = \frac{6}{9} = \frac{2}{3}$, and $s = \sqrt{1+qq} = \sqrt{1+\frac{4}{9}} = \sqrt{\frac{13}{9}} = \frac{1}{3}\sqrt{13} = 1{\cdot}2018504$. Or, by the table of tangents and secants, the secant corresponding to the tangent $\frac{2}{3}$ or ·666 &c, is $1{\cdot}2018504 = s$.

Then, $\frac{2}{3} + 1{\cdot}2018504 = 1{\cdot}868517$, whose common logarithm is ·271497, which being multiplied by 2·302585093, produces ·6251449 for its hyperbolic logarithm; also $\frac{2}{3} \times 1{\cdot}2018504 =$ ·8012336; and the sum of these two is 1·4263785.

Therefore $9 \times 1{\cdot}4263785 = 12{\cdot}8374065$ is the length of the curve required.

RULE II.*

Putting $y$ to denote the ordinate, and $q$ the quotient arising from the division of the double ordinate by

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* DEMONSTRATION.

By the last, the fluxion of the curve was $\dot{z} = \dot{y}\sqrt{1+\frac{yy}{aa}} =$, by extracting the root, $\dot{y} \times (1 + \frac{y^2}{2a^2} - \frac{y^4}{2.4a^4} + \frac{3y^6}{2.4.6a^6}$ &c); and, by taking the fluents and writing $q$ for $\frac{y}{a}$, we obtain

$$z = y \times (1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.5} + \frac{3q^6}{2.4.6.7} \text{ \&c})$$

$$= y \times (1 + \frac{1.1}{2.3}q^2{}_A - \frac{1.3}{4.5}q^2{}_B + \frac{3.5}{6.7}q^2{}_C \text{ \&c}) = \text{the length of the curve from the vertex to the ordinate; the double of which will be that of the double curve. } \textit{Q. E. D.}$$

*Corollary.*

by the parameter, or from the diviſion of double the abſciſs by the ordinate; the length of the double curve will be denoted by the infinite ſeries

$2y \times (1 + \frac{q^2}{2.3} - \frac{q^4}{2.4.5} + \frac{3q^6}{2.4.6.7} - \frac{3.5q^8}{2.4.6.8.9}$ &c); or by

$2y \times (1 + \frac{1}{2.3}q^2 A - \frac{1.3}{4.5}q^2 B + \frac{3.5}{6.7}q^2 C - \frac{5.7}{8.9}q^2 D$ &c).

Where A, B, C, &c, denote the 1ſt, 2d, 3d, &c, terms.

*Note.* This ſeries will converge no longer than till $q = 1$, that is, when the ordinate to the curve, whoſe length is required, meets the axe in the focus; for if the ordinate $y$ be beyond the focus, it will be greater than the ſemi-parameter, conſequently $q$ will be greater than 1, and the ſeries will diverge.

EXAMPLE.

Let there be taken the ſame example as before, in which the abſciſs is 2, and the ordinate 6.

Then

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*Corollary.* The hyperbolic logarithm of $q + \sqrt{1 + qq}$ is $= q \times (1 - \frac{q^2}{2.3} + \frac{3q^4}{2.4.5} - \frac{3.5q^6}{2.4.6.7}$ &c).

For, by the laſt rule, $\frac{2z}{a}$ = hyp. log. of $(q + \sqrt{1+qq})$ is $= q\sqrt{1+qq}$

$= q \times (1 + \frac{q^2}{2} - \frac{q^4}{2.4} + \frac{3q^6}{2.4.6}$ &c); and by this rule, $\frac{2z}{a}$ is $= q \times (1 + \frac{q^2}{3} - \frac{q^4}{4.5} + \frac{3q^6}{4.6.7}$ &c); but by taking the former of theſe from the latter, we have

$q \times (1 - \frac{q^2}{2.3} + \frac{3q^4}{2.4.5} - \frac{3.5q^6}{2.4.6.7} + \frac{3.5.7q^8}{2.4.6.8.9}$ &c) = hyp. log. of $(q + \sqrt{1 + qq})$.

Then $\frac{2\times 2}{6} = \frac{2}{3} = q$, which being uſed for it in the general ſeries, and the affirmative and negative terms collected, they will appear as below:

| | | | |
|---|---|---|---|
| A = | 1·00000000 | C = | 0·00493827 |
| B = | 0·07407407 | E = | 16935 |
| D = | 78385 | G = | 1215 |
| F = | 4311 | I = | 117 |
| H = | 368 | L = | 13 |
| K = | 38 | | |
| | | | — 0·0051211 |

\+ 1·0749051 ſum of the affirmative terms
— 0·0051211 ſum of the negative terms

dif. 1·069784 ſum of the whole ſeries
12 = 2y

12·837408 = length of the curve, nearly the ſame as before.

## RULE III.*

To the ſquare of the ordinate add $\frac{4}{3}$ of the ſquare of the abſciſs, and the root of the ſum will be the length of the ſingle curve nearly; the double of which will be that of the curve on both ſides of the abſciſs nearly.

That is, $\sqrt{yy + \frac{4}{3}xx} = c$ the length of the ſingle curve nearly; $y$ being the ordinate, and $x$ the abſciſs.

EX-

* DEMONSTRATION OF THIS AND RULE IV.

By the laſt rule, the curve is

$$c = y \times \left(1 + \frac{1}{2.3}q^2 - \frac{1}{2.4.5}q^4 + \frac{3}{2.4.6.7}q^6 \text{ &c}\right),$$

but $\sqrt{1 + \frac{1}{3}qq} = 1 + \frac{1}{2.3}q^2 - \frac{1}{2.4.9}q^4 + \frac{3}{2.4.6.27}q^6$ &c),

hence

EXAMPLE.

Taking again the ſame example, in which $x = 2$, and $y = 6$, we ſhall have $c = \sqrt{yy + \frac{4}{3}xx} = \sqrt{36 + \frac{16}{3}}$ $= 6{\cdot}4291$ the ſingle curve; the double of which is $12{\cdot}8582$ the length of the curve nearly.

RULE IV.*

To the ſquare of the ordinate add $\frac{2}{3}$ of the ſquare of the abſciſs, and divide the ſum by the ordinate; then ſubtract 4 times this quotient from 9 times the length of the ſingle curve, as found from the laſt rule, and $\frac{1}{5}$ of the remainder will be the length of the ſingle curve very nearly.

That is, $c = \frac{1}{5}(9\sqrt{yy + \frac{4}{3}xx} - 4 \times \frac{yy + \frac{2}{3}xx}{y})$ very nearly.

EX-

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hence $\frac{c}{y} - \sqrt{1 + \frac{1}{3}qq} = -\frac{1}{2\cdot5\cdot9}q^4 + \frac{5}{4\cdot7\cdot27}q^6$ &c: and, ſuppoſing $q$ not greater than 1, and rejecting the ſeries, $c$ will be $= y\sqrt{1 + \frac{1}{3}qq} = \sqrt{yy + \frac{2}{3}xx}$ nearly. Which is rule 3.

Again, from the 1ſt ſeries, $\frac{c}{y} - 1 - \frac{1}{8}qq = -\frac{1}{2\cdot4\cdot5}q^4 + \frac{1}{4\cdot4\cdot7}q^6$ &c; or $\frac{4}{3} \times (\frac{c}{y} - 1 - \frac{1}{8}qq) = -\frac{1}{2\cdot5\cdot9}q^4 + \frac{3}{4\cdot7\cdot27}q^6$ &c; and from above, $\frac{c}{y} - \sqrt{1 + \frac{1}{3}qq} = -\frac{1}{2\cdot5\cdot9}q^4 + \frac{5}{4\cdot7\cdot27}q^6$ &c; hence, by ſubtracting, $\frac{c}{y} - \sqrt{1 + \frac{1}{3}qq} - \frac{4}{9}(\frac{c}{y} - 1 - \frac{1}{8}qq) = \frac{1}{2\cdot7\cdot27}q^6$ &c: and, conſequently, the remaining ſeries being very ſmall, we ſhall obtain $c = \frac{1}{5}y \times (9\sqrt{1 + \frac{1}{3}qq} - 4(1 + \frac{1}{8}qq) = \frac{1}{5} \times (9\sqrt{yy + \frac{4}{3}xx} - 4 \times \frac{yy + \frac{2}{3}xx}{y})$ very nearly. Which is rule 4.*

And in this manner we may proceed to any degree of accuracy required.

EXAMPLE.

Taking ſtill the ſame example; we ſhall have $4 \times \frac{yy + \frac{2}{3}xx}{y} = 4 \times \frac{36 + \frac{8}{3}}{6} = 25\frac{7}{9}$; but by the laſt rule $c = 6{\cdot}4291$; hence $9 \times 6{\cdot}4291 = 57{\cdot}8619$; and $\frac{57{\cdot}8619 - 25\frac{7}{9}}{5} = 6{\cdot}4168 = c$, the double of which is $12{\cdot}8336$.

*Note.* It muſt be obſerved, that, as theſe two approximations are derived from the 2d rule, they muſt be uſed only in thoſe caſes in which that rule might be applied, viz. thoſe in which the abſciſs does not exceed half the ordinate.

PROBLEM V.

*To find the Area included by the Curve of a Parabola and Any Right Line, called the Baſe of the Segment or Area.*

Take $\frac{2}{3}$ of its circumſcribed parallelogram for the area.*

*Note.* The baſe of the circumſcribed parallelogram, is the ſame with the baſe of the ſegment; their

* DEMONSTRATION.

Put $y =$ an ordinate to any diameter, $x =$ its abſciſs, and $p =$ the parameter of that diameter; then the equation to the curve will be $px = yy$, and putting $s =$ the ſine of the angle made by the abſciſs and ordinate, the fluxion of the area $a$ will be $= sy\dot{x} = \frac{2sy^2\dot{y}}{p}$; hence the area $a$ is $= \frac{2sy^3}{3p} = \frac{2}{3}sxy$; that is, $\frac{2}{3}$ of the parallelogram having the ſame baſe and altitude.

*Corol.*

their altitudes are likewise the same; and therefore the parabolic area will be equal to $\frac{2}{3}$ of its altitude multiplied by its base; that is $= \frac{2}{3}ab$; putting $a$ to denote its altitude, and $b$ its base.—It may farther be observed, that if the base be perpendicular to the diameter of the figure, then the altitude $a$ will be the same with the abscis of the figure; otherwise, the altitude is equal to the abscis drawn into the natural sine of the angle made by the abscis and ordinate, or base, the radius being 1. And this is to be observed in every other figure.

EX-

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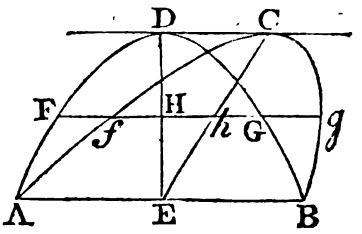


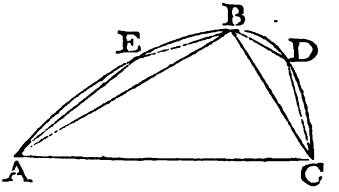

*Corol.* 1. Hence it is evident that all parabolas ADB, ACB, of the same base or equal base AB, and of equal altitudes, or between the same parallels AB, DC, are equal to one another.

*Corol.* 2. Any common sections FG, *fg*, of the parabolas ADB, ACB, are equal to each other.

For CE : C*b* :: DE : DH :: AB$^2$ : *fg*$^2$ :: AB$^2$ : FG$^2$; but AB = AE, therefore *fg* = FG.

*Corol.* 3. And hence, also, the segments FDG, *fcg*, are equal to each other.——For they are of equal bases and altitudes.

*Corol.* 4. Moreover, the frustums AFGB, A*fg*B, of equal ends and altitudes, are equal to one another.




*Corol.* 5. Let ABC be a triangle, having the same base and altitude with the parabolic segment AEBDC; then, because the triangle is half the circumscribed parallelogram, it will be $\frac{3}{4}$ of the parabolic segment; and consequently the segments AEB, BDC, together, will be $\frac{1}{4}$ of the whole portion AEBDC, or $\frac{1}{8}$ of the triangle ABC.

Again,

EXAMPLE I.

Required the area of a parabola, the abſciſs being 2, and the ordinate, perpendicular to the abſciſs, 6.

Here the altitude is 2, and the baſe or double ordinate is 12; therefore $\frac{2}{3} \times 2 \times 12 = 16$ is the area.

EXAMPLE II.

If the baſe of a parabolic ſegment be 12, and its abſciſs 2 make an angle of 30 degrees with it; what will be the area?

The ſine of 30° being half radius, the altitude will be $2 \times \frac{1}{2} = 1$; and hence $\frac{2}{3} \times 12 = 8$ is the area.

PRO-

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Again, it will appear that, if in the two ſegments AEB, BDC, be inſcribed triangles AEB, BDC, of the ſame baſes and altitudes with them, then theſe laſt triangles will be $\frac{1}{4}$ of their circumſcribed ſegments, and, conſequently, $\frac{1}{4}$ of the triangle ABC. And if, in like manner, in the laſt made ſegments be inſcribed the greateſt triangles, they will be $\frac{1}{4}$ of the triangles immediately preceding them, or $\frac{1}{4 \cdot 4} = \frac{1}{16}$ of the firſt triangle. And ſo on continually. Conſequently all the inſcribed triangles, taken together, will be expreſſed by the ſeries $b \times (1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4}$ &c); where $b$ denotes the firſt triangle; and which, when the ſeries is infinitely continued, will denote the area of the parabolic ſegment.

*Corol.* 6. Hence the infinite ſeries $1 + \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}$ &c, will be $= 1\frac{1}{3}$, as we alſo know from other principles; and conſequently the ſum of any finite number of terms of the ſeries $\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3}$ &c, is leſs than $\frac{1}{3}$; and the ſum of the infinite number of them $= \frac{1}{3}$.

PROBLEM VI.

*To find the Area of a Fruſtum, or Zone of a Parabola, included by two Parallel Right Lines and the Intercepted Curves of the Parabola.*

To one of the parallel ends, add the quotient ariſing from the diviſion of the ſquare of the other by the ſum of the ſaid ends; multiply the ſum by the altitude of the fruſtum, or diſtance of the ends; and $\frac{2}{3}$ of the product will be the area.

Or divide the difference of the cubes of the diameters, by the difference of their ſquares; and multiply the quotient by $\frac{2}{3}$ of the altitude.

That is, $(\text{D} + \frac{dd}{\text{D}+d}) \times \frac{2}{3}a$, or $(d + \frac{\text{DD}}{\text{D}+d}) \times \frac{2}{3}a$, or $\frac{\text{D}^3 - d^3}{\text{D}^2 - d^2} \times \frac{2}{3}a$ = the area; D, $d$ being the two ends, and $a$ the altitude.*

EX-

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* DEMONSTRATION.

By the property of the parabola, $\text{DD}-dd : a :: \begin{cases} \text{DD} : \frac{a\text{D}\text{D}}{\text{DD}-dd} \\ dd : \frac{add}{\text{DD}-dd} \end{cases}$ = the altitudes of two complete ſegments whoſe baſes are the ends D, $d$ of the fruſtum; and conſequently their difference, or the fruſtum, is $\frac{2}{3}a \times \frac{\text{D}^3 \sim d^3}{\text{D}^2 \sim d^2} = \frac{2}{3}a \times \frac{\text{DD} + \text{D}d + dd}{\text{D}+d} =$ $\frac{2}{3}a \times (\text{D} + \frac{dd}{\text{D}+d}) = \frac{2}{3}a \times (d + \frac{\text{DD}}{\text{D}+d})$. *Q. E. D.*

*Corollary.* Hence a parabolic fruſtum is equal to a parabola of the ſame altitude, and whoſe baſe is equal to one end of the fruſtum, increaſed by a 3d proportional to the ſum of the ends and the other. Therefore, in the one end DC, produced, of a parabolic

EXAMPLE I.

If the two parallel ends of the fruſtum of a parabola be 10 and 6, and the part of the abſciſs, perpendicular to, and connecting the middles of, thoſe ends, be 4; what will be the area?

Here the abſciſs being perpendicular to the ends, will be the altitude of the figure, and therefore

$$(10 + \frac{6 \times 6}{10 + 6}) \times \frac{4 \times 2}{3} = \frac{196}{6} = 32\frac{4}{6} = \text{the area.}$$

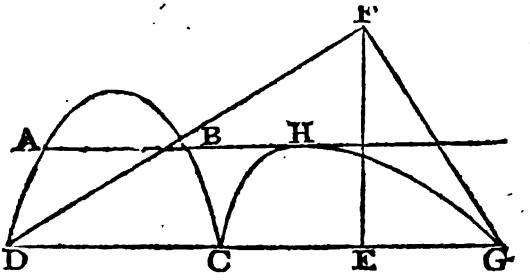
Or $(6 + \frac{10 \times 10}{10 + 6}) \times \frac{4 \times 2}{3} = \frac{196}{6} = 32\frac{4}{6} = \text{the area.}$

EXAMPLE II.

If the part of the abſciſs, connecting the middles of the two ends, make with them an angle of 48° $35\frac{5}{12}'$, required the area; the other dimenſions being as in the firſt example.

The ſine of 48° $35\frac{5}{12}'$ being ·75 or $\frac{3}{4}$ very nearly, hence $4 \times \frac{3}{4} = 3 =$ the altitude $= \frac{3}{4}$ of that in the laſt






parabolic fruſtum ABCD, take CE = the other end AB; raiſe EF perpendicular to EC and = CD; draw DF, and perpendicular to it FG meeting CE produced in G: Then if upon the baſe CG be deſcribed any parabola CHG touching the line ABH, it will be equal to the fruſtum ABCD.

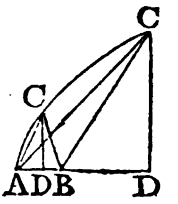
For $CG = CE + EG = AB + EG = AB + \frac{EF^2}{DE} = AB + \frac{DC^2}{DC + AB}$.

last example, and therefore the area here must be $\frac{3}{4}$ of that above; that is, $32\frac{2}{3} \times \frac{3}{4} = \frac{98}{4} = 24\frac{1}{2}$ = the area required.

## PROBLEM VII.

*To find the Area included by the Focal Distance, the Line drawn from the Focus to the Curve, and the Contained Arc of the Parabola.*

### RULE I.




Upon the axe, or focal distance AB, produced if necessary, having demitted the ordinate or perpendicular CD, cutting off the abscis AD; then

To the focal distance AB add $\frac{1}{3}$ of the abscis AD; multiply the sum by the ordinate CD; and half the product will be the area of the part ACB.

That is, $(AB + \frac{1}{3}AD) \times \frac{1}{2}DC$ = the area.*

*Note.* The focal distance AB, is $\frac{1}{4}$ of the parameter.

### EXAMPLE.

If the abscis be 2, and the ordinate 6; required the area of ACB.

By case 1 prob. 2, we have 2 : 6 :: 6 : 18 the parameter; and hence $\frac{18}{4} = 4\frac{1}{2} = AB$.

Then $(4\frac{1}{2} + \frac{2}{3}) \times 3 = 5\frac{1}{6} \times 3 = 15\frac{1}{2}$ the area required.

RULE

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* DEMONSTRATION.

Putting $AD = x$, $DC = y$, and $AB = a$; $ACB = ACD \pm BCD$ will be $= \frac{2}{3}xy \pm \frac{1}{2}y\,(\mp x \pm a) = \frac{2}{3}xy - \frac{1}{2}xy + \frac{1}{2}ay = \frac{1}{2}y\,(\frac{1}{3}x + a) = \frac{1}{2}DC \times (AB + \frac{1}{3}AD)$. *Q. E. D.*

*Corollary.* The area cut off by AC is $= \frac{1}{3}AD \times \frac{1}{2}DC$.

RULE II.

Subtract the focal distance, or distance between the focus and the beginning of the arc, from the distance between the focus and the end of the arc, and multiply the remainder by the said focal distance; then multiply the root of the product by the sum of the distance between the end of the arc and focus, and double the focal distance; and $\frac{1}{3}$ of this product will be the area.

That is, $\frac{1}{3}\sqrt{(\text{CB} - \text{BA})\,\text{AB} \times (2\text{AB} + \text{BC})}$ is the area ACB.*

EXAMPLE.

Taking here the same example as before; we have AB $= 4\frac{1}{2}$, and BC $=$ AD $+$ AB $= 2 + 4\frac{1}{2} = 6\frac{1}{2}$; and hence $\frac{1}{3}\sqrt{4\frac{1}{2} \times 2 \times 9 + 6\frac{1}{2}} = 15\frac{1}{2}$ = the area as before.

PROBLEM VIII.

*To find the Curve Surface of a Paraboloid.*

RULE I.

To the square of the ordinate, or semi-diameter of the base, add 4 times that of the axe; and the square root of the sum will be the tangent to the curve at the base, and intercepted by the axe produced; let this tangent be called $t$, viz. $t = \sqrt{yy + 4xx}$; $x$ being the axe, and $y$ the ordinate.



* DEMONSTRATION.

By the nature of the parabola, $z =$ CB is $=$ AB $+$ AD $= a + x$, or $x = z - a$, also $\frac{1}{2}y = \sqrt{ax} = \sqrt{(z-a)\,a}$; which values of $x$ and $y$, written for them in the last rule, give $\frac{1}{3}\sqrt{(z-a)\,a} \times (2a + z) =$ the area ACB. Q. E. D.

2

Then, by problem 6, find the area of the fruſtum of a parabola whoſe parallel ends are $t$ and $y$, and its altitude $= y$.

And this area multiplied by 3·14159, will be the curve ſurface of the paraboloid ABC required.*

EX-

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* DEMONSTRATION.

Calling the ordinate $y$, the abſciſs $x$, the curve $z$, and the parameter $p$; alſo $a = 3{\cdot}14159$; the equation of the generating parabola will be $px = yy$; hence $p\dot{x} = 2y\dot{y}$, and the fluxion of the ſurface $\dot{s} = 2ay\dot{z} = 2ay\sqrt{\dot{y}^2 + \dot{x}^2} = \frac{4ay\dot{y}}{p}\sqrt{\frac{1}{4}pp + yy}$; and, by taking the correct fluents, the ſurface $s$ will be $= \frac{4a}{3p} \times (\frac{1}{4}pp + yy)^{\frac{3}{2}} - \frac{1}{6}app = \frac{2}{3}ay \times \frac{(yy + 4xx)^{\frac{3}{2}} - y^3}{4xx} =$ $\frac{2}{3}ay \times \frac{(yy + 4xx)^{\frac{3}{2}} - y^3}{(yy + 4xx)^{\frac{1}{2}} - y^2} = \frac{2}{3}ay \times \frac{t^3 - y^3}{t^2 - y^2} = \frac{2}{3}ay \times \frac{tt + ty + yy}{t + y}$ $= \frac{2}{3}ay \times (t + \frac{yy}{t+y}) = \frac{2}{3}ay \times (y + \frac{tt}{t+y})$; putting $t = \sqrt{yy + 4xx}$ $=$ BE, the tangent to the curve at the extremity of the ordinate, and intercepted by the axe produced. *Q. E. D.*

*Corol.* 1. In CB produced take BG $=$ the tangent BE; erect GH perpendicular to GB and equal to BD; join DH, and perpendicular to it HI, meeting BG produced in I; take BK $= \frac{1}{3}$BI; and upon KC deſcribe a circle meeting BL perpendicular to BC in L. And the circle whoſe radius is BL, will be equal to the curve ſurface of the paraboloid BAC.

For, by the conſtruction, $BL^2 =$ KB $\times$ BC $=$ EC $\times \frac{1}{3}$EI $=$ $\frac{2}{3}$BD $\times$ (BG $+$ GI) $= \frac{2}{3}$BD $\times$ (BE $+ \frac{GH^2}{GD}$) $= \frac{2}{3}$BD $\times$ (BE $+ \frac{BD^2}{BE + BD}$) $= \frac{2}{3}y \times (t + \frac{yy}{t+y})$.

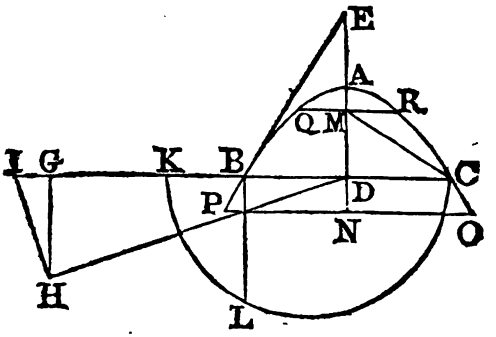
Corol.

EXAMPLE.

Required the curve ſurface of a paraboloid, whoſe axe is 20, and the diameter of its baſe 60.

Here $\sqrt{30^2 + 40^2} = 50$ is the tangent. Then, by prob. 5, $(50 + \frac{30 \times 30}{50+30}) \times \frac{2}{3} \times 30 \times 3{\cdot}14159 = \frac{4900}{80} \times 20 \times 3{\cdot}14159 = 1225 \times 3{\cdot}14159 = 3848{\cdot}451$ = the ſurface required.






*Corol.* 2. The curve ſurface of the paraboloid is to the area of its circular baſe, as IK is to BD.

For the baſe is $= a \times BD^2$, and the curve ſurface $= a \times \frac{2}{3}IB \times BD = a \times IK \times BD$.

*Corol.* 3. If M be the focus of the generating parabola, and there be taken AN = MC = AD + AM, and the double ordinates PNO, QMR be drawn; the ſurface generated by BAC, will be to the area PQRO, as $a = 3{\cdot}14159$, to 1, or as the circumference of a circle, to its diameter.

For the ſurface is $= \frac{4a}{3p} \times (\frac{1}{4}pp + yy)^{\frac{3}{2}} - \frac{1}{8}app = \frac{4a}{3p} \times (\frac{1}{4}pp + px)^{\frac{3}{2}} - \frac{1}{8}app = \frac{4}{3}a \times (\frac{1}{4}p + x)\sqrt{\frac{1}{4}pp + px} - \frac{1}{8}app = \frac{4}{3}a \times MC\sqrt{MC \times p} - \frac{1}{8}app = \frac{4}{3}a \times AN \times NO - \frac{4}{3}a \times AM \times MR = a \times (PAO - QAR) = a \times PQRO$.

RULE II.

Divide the difference between the cube of the tangent and the cube of the ordinate, by 4 times the ſquare of the axe; multiply the quotient by $\frac{2}{3}$ of the ordinate, and the product again by 3·14159 for the ſurface.

That is, $\frac{t^3-y^3}{4xx} \times \frac{2}{3}cy$ = the ſurface; where $t = \sqrt{yy+4xx}$ is the tangent, $x$ the abſciſs or axe, $y$ the ordinate or ſemi-diameter of the baſe, and $c = 3{\cdot}14159$.*

EXAMPLE.

Taking the ſame example as before, in which $x = 20, y = 30$, and therefore $t = \sqrt{40^2+30^2} = 50$; we ſhall have $\frac{50^3-30^3}{40\times 40} \times 30 \times \frac{2}{3}c = \frac{980}{16} \times 20c = 1225 \times 3{\cdot}14159 = 3848{\cdot}451$ the ſurface, as before.

PROBLEM IX.

*To find the Curve Surface of the Fruſtum of a Paraboloid, having given its Altitude and the Diameters of its Ends.*

Divide the difference of the ſquares of the ſemi-diameters of the ends, by their diſtance, or by the altitude of the fruſtum; or divide the difference of the ſquares of the whole diameters, by 4 times the altitude; and the quotient will be the parameter of the axe of the generating parabola.

Find

* This rule appears in the inveſtigation of the laſt.

Find the two ſums ariſing from the addition of the ſquare of the parameter and the ſquare of each diameter; multiply each ſum by its root, and divide the difference of the products by the parameter; multiply the quotient by 3·14159, and $\frac{1}{6}$ of the product will be the ſurface of the fruſtum.

That is, $\frac{(pp + \text{DD})^{\frac{3}{2}} - (pp + dd)^{\frac{3}{2}}}{p} \times \frac{1}{6}c =$ the ſurface; D and $d$ being the diameters of the ends, $p =$ the parameter $= \frac{\text{DD} - dd}{4a}$, $a =$ the altitude, and $c = 3{\cdot}14159$.*

EXAMPLE.

Required the curve ſurface of the fruſtum of a paraboloid, the altitude being $25\frac{7}{10}$, and the diameters of its ends 48 and 15.



* DEMONSTRATION.

In the inveſtigation of rule 1 of the laſt problem, it appears that $\frac{4c}{3p} \times (\frac{1}{4}pp + yy)^{\frac{3}{2}} - \frac{1}{6}cpp$ or $\frac{c}{6p} \times (pp + \text{DD})^{\frac{3}{2}} - \frac{1}{6}cpp$ is the ſurface of the ſegment whoſe baſe diameter is D; and $\frac{c}{6p} \times (pp + dd)^{\frac{3}{2}} - \frac{1}{6}cpp =$ that whoſe diameter is $d$; and, by taking the difference, $\frac{1}{6}c \times \frac{(pp + \text{DD})^{\frac{3}{2}} - (pp + dd)^{\frac{3}{2}}}{p}$ will expreſs the ſurface of the fruſtum, the diameters of whoſe ends are D, $d$.

And that the parameter is $= \frac{\text{DD} - dd}{4a}$, appears thus: Since $p\text{X} = \text{YY}$, and $px = yy$, therefore $p\text{X} - px = \text{YY} - yy$, and $p = \frac{\text{YY} - yy}{\text{X} - x} = \frac{\text{DD} - dd}{4a}$.

Here D $= 48$, $d = 15$, and $a = 25\frac{79}{80}$.

Therefore $\frac{DD-dd}{4a} = \frac{48^2-15^2}{4\times 25\frac{79}{80}} = \frac{2079}{103\frac{1}{20}} = 20 = p$,

Then $\sqrt{pp+DD} = \sqrt{20^2+48^2} = \sqrt{2704} = 52$,

And $\sqrt{pp+dd} = \sqrt{20^2+15^2} = \sqrt{625} = 25$.

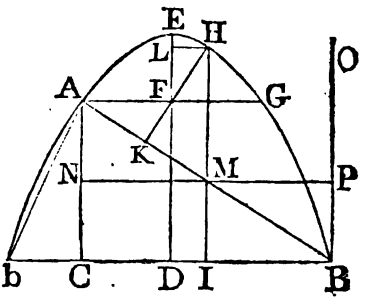
Hence $\frac{2704\times 52-625\times 25}{20\times 6} \times c = \frac{41661}{40} \times 3{\cdot}14159$

$= 4166{\cdot}1 \times {\cdot}785398 = 3272{\cdot}0472885$ the ſurface required.

## PROBLEM X.

*If a Paraboloid be Cut by a Plane, Oblique to its Axe; it is required to find the Axes of the Elliptic Section.*

It is evident that AB is the tranſverſe axe. And if bB be a double ordinate to the axe, meeting the diameter AC in c, then BC will be the other or conjugate axe of the ſection. As is proved in corollary 7 to prop. 1 ſect. 4.




### EXAMPLE.

If a paraboloid, whoſe axe is $45\frac{5}{7}$, and its baſe or greateſt double ordinate 32, be cut by a plane paſſing through the extremity of the baſe, and through the oppoſite ſide of the figure at the height of 20 above the baſe; what are the axes of the ſection?

Draw AF perpendicular to, and meeting the axe DE in F.——Then EF $=$ ED $-$ DF $= 45\frac{5}{7} - 20$ $= 25\frac{5}{7}$; and, by the nature of the parabola, ED : EF :: $DB^2$ : $AF^2 = 144$; and hence AF $=$ DC

$= 12$: therefore $CB = CD + DB = 12 + 16 = 28$ $=$ the conjugate axe. And $AB = \sqrt{AC^2 + CB^2} = \sqrt{20^2 + 28^2} = 4\sqrt{5^2 + 7^2} = 4\sqrt{74} = 34.4093010682$ the tranſverſe axe required.

EXAMPLE II.

Let the ſame paraboloid be cut through the extremity of the baſe, by a plane cutting the figure again on the ſame ſide of the axe, and at the ſame diſtance of 20 above the baſe; to find the axes of the ſection.

Here $bc = bD - cD = 16 - 12 = 4$ is the conjugate axe. And $bA = \sqrt{AC^2 + bC^2} = \sqrt{20^2 + 4^2} = 4\sqrt{5^2 + 1^2} = 4\sqrt{26} = 20.396078$ is the tranſverſe.

PROBLEM XI.

*To find the Solidity of any Segment of a Paraboloid, whoſe Baſe is either Perpendicular or Oblique to the Axe.*

RULE I.

*For Both Right and Oblique Segments.*

Multiply the baſe by half the altitude, and the product will be the content.*

*Note.*

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* DEMONSTRATION.

Let $b$ denote the baſe, $a$ the altitude, $x$ any variable abſciſs, or part of the diameter drawn through the middle of the baſe, and $s$ the ſine of the angle formed by this diameter and its ordinate, to the radius 1.

Then as $a : sx :: b : \frac{bsx}{a}$ the ſection correſponding to the abſciſs $x$, by prop. 1 ſect. 4. Therefore $\frac{sx}{a} \times b\dot{x}$ is the fluxion of

*Note.* When the base is perpendicular to the axe, it is a circle, and the altitude is equal to the whole length of the axe, contained between the base and the vertex.—But if the base be oblique to the axe, it

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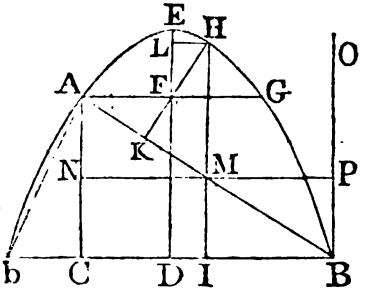
of the solid; and, by taking the fluent, the solid itself will be $\frac{bssxx}{2a}$; which, when $sx = a$, becomes $\frac{1}{2}ab$ for the whole solid, whose base is $b$ and altitude $a$. *Q. E. D.*

*Corol.* 1. A paraboloid is equal to half a prism of the same base and altitude. Also a prism, a semi-spheroid or semi-sphere, a paraboloid, and a pyramid, all of equal bases and altitudes, are to one another as the numbers 1, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$, or as 6, 4, 3, 2; and are, therefore, in a discontinued geometric proportion, whose ratio is that of 3 to 2. Which will appear by comparing the above value of the paraboloid with those of the other solids.

When the base of the paraboloid is perpendicular to the axe, or when the paraboloid is right, the semi-spheroid, or semi-sphere, will also be right; and the prism and pyramid will be the right or common cylinder and cone, the common base being a circle. But if the paraboloid be oblique, by having its base oblique to the axe, the common base of all the solids will be an ellipse; the semi-spheroid will also be oblique, by having its base oblique to the axe; and the prism and pyramid may be either right or oblique.




*Corol.* 2. If upon the diameter NP = EC = the conjugate axe of the section AE, be described a circle; the oblique paraboloid, or segment AHE, will be equal to the right paraboloid whose base is that circle, and its altitude MH. For, the diameter of the circle being the conjugate axe of the base of the oblique paraboloid, the circle will be to that base, as the conjugate is to the transverse axe, that is, as CB to AB; but HK is to HM in the same proportion of CB to AB; consequently HK × elliptic base AB will be = HM × circular base NP or EC.

It

it will be an ellipſe, and the altitude will be the perpendicular demitted to the baſe, from the vertex of the diameter drawn through the middle of the baſe; or it will be equal to the product ariſing from the multiplication

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It is alſo evident, that all ſegments, right or oblique, having the ſame vertex H, and the tranſverſe axes of whoſe baſes paſs through the ſame point M, and terminate in AC, OP, parallel to and equi-diſtant from HM, will be equal to one another.

*Corol.* 3. If B, $b$ denote the two baſes, or parallel ends, of the fruſtum of a paraboloid, either right or oblique; and $d$ the diſtance of the ends, or the altitude of the fruſtum; then $\frac{1}{2}d \times (B + b)$ will be the ſolidity of the fruſtum.

For, by this problem, $\frac{1}{2}AB$ is the ſegment whoſe baſe is B and altitude A; and $\frac{1}{2}ab$ is that whoſe baſe is $b$ and altitude $a$: therefore the fruſtum, or the difference of the ſegments, is $\frac{1}{2}AB - \frac{1}{2}ab$. But, by the nature of the paraboloid,

$B - b : d$ or $A - a :: B : A = \frac{Bd}{B-b}$, and $B - b : d :: b : a = \frac{bd}{B-b}$;

which values of A and $a$ being ſubſtituted for them, we obtain

$$\frac{1}{2}AB - \frac{1}{2}ab = \frac{dB^2 - db^2}{2B - 2b} = \frac{1}{2}d \times (B + b).$$

*Corol.* 4. Let AGBb be a right fruſtum, HI the diameter to the double ordinate AB, and HK perpendicular to AB, that is, HK the altitude of the oblique ſegment AHB. Then the value of the oblique ſegment AHB is $\frac{1}{2}n \times AB \times BC \times HK$, putting $n = .785398$.

But, by the property of the figure, $DB^2 - DC^2 : DB^2 - DI^2 ::$

$AC : HI = \frac{DB^2 - DI^2}{DB^2 - DC^2} \times AC$; hence $HM = HI - IM = HI -$

$\frac{1}{2}AC = \frac{DB + DC}{DB - DC} \times \frac{1}{4}CA = \frac{BC \times CA}{8DI}$; and, by ſimilar triangles,

$AB : CB :: HM : HK = \frac{HM \times CB}{AB} = \frac{AC \times CB^2}{AB \times 8DI}$; therefore

the oblique ſegment AHB, or $\frac{1}{2}n \times AB \times CB \times HK$, will be

$$\frac{CB^3}{8DI} \times \frac{1}{2}AC \times n = \frac{CB^4}{8B^2 - AG^2} \times AC \times \frac{1}{2}n.$$

Or,

multiplication of the part of the ſaid diameter, intercepted by the vertex and the baſe, by the ſine of the angle of inclination to the baſe.

EXAMPLE I.

Required the ſolidity of a right paraboloid whoſe axe is 30, and diameter of its baſe 40.

By the rule, $40^2 \times 15 \times .785398 = 18849.5559215$ is the ſolidity required.

EXAMPLE II.

If from the above right paraboloid be cut a part BAB, by a plane paſſing through the extremity of the baſe, and the oppoſite ſide at the height AC of $22\frac{1}{2}$ above the baſe; it is required to find the content of the oblique ſegment AHB.

Here $EF = ED - AC = 30 - 22\frac{1}{2} = 7\frac{1}{2}$; then, by the property of the parabola, $\sqrt{ED} : \sqrt{EF} :: DB : AF = DB\sqrt{\frac{EF}{ED}} = 20\sqrt{\frac{7\frac{1}{2}}{30}} = 20\sqrt{\frac{1}{4}} = 20 \times \frac{1}{2} = 10$; hence $CB = BD + DC = BD + AF = 20 + 10 = 30$ the conjugate diam. of the baſe, by the laſt problem; and

AB

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Or, by ſubſtituting the value of AC, viz. $\frac{DB^2 - DC^2}{DB^2} \times DE = \frac{CB \times 2DI}{DB^2} \times DE$, inſtead of it, the ſame ſegment AHB will be $\frac{CB^4}{8DB^2} \times DE \times n = \frac{CB^4}{bB^2} \times DE \times \frac{1}{2}n$.

*Corol.* 5. If from $DE \times bB^2 \times \frac{1}{2}n$, the value of the right ſegment bEB, be taken the above value of the oblique ſegment AHB, there will remain $\frac{bB^4 - CB^4}{bB^2} \times DE \times \frac{1}{2}n$, or $\frac{bB^4 - CB^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n$ for the value of the greater ungula bAB.

*Corol.*

AB $= \sqrt{AC^2 + CB^2} = \sqrt{22{\cdot}5^2 + 30^2} = 7{\cdot}5\sqrt{3^2+4^2}$ $= 7{\cdot}5 \times 5 = 37{\cdot}5$ the tranſverſe. Now H being the vertex of the diameter HMI drawn through M the middle of AB, and HK perpendicular to AB, the triangles HKM, MIB, ACB, will be ſimilar; but MB $= \frac{1}{2}$AB, and therefore MI $= \frac{1}{2}$AC $= 11\frac{1}{4}$; but LH $=$ DI, $= \frac{1}{2}$DB $- \frac{1}{2}$DC $= 10 - 5 = 5$; then, by the property of the parabola, $DB^2 : DB^2 - LH^2 ::$ DE : HI $= 28\frac{1}{8}$; hence HM $=$ HI $-$ IM $= 28\frac{1}{8} - 11\frac{1}{4}$ $= 16\frac{7}{8}$; and, by ſimilar triangles, AB : CB :: HM : HK $= 13\frac{1}{2}$ the altitude.

Then CB $\times$ AB $\times \frac{1}{2}$HK $\times$ ·785398 $= 30 \times 37{\cdot}5$ $\times$ 6·75 $\times$ ·785398 $=$ 7593·75 $\times$ ·785398 $=$ 5964·1173033 the ſegment AHB required.

## RULE II.

### *For the Oblique Segment.*

Divide the fourth power of CB by the ſquare of bB; multiply the quotient by $\frac{1}{2}$DE; then the product multiplied by ·785398 will be the content of the oblique ſegment AHB.

That is, $\frac{CB^4}{bB^2} \times \frac{1}{2}DE \times n =$ AHB. By corollary 4.

EX-

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*Corol.* 6. And if from the ſaid value of the oblique ſegment AHB, be taken that of the right ſegment AEG, viz. $\frac{AG^4}{bB^2} \times$ DE $\times \frac{1}{2}n$ or $\frac{AG^4}{bB^2 - AG^2} \times$ AC $\times \frac{1}{2}n$, there will remain $\frac{CB^4 - AG^4}{bB^2} \times$ DE $\times \frac{1}{2}n$ or $\frac{CB^4 - AG^4}{bB^2 - AG^2} \times$ AC $\times \frac{1}{2}n$ for that of the leſs ungula AGB.

EXAMPLE.

Taking here the laſt example, we have $\frac{CB^4}{bB^2} \times \frac{1}{2}DB \times n = \frac{30^4}{40^2} \times 15 \times .785398 = \frac{30^2 \times 9 \times 15}{16} \times .785398 = 759375 \times .785398 = 5964.1173033$, the ſame as before.

PROBLEM XII.

*To find the Content of the Fruſtum of a Paraboloid, having its Ends Parallel to each other, and either Right or Oblique to the Axe.*

Multiply the ſum of the two ends by their diſtance, or the altitude of the fruſtum; and half the product will be the content.

As is proved in corollary 3 to the laſt problem.

*Note.* The ſame things may be obſerved here as in the note to the laſt problem.

EXAMPLE.

The length of a caſk, compoſed of two equal fruſtums of a paraboloid, is 45 inches; required the content in ale and wine gallons, ſuppoſing the bung diameter to be 40, and the head diameter 20 inches.

Here $(40^2 + 20^2) \times .785398 \times 22\frac{1}{2} = 45000 \times .785398 = 35342.917352885$ is the content in inches.

Then $\frac{35342.91735}{282} = 125.329494$ ale gallons.

And $\frac{35342.91735}{231} = 152.99964$ wine gallons.

PRO-

### PROBLEM XIII.

*To find the Solidity of the Parabolic Ungula* bAB, *or* AGB, *made by a Plane paſſing through the Oppoſite Extremities of the Ends of the Fruſtum* AGBb.

Divide the difference of the 4th powers of the diameter of the baſe, and half the ſum of the diameters of the ends of the fruſtum, by the difference of the ſquares of the ſaid diameters; multiply the quotient by ·785398, and the product multiplied by half the altitude will produce the content.

That is, $\frac{bB^4 - cB^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n =$ the ungula bAB,

And $\frac{cB^4 - AG^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n =$ the ungula BAG.

As is proved in corol. 4 and 5 of prob. 10.

### EXAMPLE I.

If a veſſel, in form of the fruſtum of a paraboloid, and open at the narrow end, be ſo placed, that the liquor in it may juſt cover the bottom, whoſe diameter is 40 inches, and riſe to the height of $22\frac{1}{2}$ inches towards the top of it; required the quantity of liquor contained in the veſſel, ſuppoſing the diameter of the veſſel at the upper edge of the liquor to be 20 inches.

Here $bB = 40$, $AC = 22\frac{1}{2}$, and $AG = 20$; conſequently $BC = 20 + 10 = 30$.

Hence $\frac{40^4 - 30^4}{40^2 - 20^2} \times 22\frac{1}{2} \times \frac{·785398}{2} = \frac{17500}{96} \times 90 \times ·785398 = 12885·438618$ cubic inches; which being divided by 282 or by 231, will give 45·693044 ale gallons, or 55·781119 wine gallons.

EX-

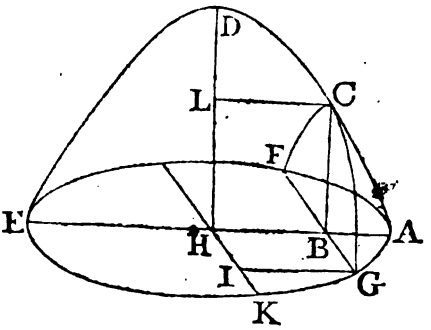
EXAMPLE II.

If the veſſel have its dimenſions inverted, that is, if AGB be the part filled, AB being the ſurface of the liquor, the bottom diameter AG being 20, the diameter bB at the top of the liquor 40, and the altitude AC $22\frac{1}{2}$ inches; required the quantity of liquor.

Here $\frac{CB^4 - AG^4}{bB^2 - AG^2} \times AC \times \frac{1}{2}n = \frac{30^4 - 20^4}{40^2 - 20^2} \times 22\frac{1}{2} \times \frac{\cdot 785398}{2} = \frac{6500}{96} \times 90 \times \cdot 785398 = 4786 \cdot 020058$ cubic inches $= 16 \cdot 971702$ ale gallons $= 20 \cdot 718701$ wine gallons.

PROBLEM XIV.

*To find the Solidity of the Slice* ABC *cut off a Paraboloid* ADE *by a Plane* FCG *Parallel to the Axe* DH.




Multiply the baſe of the ſlice by the ſquare of the diameter of the baſe of the whole paraboloid, and divide the product by the ſquare of the chord of the baſe of the ſlice; from the quotient ſubtract $\frac{1}{3}$ of the product of the ſaid chord, and its diſtance from the center of the baſe of the paraboloid; and the difference multiplied by half the altitude of the ſlice will be the ſolidity.

That

That is, $(\frac{\text{FAG} \times \text{EA}^2}{\text{FG}^2} - \frac{\text{FG} \times \text{HB}}{3}) \times \frac{1}{2}\text{BC}$ is the ſlice ABC.*



* DEMONSTRATION.

By corollary 4 prop. 1 ſect. 4 the ſection FCG is a parabola, having the ſame parameter with the generating parabola EDA, and therefore $\frac{1}{3}$FG × CB × $\dot{\text{BA}}$ is the fluxion of the ſolid: but, putting $a = \text{DH}$, $r = \text{HA}$, $x = \text{AB}$, and $y = \text{BG}$, by the property of the parabola, it will be $rr : yy :: a : \text{CB} = \frac{ayy}{rr}$; and, by the property of the circle, $r - x = \sqrt{rr - yy}$; hence $\dot{\text{BA}} = \dot{x} = \frac{yy}{\sqrt{rr - yy}}$; then, by ſubſtituting theſe values, the fluxion of the ſolid will become $\frac{4a}{3rr} \times \frac{y^4\dot{y}}{\sqrt{rr - yy}}$; whoſe fluent is $a \times$ area BAG $- \frac{ay^3\sqrt{rr - yy}}{3rr} =$ the ſolid required.

But, putting A for the altitude BC, it will be $yy : rr :: \text{A} : a = \frac{\text{A}rr}{yy}$; which being ſubſtituted for it, in the above found value of the ſolid, gives $\text{A} \times (\frac{rr}{yy} \times \text{BAG} - \frac{1}{3}y\sqrt{rr - yy}) = \frac{1}{2}\text{BC} \times (\frac{\text{EA}^2}{\text{FG}^2} \times \text{FAG} - \frac{1}{3}\text{FG} \times \text{HB})$ for the ſlice ACB. *Q. E. D.*

*Corol.* 1. The ſlice ABC is equal to a paraboloid of the ſame altitude, and whoſe baſe is equal to the difference between $\frac{2}{3}$ of the rectangular area HBGI, and a fourth proportional to the areas $\text{BG}^2$, $\text{HA}^2$, FAGF.

*Corol.* 2. From the demonſtration it appears that the ſlice is alſo equal to $a \times (\text{BAG} - \frac{y^3\sqrt{rr - yy}}{3rr})$.

*Corol.* 3. When B coincides with H, A will be $= a$, and $r = y$; and therefore the rule becomes $a \times$ HAK for half the paraboloid.

*Corol.*

EXAMPLE I.

If a caſk, compoſed of two equal paraboloidal fruſtums, having its length = 40, its bung diameter 32, and its head diameter 24 inches, when laid with its axe parallel to the horizon, want ſo much of being full, that there are $3\frac{1}{3}$ or 3·2 inches of the bung diameter left dry; it is required to find how many ale gallons will be neceſſary to fill the caſk.

Here 3·2 is the verſed ſine of the baſe of the ſlice, and 3·2 ÷ 32 = ·1 = the tabular verſed ſine, to which, in the table, correſponds the area ·04087528. And hence ·04087528 × $32^2$ = 41·85628672 = the baſe of the ſlice.

But,

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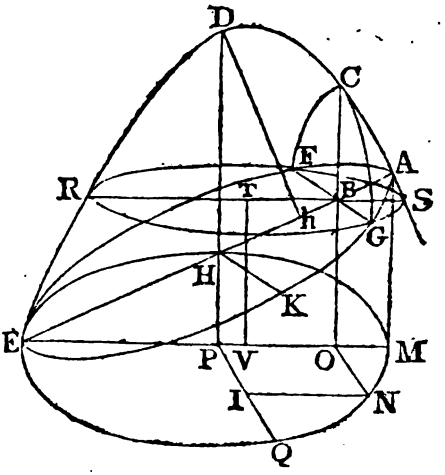
*Corol.* 4. From $a \times$ HAK = half the paraboloid, ſubtract $a \times$ BAG $- \frac{ay^3\sqrt{rr-yy}}{3rr}$ = the ſlice ABC, and there will remain $a \times$ HBGK $+ \frac{ay^3\sqrt{rr-yy}}{3rr} = a \times$ HBGK $+ \frac{1}{3}yy\sqrt{\text{DL} \times \text{LH}}$ for the value of the complemental ſlice DHBC; LC being drawn parallel to HA.

*Corol.* 5. After the ſame manner may be found the ſlice parallel to the axe of an oblique ſegment ADE.

Thus, ſince $\frac{1}{3}$FG × CB × AB × s.∠ABC is the fluxion of the ſlice ABC; by putting $a$ = DH the diameter of the double ordinate AE, $t$ = HA the ſemi-tranſverſe axe of the ſection AFEGA, $c$ = HK = $\frac{1}{2}$EM the ſemi-conjugate, $x$ = AE, and $y$ = BG; we ſhall have $cc : yy :: a : \frac{ayy}{cc}$ = BC; but $cc$:




yy

But, by the property of the circle, $2\sqrt{(32-3\cdot2)\times3\cdot2} = 2\sqrt{28\cdot8\times3\cdot2} = 19\cdot2$ is the chord of the baſe of the ſlice; and $16-3\cdot2 = 12\cdot8$ is its diſtance from the center or middle of the bung diameter. Alſo, by the nature of the parabola, $16^2-12^2 : 9\cdot6^2 :: 20 : 16\frac{16}{35}$ the altitude of the ſlice, or half the length of the empty part.

Then $(\frac{41\cdot85628672\times32^2}{19\cdot2\times19\cdot2} - \frac{19\cdot2\times12\cdot8}{3})\times\frac{24^2}{35} = (\frac{41\cdot85628672\times5^2}{3^2} - 6\cdot4\times12\cdot8)\times\frac{24^2}{35} = \frac{8^2}{7}\times(41\cdot85628672\times5 - \cdot4\times19\cdot2^2) = 565\cdot2611072$ cubic inches $= 2\cdot004472$ ale gallons, the content of the empty part required.



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$yy :: tt : tt-(t-x)^2$; hence $t-x = \frac{t\sqrt{cc-yy}}{c}$, and $\dot{x} = \dot{AB} = \frac{t\dot{y}\dot{y}}{c\sqrt{cc-yy}}$; alſo s. $\angle ABC$ is $= \frac{c}{t}$; conſequently the above fluxion will become $= \frac{4}{3}y\times\frac{ayy}{cc}\times\frac{t\dot{y}\dot{y}}{c\sqrt{cc-yy}}\times\frac{c}{t} = \frac{4ay^4\dot{y}}{3cc\sqrt{cc-yy}}$; the fluent of which, viz. $\frac{ac}{t}\times$ elliptic area BAG $- \frac{ay^3\sqrt{cc-yy}}{3cc}$, will be the value of the ſlice BCA.

Or, if D$b$ be drawn perpendicular to AE; ſince, by ſimilar triangles, it is $t : c :: a : \frac{ac}{t} =$ D$b$, the ſame ſlice BCA will be expreſſed by D$b\times$ BAG $- \frac{ay^3\sqrt{cc-yy}}{3cc} =$ D$b\times$ BAG $- \frac{DH\times FG^3\sqrt{EM^2-FG^2}}{12EM^2}$.

And if CB be produced to meet EM in O; and ON be perpendicular to EM, and meet the circle ENM in N; then will the circle ENM be to the ellipſe EGA, as well as the ſegment NMO to the ſegment GAB, as EM to EA, or as $c$ to $t$; and therefore the

EXAMPLE II.

If the ſame caſk be placed in the ſame manner, and $6\frac{2}{5}$ inches of the bung diameter be dry; required how many ale gallons the caſk wants of being full.

Here the dry inches being more than 16 — 12 or 4, the difference of the bung and head ſemi-diameters, it ſhews that $6\frac{2}{5} - 4 = 2\frac{2}{5}$ inches of the head diameter alſo are dry; and therefore the empty part of the caſk will be double the difference of two ſlices, the verſed ſines of whoſe baſes are $6\frac{2}{5}$ and $2\frac{2}{5}$; the whole diameters being 32 and 24.

Now the tabular verſed ſines will be $6{\cdot}4 \div 32 = {\cdot}2$, and $2{\cdot}4 \div 24 = {\cdot}1$; whoſe tabular areas are ·11182380 and ·04087528; hence $32^2 \times {\cdot}1118238 =$

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the value of the ſlice BCA will alſo be $a \times \text{MNO} - \frac{a y^3 \sqrt{cc - yy}}{3cc} = \text{DH} \times \text{MNO} - \frac{\text{DH} \times \text{FG}^3 \sqrt{\text{EM}^2 - \text{FG}^2}}{12\text{EM}^2}$.
Where the circular area MNO, is the projection of the elliptic area AEG, by lines parallel to the diameter DH, upon a plane perpendicular to the ſame.

Moreover, ſince $yy : cc :: \text{BC} = \text{A} : a = \text{DH} = \frac{Acc}{yy}$, by ſubſtituting this value of $a$ inſtead of it, the ſolidity of the ſame ſlice will become
$\frac{Ac^3}{3yy} \times \text{AGB} - \frac{1}{3}\text{A}y\sqrt{c^2 - y^2} = (\frac{\text{PM}^2}{\text{ON}^2} \times \text{MNO} - \frac{1}{3}\text{PO} \times \text{ON}) \times \text{BC}$.

*Corol.* 6. Hence the ſlice FGAC, is equal to a paraboloid whoſe altitude is BC, and its baſe equal to the difference between $\frac{1}{4}$ of the rectangular area PONI, and the fourth proportional to the areas $\text{ON}^2$, $\text{PM}^2$, 2OMN.

*Corol.* 7. When B coincides with H, the rule becomes barely $\text{DH} \times \text{PMQ} = \text{D}b \times \text{AHK}$, for the value of DAH or DEH or half the oblique ſegment.

*Corol.*

$= 114{\cdot}5075712$ = the base of the greater slice, and $24^2 \times {\cdot}04087528 = 23{\cdot}54416128$ = that of the less.

Again, $16 - 6{\cdot}4 = 9{\cdot}6$ is the distance of the chord of the base of the greater slice from the middle of the bung diameter, and $12 - 2{\cdot}4 = 9{\cdot}6$ is the distance of that of the less from the middle of the head diameter; and hence the former chord will be $2\sqrt{16^2 - 9{\cdot}6^2} = 2\sqrt{25{\cdot}6 \times 6{\cdot}4} = 25{\cdot}6$, and the latter $= 2\sqrt{12^2 - 9{\cdot}6^2} = 2\sqrt{21{\cdot}6 \times 2{\cdot}4} = 14{\cdot}4$.

Moreover, by the nature of the parabola, as $16^2 - 12^2$ or $28 \times 4 : 20 :: \begin{cases} 12{\cdot}8^2 : 29\frac{9}{35} \\ 7{\cdot}2^2 : 9\frac{9}{35} \end{cases}$ the altitudes of the slices.

Consequently,



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*Corol.* 8. By taking the slice from the semi-segment, we obtain $\text{DH} \times \text{PONQ} + \text{DH} \times \frac{y^3\sqrt{cc - yy}}{3cc} = \text{DH} \times \text{PONQ} + \text{DH} \times \frac{\text{ON}^3 \times \text{PO}}{3\text{PM}^2}$ for the value of the complemental slice DCBH.

*Corol.* 9. And by adding this last to the semi-segment, there results $\text{DH} \times \text{EONQ} + \text{DH} \times \frac{\text{PO} \times \text{ON}^3}{3\text{PM}^2} = \frac{\text{BC} \times \text{PM}^2}{\text{ON}^2} \times \text{EONQ} + \text{BC} \times \frac{1}{3}\text{PO} \times \text{ON}$, for the value of the slice EDCBE greater than the semi-segment.

*Corol.* 10. If RGSFR be a circle perpendicular to BC; and if from its center T be demitted the perpendicular TV; then, by taking the oblique slice ABC from the right slice SBC, we shall obtain

$$\text{BC} \times \frac{\text{RS}^2 \times \text{BSG} - \text{EM}^2 \times \text{OMN} + \text{FG}^3 \times \frac{1}{2}\text{PV}}{\text{FG}^2}$$

for the value of the ungula, or cuneus, AFGSA, contained between the two sections RFSGR, EFAGE. And the same will hold, though both the sections be oblique.

*Corol.* 11. And, in the same manner,

$$\text{BC} \times \frac{\text{EM}^2 \times \text{EONQ} - \text{RS}^2 \times \text{BRG} + \text{FG}^3 \times \frac{1}{2}\text{PV}}{\text{FG}^2}$$

will be the value of the opposite ungula or cuneus EGFRE.

Conſequently, $(\frac{114.5075712 \times 32^2}{25.6^2} - \frac{25.6 \times 9.6}{3}) \times \frac{32^2}{35}$ $= (\frac{114.5075712 \times 5^2}{4^2} - 25.6 \times 3.2) \times \frac{32^2}{35} = \frac{8^2}{7} \times$ $(114.5075712 \times 5 - \frac{4}{5} \times 25.6^2) = 2837.886683$ is double the greater ſlice.

And $(\frac{23.54416128 \times 24^2}{14.4^2} - \frac{14.4 \times 9.6}{3}) \times \frac{18^2}{35} =$ $(\frac{23.54416128 \times 5^2}{3^2} - \frac{14.4 \times 9.6}{3}) \times \frac{18^2}{35} = \frac{6^2}{7} \times$ $(23.54416128 \times 5 - .4 \times 14.4^2) = 178.852147$ is double the leſs ſlice.

Therefore $2837.886683 - 178.852147 =$ $2659.034536$ inches $= 9.4292$ ale gallons is the quantity required.

## PROBLEM XV.

### *To find the Solidity of a Parabolic Spindle.*

Multiply the area of the greateſt circle, or middle ſection, by the length, and $\frac{8}{15}$ of the product will be the content.*

That

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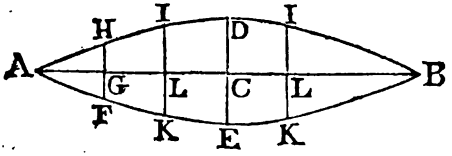
* DEMONSTRATION.

Putting $a =$ DC, $b =$ CA, $c = 3.14159$, $x =$ AG, and $y =$ GH; we ſhall have, by the property of the parabola, as $AC^2$ : AG × GE :: CD : GH $= a \times \frac{2bx - xx}{bb} = \frac{2bx - xx}{p} = y$, putting $p = \frac{bb}{a}$ the parameter of DC; and hence the fluxion of the ſolid $cyy\dot{x}$, will be $\frac{c\dot{x}}{pp} \times (2bx - xx)^2$; the fluent of which, $cx^3 \times \frac{\frac{4}{3}bb - bx + \frac{1}{5}xx}{pp}$, will be a general expreſſion for the ſegment

That is, $\frac{8}{15}n \times \text{DE}^2 \times \text{AB}$ = the whole ſolid ADBEA; AB being the length or axe, DE the greateſt diameter, or double the abſciſs of the generating parabola ADB, and $n = 785398$.




EXAMPLE.

Required the ſolidity of a parabolic ſpindle, whoſe length is 80, and greateſt diameter 32.

By the rule, $785398 \times 32^2 \times 80 \times \frac{8}{15} = 785398 \times 43690\frac{2}{3} = 34314.5693576$ is the content required.

PROBLEM XVI.

*To find the Content of the Segment* AFH *of a Parabolic Spindle.*

To $\frac{4}{3}$ of the ſquare of the length of the ſemi-ſpindle, add $\frac{1}{3}$ of the ſquare of the altitude of the

 ſegment;

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ſegment AFH; which, when $x = b$, gives $\frac{8cb^5}{15pp} = \frac{8}{15}caab$ for the value of the ſemi-ſpindle ADE. *Q. E. D.*

*Corol.* 1. A parabolic ſemi-ſpindle, is to a cylinder of the ſame baſe and altitude, as 8 to 15; and to a paraboloid of the ſame baſe and altitude, as 8 to $7\frac{1}{2}$, or as 16 to 15.

*Corol.* 2. It appears from the demonſtration, that the ſegment AFH is expreſſed by $cx^3 \times \frac{\frac{4}{3}bb - bx + \frac{1}{3}xx}{pp} =$

$$\frac{\frac{4}{3}\text{AC}^2 - \text{AC} \times \text{AG} + \frac{1}{3}\text{AG}^2}{\text{AC}^4} \times c \times \text{AG}^3 \times \text{DC}^2.$$

*Corol.*

ſegment; from the ſum take the product of the ſaid length and altitude; and call the difference P.

Multiply the cube of the altitude of the ſegment by the ſquare of the greateſt diameter of the ſpindle; divide the product by the fourth power of the length of the ſemi-ſpindle; and call the quotient Q.

Then the product of P and Q multiplied by ·785398, will give the content of the ſegment.

That is, $\frac{\frac{4}{3}AC^2 - AC \times AG + \frac{1}{3}AG^2}{AC^4} \times AG^3 \times DE^2 \times$ ·785398, is the ſegment AFH, by corollary 2 of the laſt problem.

EXAMPLE.

If the diameter of the baſe of the ſegment of a parabolic ſpindle be 24, and its altitude 20; what will be the content, ſuppoſing the length of the whole ſpindle to be 80?

Here

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*Corol.* 3. If from the value of the ſemi-ſpindle ADE, be taken that of the ſegment AFH, there will remain $\frac{8cb^5}{15pp} - cx^3 \times \frac{\frac{4}{3}bb - bx + \frac{1}{3}xx}{pp} = \frac{cz}{pp}$ or $\frac{ca^2z}{b^4} \times (b^4 - \frac{2}{3}b^2z^2 + \frac{1}{5}z^4)$ for the value of the fruſtum HDEF; putting $z$ inſtead of $b - x$ or GC.

And if inſtead of $z$, in the two laſt terms of this expreſſion, be ſubſtituted its value $b\sqrt{\frac{a-y}{a}}$, the value of the ſaid fruſtum will be denoted by

$$fz \times \frac{8aa + 4ay + 3yy}{15} = c \times GC \times \frac{8DE^2 + 4DE \times FH + 3FH^2}{60}.$$

*Corol.* 4. If to, or from, the fruſtum

$$HDEF = cA \times \frac{8D^2 + 4Dd + 3d^2}{60},$$

be

Here AC = 40, AG = 20, and GH = 12; then GC = 40 − 20 = 20; and, by the parabola, $AC^2 - GC^2 : AC^2 :: GH : CD = 16$; the double of which gives DE = 32.

Then $(\frac{4 \times 40^2}{3} - 40 \times 20 + \frac{20^2}{5}) \times \frac{20^3 \times 32^2}{40^4} \times .785398 = (\frac{4}{3} - \frac{1}{2} + \frac{1}{20}) \times 5 \times 32^2 \times .785398 = 4522\frac{2}{3} \times .785398 = 3552.094093659$ is the content required.

## PROBLEM XVII.

*To find the Content of the Fruſtum of a Parabolic Spindle, one of the Ends of the Fruſtum paſſing through the Center of the Spindle.*

Add into one ſum, 8 times the ſquare of the diameter of the greater end, 3 times the ſquare of the diameter of the leſs end, and 4 times the product of the

---

be added, or ſubtracted, the fruſt. DIKE $= ca \times \frac{8D^2 + 4D\delta + 3\delta^2}{60}$, there will result $c \times \frac{8D^2(A \pm a) + dA(4D + 3d) \pm \delta a(4D + 3\delta)}{60}$ for the value of the fruſtum HIKF, neither of whoſe ends paſs through the center of the ſpindle; the upper or under ſigns being uſed, according as the ſaid center is within or without the fruſtum, and in which D repreſents the diameter through the center of the ſpindle, $d$ the diameter of the leſs end of the fruſtum, and $\delta$ that of the greater; alſo A the diſtance of $d$ from the center of the ſpindle, and $a$ that of $\delta$ from the ſame. Or in a caſk of this form, ſtanding upon one end, D will be the bung diameter, $d$ the head diameter, 2A the length of the caſk, $A \pm a = w$ the wet inches of 2A, and $\delta$ the diameter at the ſurface of the liquor, the value of which diameter is as determined in the following corollary.

*Corol.* 5. Since, by the property of the parabola, $A^2 : a^2 :: D - d : D - \delta$, the value of $\delta$ will be $\frac{DA^2 - a^2(D - d)}{AA}$.

the diameters; multiply the ſum by the length; and the product multiplied by ·05236, or $\frac{1}{19}$ of ·785398, will be the content.

That is, $(3DE^2 + 4DE \times FH + 3FH^2) \times CG \times ·05236$ is the fruſtum DEFH, by corollary 3 to prob. 15.

EXAMPLE.

Required the content of a caſk in the form of the middle zone of a parabolic ſpindle, the bung diameter being 32 inches, the head diameter 24, and length 40 inches.

The caſk is evidently two equal fruſtums, whoſe greater diameters are 32, and leaſt 24, and their lengths each 20; therefore

$(8 \times 32^2 + 4 \times 32 \times 24 + 3 \times 24^2) \times 40 \times ·05236$
$= 8^3 \times (8 \times 4^2 + 3 \times 4^2 + 3 \times 3^2) \times 5 \times ·05236$
$= (11 \times 16 + 27) \times 8^3 \times 5 \times ·05236 = 519680$
$\times ·05236 = 27210·38117$ inches $=$ 96·4907 ale gallons, the content required.

PROBLEM XVIII.

*To find the Content of the Fruſtum of a Parabolic Spindle, neither of whoſe Ends paſs through the Center of the Spindle.*

Multiply the diameter of each end, by its diſtance from the diameter in the middle of the ſpindle; and multiply each product by the ſum of 3 times the ſaid diameter of the end, and 4 times the ſaid middle diameter; to or from the product belonging to the leſs diameter, add or ſubtract that which belongs to the greater, according as the center of the ſpindle is within or without the fruſtum; to the ſum or difference, add 8 times the product ariſing from the multiplication of the length of the fruſtum by the

ſquare

ſquare of the diameter in the middle of the ſpindle; and this ſum multiplied by ·05236, or $\frac{1}{19}$ of ·785398, will produce the content of the fruſtum.

That is, $[8DE^2 \times GL + (4DE + 3FH)\,FH \times GC \pm (4DE + 3IK)\,IK \times LC] \times ·05236$ is the fruſtum HIKF; uſing the upper or under ſign, according as GL is greater or leſs than GC; as is proved in corollary 4 to prob. 15.

*Note.* The value of the greater diameter IK is $\frac{DE \times GC^2 - LC^2 \times (DE - FH)}{GC^2}$; as in corollary 5 to the ſame problem.

EXAMPLE I.

If the liquor in the caſk, in the example to the laſt problem, when ſtanding upon its end, with its axe perpendicular to the horizon, riſe to the height of 30 inches; what quantity of liquor will there be?

Here, by the note, $IK = \frac{32 \times 20^2 - 10^2 \times (32 - 24)}{20 \times 20}$ $= 30$.

Then, by the rule, $[8 \times 32^2 \times 30 + (4 \times 32 + 3 \times 24)\,24 \times 20 + (4 \times 32 + 3 \times 30)\,30 \times 10] \times ·05236$ $= [8 \times 16^2 + (128 + 72)\,4 + (64 + 45)\,5] \times 120 \times ·05236 = 407160 \times ·05236 = 21318·84775$ cubic inches $= 75·59875$ ale gallons, the quantity required.

EXAMPLE II.

If in the ſame caſk, placed as before, the liquor riſe but to the height of 10 inches; how much will be in it?

Here all the dimenſions are the ſame quantities as in the laſt example, excepting GL, which here is only 10, inſtead of 30; and the caſk is leſs than half full.

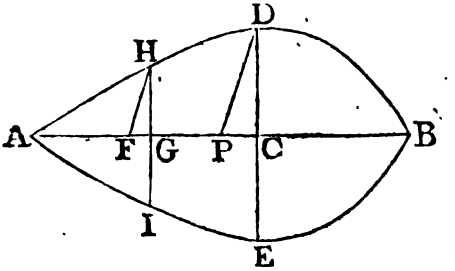
Wherefore,

Wherefore, by the rule, $[8 \times 32^2 \times 10 + (4 \times 32 + 3 \times 24)24 \times 20 - (4 \times 32 + 3 \times 30)30 \times 10] \times .05236 = [8 \times 16^2 + (128+72)12 - (64+45)15] \times 40 \times .05236 = 112520 \times .05236 = 5891.533423$ cubic inches $= 20.89196$ ale gallons, the quantity required.

## PROBLEM XIX.

*To find the Content of the Univerſal Parabolic Spindle, or Solid generated by the Revolution of a Parabolic Segment about its Baſe, being an Ordinate to Any Diameter.*

Multiply the area of the greateſt ſection by the length, and $\frac{8}{15}$ of the product will be the content; as in the common ſpindle.




That is, DE$^2$ × AB × $\frac{8}{15}n$ = the ſolid ADBEA, generated from the revolution of the parabolic ſegment ADB about its baſe AB, which is a double ordinate to the diameter DP, and DCE being perpendicular to AB.*

EX-

* DEMONSTRATION.

Let GH be parallel to CD, and HF parallel to DP; and put $a$ = DP, $\frac{1}{2}b$ = AP = $\frac{1}{2}$AB, $\frac{1}{2}c$ = CD = $\frac{1}{2}$DE, $z$ = AF, and $p = 4n = 3.14159$. Then, by the property of the parabola, AP$^2$ : AF × FB :: PD : FH :: by ſim. triangles, CD : GH = CD

EXAMPLE.

If an oblique parabola, whoſe baſe is inclined to its diameter in an angle of 30 degrees, be turned about its baſe; required the value of the ſolid generated by its revolution; the baſe of the parabola, or length of the ſpindle, being 80, and the diameter from the vertex to the baſe 32.

Here AB = 80, DP = 32, and the angle DPC = 30°; then, as radius = 1 : $\frac{1}{2}$ = ſine ∠P or 30° :: PD = 32 : DC = 16 the radius of the greateſt circle of the ſpindle.

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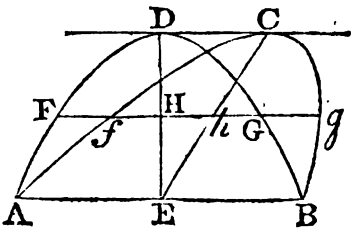
$\frac{CD \times AF \times FB}{AP^2} = 2c \times \frac{bz - zz}{bb}$ :: PC = $\sqrt{PD^2 - DC^2} = d$ : FG = $\frac{GH \times PC}{CD} = 4d \times \frac{bz - zz}{bb}$; hence AG = AF + FG = $z \times \frac{bb + 4bd - 4dz}{bb}$; and the fluxion $p \times GH^2 \times AG$ of the ſolid will be $p\dot{z} \times \frac{bb + 4bd - 8dz}{bb} \times \frac{4cc}{b^4} \times (bz - zz)^2 = \frac{4pc^2z^2\dot{z}}{b^6} \times [b^3 \times (b + 4d) - 2b^2z \times (b + 8d) + bz^2 \times (b + 20d) - 8dz^3]$; the fluent of which, viz. $\frac{4pc^2z^3}{b^6} \times [\frac{1}{3}b^3 \times (b + 4d) - \frac{1}{2}b^2z \times (b + 8d) + \frac{1}{5}bz^2 \times (b + 20d) - \frac{4}{7}dz^3]$ will be a general expreſſion for the value of the ſegment; which, when $z$ becomes $= b$, will be $\frac{2}{15}pccb = \frac{8}{15}nbcc$ for the content of the whole ſolid ADBEA. *Q. E. D.*

*Corol'ary.* If any parabolas ADB, ACB, having the ſame baſe AB and between the ſame parallels AB, CD, be turned about their common baſe; they will generate equal ſpindles.

For the lengths and greateſt diameters will be the ſame in all of them.

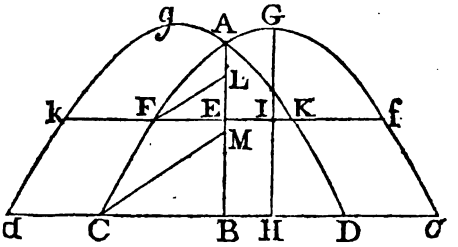


Therefore $32^2 \times 80 \times \frac{8}{15} \times .785398 = .785398 \times 43690\frac{2}{3} = 34314.5693576$, as in the example to prob. 15.

## PROBLEM XX.

*To find the Value of an Universal Paraboloid, or a Body Generated by the Revolution of a Parabola about Any Diameter.*

### INVESTIGATION.




Suppose AB to be the diameter about which the parabola CAc revolves, CAD a vertical section of the solid generated by AC, and *dg*AG*c* that of the solid generated by AG*c*; also let GIH be perpendicular, and FEI*f* parallel to c*c*. Then, by the property of the parabola, $CH^2 = bb : CH^2 - FI^2 = CH^2 - (FE \mp EI)^2 = bb - (y \mp d)^2 :: GH = a : EB = x :: -2\dot{y} \times (y \mp d) : \dot{x} = -2a\dot{y} \times \frac{y \mp d}{bb}$; hence the fluxion of the solid $2cyy\dot{x}$ will become $-2acy^2\dot{y} \times \frac{y \mp d}{bb}$; and the correct fluent gives

$$ac \times \frac{(b \pm d)^3 \times (3b \mp d) - y^3 \times (3y \mp 4d)}{6bb} =$$

$$\frac{ac}{6bb} \times \left[(b \pm d)^3 \times (3b \mp d) - \left(by\frac{a-x}{a} \pm d\right)^3 \times \left(3b\sqrt{\frac{a-x}{a}} \mp d\right)\right]$$

for the value of the solid required; in which the upper signs respect the solid *dkfc* generated by *cf*, and the under ones the solid DKFC generated by CF.

*Corol.* 1. When E arrives at A, then $y = 0$, and the expreſſion becomes $ac \times (b \pm d)^3 \times \frac{3b \mp d}{6bb} = c \times GH \times CB^3 \times \frac{cB + 2CH}{6cH} =$, by ſubſtituting for GH its value $\frac{CH^2 \times BA}{CB \times Bc}$, $c \times CB^2 \times AB \times \frac{Bc + 2CH}{6Bc}$ for the ſolid CAD.

And if CM be an ordinate to the diameter AB; we ſhall have $Bc : BC :: AB : AM = \frac{AB \times BC}{Bc}$; hence $AB \times \frac{Bc + 2CH}{6Bc}$ will be $= \frac{2AB + AM}{6}$, and conſequently the above value of the ſolid CAD becomes $c \times CB^2 \times \frac{2AB + AM}{6}$.

*Corol.* 2. When B coincides with H, $d$ will be $= 0$, and then the rule becomes $ac \times \frac{b^4 - y^4}{2bb} = c \times \frac{1}{2}HI \times (bb + yy)$ for the fruſtum CF$fc$ generated about HI, as in problem 12. Or $c \times CH^2 \times \frac{1}{2}GH$ for the whole paraboloid, as in prob. 11.

### THE INVESTIGATION OTHERWISE.

Suppoſing FL to be an ordinate to the diameter AB, whoſe parameter is $p$; and putting $m$ and $n$ for the ſine and coſine of the angle L, to the radius 1; we ſhall have $FL = \sqrt{p \times AL} = \sqrt{px}$, where $x$ is now $= AL$; then $FE = m\sqrt{px}$, and $LE = n\sqrt{px}$; hence $AE = AL + LE = x + n\sqrt{px}$. Then the fluxion of the ſolid $c \times FE^2 \times AE$ is $cm^2px \times (\dot{x} + \frac{1}{2}n\dot{x}\sqrt{\frac{p}{x}})$; whoſe fluent gives $cm^2px \times (\frac{1}{2}x + \frac{1}{3}n\sqrt{px}) = \frac{1}{6}cyy \times (3x + 2n\sqrt{px}) = \frac{1}{6}c \times FE^2 \times (2AE + AL)$ for the value of the ſolid AFK, generated by the revolution of AF. And $\frac{1}{6}c \times CB^2 \times (2AB + AM)$ for the ſolid CAD, as above.

*Scholium.*

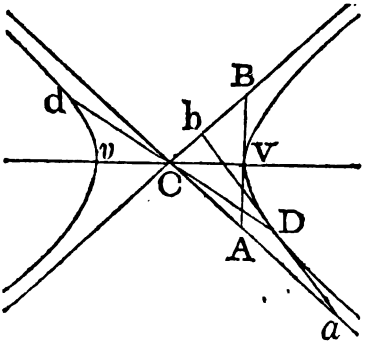
*Scholium.* It is evident that the ſurface of this ſolid might be found by the method by which we determined that of an elliptic ſpindle; and the ſurface of the parabolic ſpindle might eaſily be determined alſo, if there appeared any occaſion for it; but not by the ſame method.

## SECTION VII.

### OF HYPERBOLIC LINES, AREAS, SURFACES, AND SOLIDITIES.

#### DEFINITIONS.

TO what hath been ſaid of the hyperbola among the conic ſections in general, may be added the following obſervations.




If AVB touch the hyperbola in the vertex V, and be equal to the conjugate axe, viz. AV and VB each equal to half the conjugate axe; and if from the center

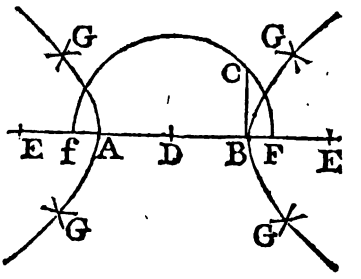
center C, through the extremities A, B, right lines CA, CB, be drawn; thoſe lines will be what are called aſymptotes to the hyperbola; that is, they are lines which continually approach to the curve.

And, like as the conjugate axe is a tangent to the curve at the vertex of the tranſverſe, and bounded by the aſymptotes; ſo, if to the extremity D, of any diameter *d*D, be drawn a tangent *a*D*b*, and terminated by the aſymptotes, then *ab* will be the conjugate to *d*D.

When the aſymptotes form a right angle at the center, the hyperbola is ſaid to be right-angled; as alſo equilateral, becauſe its axes are then equal to each other.

## PROBLEM I.

*To Conſtruct an Hyperbola, having given the Tranſverſe and Conjugate Axes* AB *and* 2BC.




The ſemi-conjugate BC being erected perpendicular to AB, and D being the middle of AB, or the center of the hyperbola; with the center D and radius DC, deſcribe an arc meeting AB, both ways produced, in F and *f*; which will be the two foci.

Then, aſſuming ſeveral points E in the tranſverſe produced, with the radii AE, BE, and centers *f*, F, deſcribe arcs interſecting in the ſeveral points G; through all which points draw the hyperbolic curve.

## PROBLEM II.

*In an Hyperbola, to find any Two Conjugate Diameters, an Ordinate to one of them, and its Abſciſs, one from another; viz. having any three of them given, to find the Fourth.*

### CASE I.

*To find the Ordinate.*

As any diameter :
Is to its conjugate ::
So is the mean propor. between the abſciſſes :
To the ordinate.*

That is, as $d : c :: \sqrt{(d + x)\, x} : y = \frac{c}{d}\sqrt{(d + x)\, x}$; where $d$ denotes the diameter, $c$ its conjugate, $y$ an ordinate to the diameter, and $x$ its abſciſs, or its diſtance from the vertex of the diameter, meaſured upon it.

*Note.* In the hyperbola, the leſs abſciſs added to the diameter, gives the greater abſciſs.

### EXAMPLE.

If the diameter be 24, its conjugate 21, and the leſs abſciſs 8; what is the ordinate?

By the rule, $\frac{c\sqrt{(d + x)\, x}}{d} = \frac{21\sqrt{32 \times 8}}{24} = \frac{7 \times 16}{8} =$ 14 the ordinate required.

CASE

* The values of the ſeveral quantities, in the four caſes of this problem, are eaſily found from the general property of the curve, viz. $dd : cc :: (d+x)x : yy$. The demonſtration of which, together with that of the preceding problem, belongs to conic ſections.

CASE II.

*To find the Abſciſſes.*

As the conjugate :
Is to the diameter : :
So is the root of the ſum of the ſquares of the ordinate and ſemi-conjugate :
To the diſtance between the foot of the ordinate and the center.

Then, to and from this diſtance, add and ſubtract the ſemi-diameter, and the ſum and difference will give the two abſciſſes.

That is, $\frac{d\sqrt{\frac{1}{4}cc+yy}}{c} \pm \frac{1}{2}d = x$ the greater or leſs abſciſs, according as the upper or under ſign is uſed.

EXAMPLE.

The diameter and its conjugate being 24 and 21, required the two abſciſſes to the ordinate 14.

Here $\frac{d\sqrt{\frac{1}{4}cc+yy}}{c} \pm \frac{1}{2}d = \frac{24\sqrt{\frac{1}{4}\times 21^2+14^2}}{21} \pm 12$

$= \frac{4\sqrt{21^2+28^2}}{7} \pm 12 = 4\sqrt{3^2+4^2} \pm 12 = 20 \pm 12 = 32$ and 8, the two abſciſſes required.

CASE III.

*To find the Conjugate.*

As the mean propor. between the abſciſſes :
Is to the ordinate : :
So is the diameter :
To its conjugate.

That is, $\sqrt{(d+x)x} : y :: d : \frac{dy}{\sqrt{(d+x)x}} = c.$

EXAMPLE.

What is the conjugate to the diameter 24, the leſs abſciſs being 8, and its ordinate 14?

Here $\frac{dy}{\sqrt{(d+x)x}} = \frac{24 \times 14}{\sqrt{32 \times 8}} = \frac{24 \times 14}{16} = 21$, the conjugate.

CASE IV.

*To find the Diameter.*

To or from the root of the ſum of the ſquares of the ordinate and ſemi-conjugate, add or ſubtract the ſemi-conjugate, according as the leſs or greater abſciſs is uſed: Then, as the ſquare of the ordinate, is to the product of the abſciſs and conjugate; ſo is the ſum or difference, above found, to the diameter.

That is, $cx \times \frac{\sqrt{\frac{1}{4}cc + yy} \pm \frac{1}{2}c}{yy} = d.$

EXAMPLE.

The leſs abſciſs being 8, and its ordinate 14, required the diameter, ſuppoſing the conjugate to be 21.

Here

$$cx \times \frac{\sqrt{\frac{1}{4}cc + yy} + \frac{1}{2}c}{yy} = 21 \times 8 \times \frac{\sqrt{\frac{1}{4} \times 21^2 + 14^2} + 10\frac{1}{2}}{14 \times 14}$$

$$= 3 \times \frac{\sqrt{21^2 + 28^2} + 21}{7} = 3 \times (\sqrt{3^2 + 4^2} + 3) =$$

$3 \times (5 + 3) = 24$, the diameter required.

PRO-

## PROBLEM III.

*Having given any Two Abscisses* X, $x$, *and their Ordinates* Y, $y$; *to find the Diameter* $d$ *to which they belong.*

Multiply each abscis by the square of the ordinate belonging to the other; multiply also the square of each abscis by the square of the other's ordinate; then divide the difference of the latter products by the difference of the former; and the quotient will be the diameter to which the ordinates belong.*

That is, $\frac{\text{XX}yy \backsim xx\text{YY}}{x\text{YY} \backsim \text{X}yy} = d$.

### EXAMPLE.

If two abscisses be 1 and 8, and their two ordinates $4\frac{3}{8}$ and 14; what is the diameter to which they belong?

Here $\frac{\text{XX}yy - xx\text{YY}}{x\text{YY} - \text{X}yy} = \frac{8^2 \times 4\frac{3}{8} \times 4\frac{3}{8} - 1^2 \times 14^2}{1 \times 14^2 - 8 \times 4\frac{3}{8} \times 4\frac{3}{8}} = \frac{35 \times 35 - 14 \times 14}{14 \times 14 - 35 \quad 4\frac{3}{8}} = \frac{5 \times 5 - 2 \times 2}{2 \times 2 - 5 \times \frac{5}{8}} = \frac{21 \times 8}{7} = 24$, the diameter required.



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* DEMONSTRATION.

By the nature of the hyperbola, $d\text{X} + \text{XX} : dx + xx :: \text{YY} : yy$, or $d\text{X}yy + \text{XX}yy = dx\text{YY} + xx\text{YY}$; hence $d = \frac{\text{XX}yy - xx\text{YY}}{x\text{YY} - \text{X}yy}$. *Q. E. D.*

### PROBLEM IV.

*Given any Three Equidiſtant Ordinates a, b, c, with their Common Diſtance, or Common Difference d of their Abſciſſes; to find the Diameter* D *to which they belong, together with its Conjugate* C.

* From the ſum of the ſquares of the extreme ordinates, take double the ſquare of the middle one; and call the difference A.

From the ſquare of the difference between A and double the ſaid ſquare of the middle ordinate, take 4 times the ſquare of the product of the extreme ordinates; and call the ſquare root of the remainder B.

Then, as A is to B, ſo is the given common diſtance, to the diameter to which the ordinates belong.

And

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* DEMONSTRATION.

If $x$ be the abſciſs to the leaſt ordinate $a$, by the nature of the hyperbola we ſhall have theſe three equations,

$$aa = \frac{CC}{DD} \times (D + x) \times x = \frac{CC}{DD} \times (Dx + xx),$$

$$bb = \frac{CC}{DD} \times (D + x + d) \times (x + d) = \frac{CC}{DD} \times (Dx + xx + 2dx + Dd + dd),$$

$$cc = \frac{CC}{DD} \times (D + x + 2d) \times (x + 2d) = \frac{CC}{DD} \times (Dx + xx + 4dx + 2Dd + 4dd).$$

And by ſubtracting the double of the ſecond from the ſum of the firſt and third, we obtain this equation,

$aa - 2bb + cc = \frac{CC}{DD} \times 2dd$; and hence $\frac{CC}{DD} = \frac{aa - 2bb + cc}{2dd}$.

Again, taking the firſt equation from the ſecond, leaves $bb - aa = \frac{CC}{DD} \times (2dx + Dd + dd)$; hence $x = \frac{DD}{CC} \times \frac{bb - aa}{2d} - \frac{D + d}{2} = d \times$

And as the common distance of the ordinates, is to the root of $\frac{1}{2}$A, so is the diameter D, to its conjugate c. That is,

$$D = \frac{d\sqrt{(-aa+4bb-cc)^2-4aacc}}{aa-2bb+cc}, \text{ and } c = \frac{D\sqrt{\frac{1}{2}A}}{d}.$$

EXAMPLE.

Required the diameter, and its conjugate, to which belong the three ordinates $4\frac{3}{8}$, 14, and $\frac{21}{8}\sqrt{65}$, their common distance being 7.

Here the diam. $D = \frac{d\sqrt{(4bb-aa-cc)^2-4aacc}}{aa-2bb+cc} =$

$$\frac{7\sqrt{(28^2-\frac{35^2}{8^2}-\frac{65\times 21^2}{8^2})^2-\frac{4\times 65\times 35^2\times 21^2}{8^4}}}{\frac{35^2}{8^2}-2\times 14^2+\frac{65\times 21^2}{8^2}} =$$

$$\frac{7\sqrt{(32^2-5^2-65\times 3^2)^2-65\times 10^2\times 3^2}}{5^2-2\times 16^2+65\times 3^2} = \frac{7\times 336}{98} = 24.$$



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$$d\times\frac{bb-aa}{aa-2bb+cc}-\frac{D+d}{2} = d\times\frac{2bb-\frac{1}{2}aa-\frac{1}{2}cc}{aa-2bb+cc}-\frac{1}{2}D.$$

Consequently, $aa = \frac{cc}{DD}\times(D+x)\times x = \frac{aa-2bb+cc}{2dd}\times$

$[d^2\times(\frac{2bb-\frac{1}{2}aa-\frac{1}{2}cc}{aa-2bb+cc})^2-\frac{1}{4}D^2]$; and hence

$$\frac{d\sqrt{(-aa+4bb-cc)^2-4aacc}}{aa-2bb+cc} = \frac{dB}{A} = \text{the diameter D.}$$

Also from the equation $\frac{cc}{DD} = \frac{aa-2bb+cc}{2dd}$, is deduced

$$\frac{D}{d}\sqrt{\frac{aa-2bb+cc}{2}} = \frac{D\sqrt{\frac{1}{2}A}}{d} = \text{the conjugate c. } \textit{Q. E. D.}$$

*Corollary.* Hence the sum of the squares of the extreme ordinates, is greater than double the square of the middle one.

And

$$\frac{24}{7}\sqrt{\frac{35^2}{2\times 8^2}-14^2+\frac{65\times 21^2}{2\times 8^2}}=8\sqrt{\frac{5^2}{2}-16^2+\frac{65\times 3^2}{2}}$$

$$=\tfrac{3}{2}\sqrt{50-32^2+9\times 130}=\tfrac{3}{2}\sqrt{196}=\tfrac{3}{2}\times 14=21,$$

the conjugate c.

## PROBLEM V.

*To find the Length of an Arc of an Hyperbola, the Arc beginning at the Vertex.*

### RULE I.*

Put $a$ = the semi-tranſverſe axe, $c$ = the ſemi-conjugate, $q=\frac{cc+aa}{c^4}$, $y$ = an ordinate to the axe drawn from the end of the arc required, A = the hyperbolic logarithm of $\frac{y+\sqrt{cc+yy}}{c}$ = 2·302585093 × common log. of $\frac{y+\sqrt{cc+yy}}{c}$, $B=\frac{y\sqrt{cc+yy}-ccA}{2}$, $C=\frac{y^3\sqrt{cc+yy}-3ccB}{4}$, $D=\frac{y^5\sqrt{cc+yy}-5ccC}{6}$, $E=\frac{y^7\sqrt{cc+yy}-7ccD}{8}$, &c.

Then

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* DEMONSTRATION.

Putting $x$ = the abſciſs of the ordinate $y$, we ſhall obtain, from the nature of the curve, $x=\frac{a\sqrt{cc+yy}}{c}-a$, and therefore $\dot{x}=\frac{ay\dot{y}}{c\sqrt{cc+yy}}$; hence $\dot{z}=\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}=\dot{y}\sqrt{\frac{cc+\frac{cc+aa}{cc}yy}{cc+yy}}=$

$$\frac{c\dot{y}}{\sqrt{cc+yy}}\sqrt{1+\frac{cc+aa}{c^4}yy}=\frac{c\dot{y}}{\sqrt{cc+yy}}\sqrt{1+qyy}=\frac{c\dot{y}}{\sqrt{cc+yy}}$$

$$\times\left(1+\tfrac{q}{2}y^2-\frac{q^2}{2.4}y^4+\frac{3q^3}{2.4.6}y^6-\frac{3.5q^4}{2.4.6.8}y^8\ \&c\right).$$

But

Then will the length of the arc be expreſſed by $c \times (A + \frac{q}{2}B - \frac{q^2}{2.4}C + \frac{3q^3}{2.4.6}D - \frac{3.5q^4}{2.4.6.8}E$ &c).

EXAMPLE.

Required the length of the curve correſponding to the ordinate 10, the tranſverſe and conjugate axes being 80 and 60.

Here $a = 40$, $c = 30$, and $y = 10$; hence $q = \frac{cc + aa}{c^4} = \frac{2500}{810000} = \frac{1}{324}$; $\sqrt{cc + yy} = 10\sqrt{10} = 31{\cdot}62277662$; and $\frac{y + \sqrt{cc + yy}}{c} = 1{\cdot}387425887$; whoſe hyperbolic logarithm is $\cdot 3274501 = A$;

alſo $\frac{y\sqrt{cc + yy} - ccA}{2} = 10{\cdot}76133 = B.$

$\frac{y^3\sqrt{cc + yy} - 3ccB}{4} = 641{\cdot}796405 = C.$

$\frac{y^5\sqrt{cc + yy} - 5cCC}{6} = 45698{\cdot}97\frac{1}{3} = D.$

and $\frac{y^7\sqrt{cc + yy} - 7ccD}{8} = 3540529{\cdot}3125 = E.$

Then

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But the fluent of $\frac{\dot{y}}{\sqrt{cc + yy}}$ is the hyp. log. of $\frac{y + \sqrt{cc + yy}}{c} = A$;

that of $\frac{y^2\dot{y}}{\sqrt{cc + yy}}$ is $\frac{y\sqrt{cc + yy} - ccA}{2} = B$;

that of $\frac{y^4\dot{y}}{\sqrt{cc + yy}}$ is $\frac{y^3\sqrt{cc + yy} - 3ccB}{4} = C$;

that of $\frac{y^6\dot{y}}{\sqrt{cc + yy}}$ is $\frac{y^5\sqrt{cc + yy} - 5cCC}{6} = D$, &c.

Theref. $z = c \times (A + \frac{q}{2}B - \frac{q^2}{2.4}C + \frac{3q^3}{2.4.6}D - \frac{3.5q^4}{2.4.6.8}E$ &c).

Then $+ \quad A = \cdot 327450, \quad - \frac{q^2}{2.4} C = \cdot 000764$

$+ \frac{q}{2} B = \cdot 016607, \quad - \frac{3 \cdot 5 q^4}{2.4.6.8} E = \cdot 000012$

$+ \frac{3q^3}{2.4.6} D = \cdot 000084 \quad - \cdot 000776$

$+ \cdot 344141$

$- \cdot 000776$

$\cdot 343365$ the sum of the series,

which multip. by $c = 30$

produces 10·30095 the length of the arc.

RULE II.*

Using the same symbols as in the last rule, the arc will be expressed by the series $y \times (1 + \frac{a^2}{6c^4} y^2 - \frac{a^4 + 4a^2c^2}{40c^8} y^4 + \frac{a^6 + 4a^4c^2 + 8a^2c^4}{112c^{12}} y^6 - \frac{5a^8 + 24a^6c^2 + 48a^4c^4 + 64a^2c^6}{1152c^{16}} y^8$ &c).

or

* DEMONSTRATION.

For, by the demonst. of the last rule, $\dot{z}$ is $= \dot{y} \sqrt{\frac{cc + \frac{aa + cc}{cc} yy}{cc + yy}}$ =, by extracting the root of the numerator and denominator, and then dividing the one by the other, $\dot{y} \times$ $(1 + \frac{a^2}{2c^4} y^2 - \frac{a^4 + 4a^2c^2}{2.4c^8} y^4 + \frac{a^6 + 4a^4c^2 + 8a^2c^4}{4.4c^{12}} y^6 - \frac{5a^8 + 24a^6c^2 + 48a^4c^4 + 64a^2c^6}{4.4.\ldots^{16}} y^8$ &c). And, by taking the fluents, we obtain the series as in the rule.

or $y \times (1 + \frac{a^2y^2}{6c^4}\text{A} - \frac{a^2+4c^2}{c^4}.\frac{3y^2}{20}\text{B} + \frac{a^4+4a^2c^2+8c^4}{a^2+4c^2}.\frac{5y^2}{14c^4}\text{C}$
$- \frac{5a^6+24a^4c^2+48a^2c^4+64c^6}{a^4+4a^2c^2+8c^4}.\frac{7y^2}{72c^4}\text{D}$ &c).

Putting A, B, C, &c, for the 1ſt, 2d, 3d, &c, term.

EXAMPLE.

Taking the ſame example as to the laſt rule, in which $a = 40$, $c = 30$, and $y = 10$; we ſhall obtain

| | | | |
|---|---|---|---|
| A | = 1 | C | = ·003170 |
| B | = 0·032922 | E | = 59 |
| D | = 398 | | − ·003229 |
| | + 1·033320 | | |
| | − ·003229 | | |

the ſum 1·030091 which multiplied
by $y = 10$
produces 10·30091 the arc, nearly as before.

RULE III.*

1. To 21 times the ſquare of the conjugate, add 9 times the ſquare of the tranſverſe; and to the ſame 21 times

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* DEMONSTRATION.

Firſt $x = \frac{a\sqrt{c^2+y^2}}{c} - a$, uſing the ſame ſymbols as in the laſt rules, $= \frac{ay^2}{2c^2} - \frac{ay^4}{2.4c^4} + \frac{3ay^6}{2.4.6c^6} - \frac{3.5ay^8}{2.4.6.8c^8}$ &c.

Then proceed as in rule 5 for the elliptic arc.

Thus $y \times \frac{\text{A}+(\text{B}+1)x}{\text{A}+\text{B}x} = y \times \frac{\text{A}+(\text{B}+1)\times(\frac{ay^2}{2c^2} - \frac{ay^4}{8c^4}\text{ &c})}{\text{A}+\text{B}\times(\frac{ay^2}{2c^2} - \frac{ay^4}{8c^4}\text{ &c})}$

$= y$

21 times the ſquare of the conjugate, add 19 times the ſquare of the tranſverſe; and multiply each ſum by the abſciſs.

2. To each of theſe two products add 15 times the product of the tranſverſe and ſquare of the conjugate.

3. Then as the leſs ſum is to the greater, ſo is the double ordinate to the length of the curve, nearly.

That is, $\frac{15c^2t + (19t^2 + 21c^2)x}{15c^2t + (9t^2 + 21c^2)x} \times y =$ the length of the curve nearly; where $t$ is the tranſverſe axe, $c$ the conjugate, $x$ the abſciſs, and $y$ the ordinate correſponding to the arc.

EXAMPLE.

Taking, again, the ſame example, we have $t = 80$, $c = 60$, $y = 10$, and $x = \frac{a\sqrt{c^2 + y^2}}{c} - a$
$= a \times \frac{\sqrt{c^2+y^2}-c}{c} = 40 \times \frac{\sqrt{1000}-30}{30} = 4 \times \frac{10\sqrt{10}-30}{3}$
$= 2{\cdot}16370216.$

Then

---

$= y \times (1 + \frac{ay^2}{2Ac^2} - \frac{Aa + 2Ba^2}{8A^2c^4}y^4$ &c); which put $=$
$y \times (1 + \frac{a^2y^2}{6c^4} - \frac{a^4 + 4a^2c^2}{40c^8}y^4$ &c) the value of the arc in the laſt rule. Then, by equating the correſponding terms, we have $\frac{a}{2Ac^2} = \frac{a^2}{6c^4}$,

or $A = \frac{3c^2}{a} = \frac{3}{2}p$; and $\frac{Aa + 2Ba^2}{8A^2c^4} = \frac{a^4 + 4a^2c^2}{40c^8}$,

hence $B = \frac{9a^2 + 21c^2}{10aa} = \frac{9a + \frac{21cc}{a}}{10a} = \frac{9t + 21p}{10t}$,

putting $p$ for the parameter, and $t$ for the whole tranſverſe axe.

Conſequently

Then $y \times \frac{15c^2t + (19t^2 + 21c^2)x}{15c^2t + (9t^2 + 21c^2)x}$

$= 10 \times \frac{15 \times 60^2 \times 80 + (19 \times 80^2 + 21 \times 60^2)2{\cdot}16}{15 \times 60^2 \times 80 + (9 \times 80^2 + 21 \times 60^2)2{\cdot}16}$

$= 10 \times \frac{11866{\cdot}70516488}{11520{\cdot}51281928} = 10{\cdot}3005$, the length of the arc required, very near.

## PROBLEM VI.

### *To find the Area of a Segment cut off an Hyperbola by a Double Ordinate.*

### RULE I.

From the product of the ordinate and its distance from the center, subtract the product arising from the multiplication of the product of the semi-axes, by the hyperbolic logarithm of the quotient resulting from the division of the sum of the products of the semi-transverse multiplied by the ordinate, and the semi-conjugate multiplied by the said distance of the ordinate from the center, by the product of the semi-

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Consequently $y \times \frac{A + (B+1)x}{A + Bx}$ is $= y \times \frac{\frac{1}{2}p + \frac{19t + 21p}{10t}x}{\frac{1}{2}p + \frac{9t + 21p}{10t}x} =$

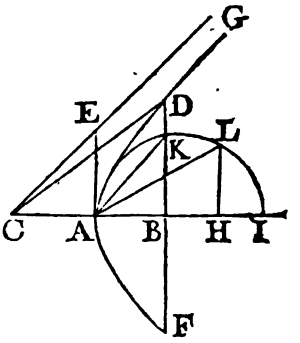
$y \times \frac{15p + \frac{19t + 21p}{t}x}{15p + \frac{9t + 21p}{t}x} = \frac{15c^2t + (19t^2 + 21c^2)x}{15c^2t + (9t^2 + 21c^2)x} \times y.$

*Q. E. D.*

*Corollary.* A right line may be found nearly equal to an hyperbolic arc, by proceeding in the same manner as in the corollary to rule 5 for the arc of the ellipse, taking here PG $= \frac{1}{2}$AP $+ \frac{19AK + 21AP}{10AK} \times$ AD.

ſemi-axes; and the remainder will be the area cut off by the double ordinate.*

That is, CB × BD — CA × AE × hyp. logarithm of $\frac{CA \times BD + CB \times AE}{CA \times AE} = vy - ac \times$ hyp. log. of $\frac{ay + cv}{ac}$ = the area ADF; putting $a$ = the ſemi-tranſverſe axe CA, $c$ = AE the ſemi-conjugate, $y$ = BD the ordinate, and $v$ = CB its diſtance from the center.




*Note*

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* DEMONSTRATION.

The fluxion of the area ABD is $y\dot{v} = \frac{c\dot{v}\sqrt{vv - aa}}{a}$, by the nature of the curve; the fluent of which is $\frac{cv\sqrt{vv-aa}}{2a} - \frac{ac}{2} \times$ h. l. of $\frac{v+\sqrt{vv-aa}}{a}$ = the area ABD; the double of which will be $\frac{cv\sqrt{vv-aa}}{a} - ac \times$ h. l. of $\frac{v+\sqrt{vv-aa}}{a} = vy - ac \times$ h. l. of $(\frac{v}{a} + \frac{y}{c}) = vy - ac \times$ h. l. of $\frac{ay+cv}{ac}$ the whole ſegment ADF. *Q. E. D.*

*Corol.* 1. The ſame ſeg. ADF or $\frac{cv\sqrt{vv-aa}}{a} - ac \times$ h. l. of $\frac{v+\sqrt{vv-aa}}{a}$, is alſo $= \frac{ay\sqrt{cc+yy}}{c} - ac \times$ h. l. of $\frac{y+\sqrt{cc+yy}}{c}$ $= ac \times (q\sqrt{1+qq} -$ h. l. of $(q + \sqrt{1+qq})$ $= ac \times (\varrho\sqrt{\varrho\varrho - 1} -$ h. l. of $\varrho + \sqrt{\varrho\varrho - 1})$; putting $q = \frac{y}{c}$, and $\varrho = \frac{v}{a}$.

*Corol.*

*Note* 1. The ſame rules will ſerve for any other two conjugate diameters, and their abſciſs and ordinate; multiplying the reſult by the ſine of the angle formed by the abſciſs and ordinate, to the radius 1.

*Note* 2. The hyperbolic logarithm, is equal to the common logarithm multiplied by 2·302585093.

EXAMPLE.

Required the area of an hyperbola, whoſe baſe is 24, and altitude 10; the tranſverſe axe being 30.

Here

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*Corol.* 2. If from the triangle CBD $= \frac{1}{2}vy$, be taken the ſemiſegment ABD $= \frac{1}{2}vy - \frac{1}{2}ac \times$ h. l. of $\frac{ay + cv}{ac}$, there will remain $\frac{1}{2}ac \times$ h. l. of $\frac{ay + cv}{ac}$ for the value of the hyperbolic ſector CAD. When $y$ or $v$ are ſuppoſed infinite; then the ſector becomes the ſpace CAG between the aſymptote and the curve infinitely produced, and the expreſſion for it becomes infinite.

*Corol.* 3. When the hyperbola is equilateral, then $a$ is $= c$, and the area ADF $= vy - aa \times$ h. l. of $\frac{v + y}{a}$.

*Corol.* 4. When A and C coincide, the hyperbola becomes a triangle; and becauſe CA and AE are then each $= 0$, the rule will become barely CB $\times$ BD or AB $\times$ BD for the area.

*Corol.* 5. Similar ſegments of any hyperbolas, are to each other as the rectangles of their axes. Where by ſimilar ſegments, are meant thoſe whoſe ordinates or abſciſſes, are as their like axes or diameters. For, by corollary 1, the ſegment being expreſſed by $ac \times (q\sqrt{1 + qq} -$ hyp. log. of $q + \sqrt{1 + qq})$ or $ac \times \varrho\sqrt{\varrho\varrho - 1} -$ hyp. log. of $\varrho + \sqrt{\varrho\varrho - 1}$; and $q = \frac{y}{c}$, or $\varrho = \frac{v}{a}$, being the ſame in all ſimilar ſegments, thoſe ſegments will be as $ac$.——And hence, ſimilar ſegments of equilateral hyperbolas, are as the ſquares of their axes, or of their like diameters.

Here $a = 15$, $y = 12$, and $v = 25$.

Hence $c = \frac{ay}{\sqrt{vv - aa}} = \frac{15 \times 12}{\sqrt{40 \times 10}} = \frac{180}{20} = 9$.

Then $\frac{ay + cv}{ac} = \frac{15 \times 12 + 9 \times 25}{15 \times 9} = \frac{4 + 5}{3} = 3$;

whoſe

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Hence alſo, hyperbolas of the ſame tranſverſe axe and abſciſs, are to one another as their conjugate axes; but if their baſes or ordinates and conjugate axes be the ſame, they will be as their tranſverſe axes. For when $a$ and $v$ are the ſame, the quantity $\frac{cv\sqrt{vv - aa}}{a} - ac \times$ h. l. of $\frac{v + \sqrt{vv - aa}}{a}$ is as $c$; and when $c$ and $y$ are the ſame, then $\frac{ay\sqrt{cc + yy}}{c} - ac \times$ h. l. of $\frac{y + \sqrt{cc + yy}}{c}$ is as $a$.

And conſequently, having the quadrature of any one hyperbola, from it we may find that of any other hyperbola; viz. knowing the area anſwering to every abſciſs in any one hyperbola, we can find the area anſwering to any abſciſs in any other hyperbola. For we can find a ſimilar abſciſs in the ſquared hyperbola, with the correſponding area. Then, as the rectangle of the axes of the ſquared hyperbola, is to the rectangle of thoſe of the hyperbola propoſed, ſo is the area of the former, to that of the latter.——Thus, let $t$, $c$ be the tranſverſe and conjugate axes of any propoſed hyperbola, and $x$ its abſciſs; alſo T, C the axes of the ſquared hyperbola: Then, $t : T :: x : \frac{Tx}{t}$ = the ſimilar abſciſs in the ſquared hyperbola, whoſe correſponding area call A: Then, as $TC : tc :: A : \frac{tcA}{TC}$ = the area of the hyperbola propoſed.——Now, it will be moſt convenient to have the ſquared hyperbola an equilateral one, whoſe axes are each an unit. For then we need only divide the abſciſs of the hyperbola propoſed, by its tranſverſe axe, and multiply the product of its axes by the area anſwering to this quotient in the ſquared, or equilateral, hyperbola. And this is ſimilar to finding the ſegments of circles, and ellipſes, from thoſe of a circle whoſe ſegments are already calculated, and arranged in tables.

*Corol.* 6. If $a$ and $c$ be any other two conjugate diameters, and $s$ the ſine of the angle formed by $a$ and its ordinates, to the radius

whose hyperbolic log. is $1{\cdot}098612288\frac{2}{3}$; which being multiplied by $ac = 15 \times 9$, produces $148{\cdot}31265897$; and this being taken from $vy = 25 \times 12 = 300$,

leaves the area required $= 151{\cdot}68734103$.



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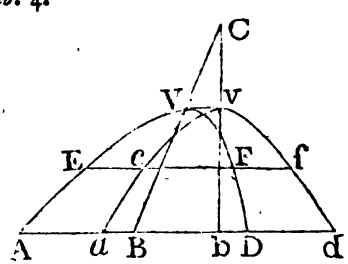
1; it is very evident that $\frac{scv\sqrt{vv - aa}}{a} - sac \times$ h. l. of $\frac{v + \sqrt{vv - aa}}{a}$ will denote the oblique segment cut off by a double ordinate to the diameter $a$. And if, in this expression, for $sa$ and $sv$ be substituted respectively A and V, the said oblique hyperbola will be expressed by $\frac{cV\sqrt{VV - AA}}{A} - Ac \times$ h. l. of $\frac{V + \sqrt{VV - AA}}{A}$; which also expresses a right hyperbola, whose transverse and conjugate axes are A, $c$, and the distance of its base from the center V.

And hence is evident the following construction, which SCHOOTEN first demonstrated, in a very operose manner, in his *Exercitationes Mathematicæ*, *Lib.* 4.




Viz. Let AD be a double ordinate to the semi-diameter CV, of the oblique hyperbola AVD. Draw V$v$ parallel and C$vb$ perpendicular to AD; and with the semi-transverse C$v$ and semi-conjugate = the semi-conjugate of CV, describe the right hyperbola $avd$. And the right hyperbola $avd$ will be = AVD the oblique one proposed; also AD = $ad$; and every other pair of corresponding ordinates EF, $ef$, equal to each other.

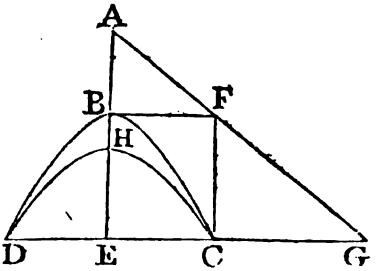
*Corol.* 7. Hence it appears that all hyperbolas, having the same center, and equal bases, and between the same parallels A$d$, V$v$, infinitely produced, are equal to each other; as are also their corresponding sections EF, $ef$, parallel to the bases; and likewise the intercepted frustums AEFD, $aefd$.

*Corol.*

RULE II.*

Putting $x$ = the abſciſs AB, and the reſt of the letters as in the firſt rule, alſo $z = \frac{x}{2a+x}$. Then $\frac{2xy}{}$

*Corol.* 8. Let AB be the ſemi-tranſverſe axe of the hyperbola DBC; draw CF parallel, and BF perpendicular to AB; and draw AFG meeting DC produced in G; then upon the baſe CD of the hyperbola deſcribe a parabola DHC, having its parameter equal to the conjugate axe of the hyperbola. And (EG − CH) × AB will be equal to the hyperbolic area EBC.




For the area EBC = A = $\frac{1}{2}vy - \frac{1}{2}ac \times$ h. l. of $\frac{ay+cv}{ac}$,

and $a \times$ the curve CH = $ac = \frac{1}{2}vy + \frac{1}{2}ac \times$ h. l. of $\frac{ay+cv}{ac}$,

by addition, A + $ac = vy$, and therefore A = $vy - ac =$

$a \times (\frac{vy}{a} - c) = (\frac{AE \times EC}{AB} - CH) \times AB = (EG - CH) \times AB$.

So that the quadrature of the hyperbola depends upon the rectification of the parabola.

*Corol.* 9. By ſubtracting the ſeg. $vy - ac \times$ h. l. of $\frac{ay+cv}{ac}$

from the ſegment $VY - ac \times$ h. l. of $\frac{aY+cV}{ac}$,

there will reſult $VY - vy - ac \times$ h. l. of $\frac{aY+cV}{ay+cv}$

for the fruſtum included by two parallel double ordinates $2Y$, $2y$; their diſtances from the center being V, $v$.

* DEMONSTRATION.

The fluxion of the area is

$\dot{y}\dot{x} = \frac{c\dot{x}\sqrt{2ax+xx}}{a}$. Then inſtead of $\frac{x}{2a+x}$ write $z$, in order that the fluent quantity $z$ may be leſs than 1, and by that means

$2xy \times (\frac{1}{3} - \frac{1}{1\cdot3\cdot5}z - \frac{1}{3\cdot5\cdot7}z^2 - \frac{1}{5\cdot7\cdot9}z^3 - \frac{1}{7\cdot9\cdot11}z^4$ &c) or

$2xy \times (\frac{1}{3} - \frac{1}{5}Az - \frac{1}{7}Bz - \frac{3}{9}Cz - \frac{5}{11}Dz$ &c)



---

means the series be made to converge by its powers, and the said fluxion

$\dot{y}\dot{x} = \frac{c\dot{x}\sqrt{2ax + xx}}{a}$ will be $= \frac{4acz^{\frac{1}{2}}\dot{z}}{\overline{1-z|^3}} = 2acz^{\frac{1}{2}}\dot{z} \times$ $(1.2 + 2.3z + 3.4z^2 + 4.5z^3$ &c);

and the fluent is $4acz^{\frac{3}{2}} \times (\frac{1.2}{3} + \frac{2.3}{5}z + \frac{3.4}{7}z^2 + \frac{4.5}{9}z^3$ &c) = the semi-segment ABD.

Now although this series converge by the powers of $z$, yet the coefficients actually diverge: in order, then, to make the coefficients converge, let the series be multiplied by some convenient quantity less than 1, as $1 - z$, and the quantity $4acz^{\frac{3}{2}}$ divided by the same, and the result will be

$\frac{8acz^{\frac{3}{2}}}{1-z} \times \frac{1.1}{1.3} + \frac{2.2}{3.5}z + \frac{3.3}{5.7}z^2 + \frac{4.4}{7.9}z^3$ &c) for the said area ABD. And, to make the series converge still quicker, let the same operation be performed upon this series, and there will result

$\frac{8acz^{\frac{3}{2}}}{\overline{1-z|^2}} \times (\frac{1}{3} - \frac{1}{1\cdot3\cdot5}z - \frac{1}{3\cdot5\cdot7}z^2 - \frac{1}{5\cdot7\cdot9}z^3 - \frac{1}{7\cdot9\cdot11}z^4$ &c) for the value of the said area.

But $\frac{8acz^{\frac{3}{2}}}{\overline{1-z|^2}} = \frac{2cx\sqrt{2ax + xx}}{a} = 2xy$; therefore

$2xy \times (\frac{1}{3} - \frac{1}{1\cdot3\cdot5}z - \frac{1}{3\cdot5\cdot7}z^2 - \frac{1}{5\cdot7\cdot9}z^3$ &c) = the area ABD. *Q. E. D.*

*Corol.* 1. An hyperbola is always less than a parabola of the same base and altitude; for $2xy \times (\frac{1}{3} - \frac{1}{1\cdot3\cdot5}z - \frac{1}{3\cdot5\cdot7}z^2$ &c)

is

will be the area of the ſemi-ſegment ABD (Fig. to rule 1); putting A, B, C, &c, for the firſt, ſecond, third, &c, term.

EXAMPLE.

Taking here the laſt example, in which $x = 10$, $y = 12$, and $2a = 30$;

$z$ will be $= \frac{x}{2a + x} = \frac{10}{30 + 10} = \frac{10}{40} = \frac{1}{4}$.

Then

---

is always leſs than $2xy \times \frac{1}{3}$ or $\frac{2}{3}xy$, the parabola. But as the altitude is diminiſhed, the hyperbola approaches nearer and nearer to the meaſure of the parabola; till at laſt they vaniſh in a ratio of equality.

*Carol.* 2. An hyperbola is evidently, for any finite altitude, always greater than a triangle of the ſame baſe and altitude. But as the altitude is increaſed, the triangle approaches nearer and nearer to the value of the hyperbola, till at length, when the altitude is infinite, they become accurately equal to each other. Which I demonſtrate, by the method of increments, after Mr. *Emerſon*'s notation, thus:

When the altitude $x$ is infinite, then $z = \frac{x}{2a + x} = \frac{x}{x} = 1$, and the general ſeries expreſſing the value of the area will become $2xy \times (\frac{1}{3} - \frac{1}{1 \cdot 3 \cdot 5} - \frac{1}{3 \cdot 5 \cdot 7} - \frac{1}{5 \cdot 7 \cdot 9}$ &c)—Now to find the ſum $s$ of any number $n$ of the negative terms; ſince it is evident that the $n$th term is $\frac{1}{2n - 1 \cdot 2n + 1 \cdot 2n + 3}$; or, putting $v = 2n - 1$, then $\dot{v} = 2\dot{n} = 2$, and the $n$th term $= \frac{1}{v \cdot v + 2 \cdot v + 3} = \frac{1}{\underset{1\,2}{vvv}}$; the $n + 1$th term, or $s$, will be $\frac{1}{\underset{1\,2\,3}{vvv}}$; and the integral $s = A - \frac{1}{\underset{1\,2}{2vvv}} = A - \frac{1}{\underset{1\,2}{4vv}}$: But when $n$ is $= 1$, then $s$ ought to be $\frac{1}{\underset{1\,2}{vvv}} =$, in this caſe, $A - \frac{1}{\underset{1\,2}{4vv}}$; hence

$A =$

Then $A = \frac{1}{3} = .33$ &c.

| | | |
|---|---|---|
| $B =$ | $\frac{1}{3}Az =$ | $.01666667$ |
| $C =$ | $\frac{1}{7}Bz =$ | $51524$ |
| $D =$ | $\frac{3}{9}Cz =$ | $4960$ |
| $E =$ | $\frac{5}{11}Dz =$ | $5639$ |
| $F =$ | $\frac{7}{13}Ez =$ | $76$ |
| $G =$ | $\frac{9}{15}Fz =$ | $11$ |
| $H =$ | $\frac{11}{17}Gz =$ | $2$ |

the ſum of theſe negative terms is $.017318039$
which being taken from the firſt term $.333333333$
leaves $.316015294$
and this multiplied by $2xy = 240$

produces $75.84367056$
$=$ the ſemi-ſegment,
the double of which is $151.68734112$
$=$ the area required.



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$A = \frac{1}{4vv} + \frac{1}{vvv} = \frac{v+4}{4vvv} = \frac{1+4}{4.1.3.5} = \frac{5}{60} = \frac{1}{12}$; and conſequently the correct integral is

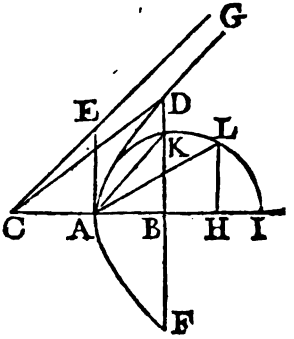
$s = \frac{1}{12} - \frac{1}{4vv} = \frac{1}{12} - \frac{1}{4.2n+1.2n+3}$; which, when $n$ is infinite, becomes accurately $= \frac{1}{12}$; that is, the ſum of the infinite ſeries $\frac{1}{1.3.5} + \frac{1}{3.5.7}$ &c is $= \frac{1}{12}$; which being taken from $\frac{1}{3}$, leaves $\frac{1}{3} - \frac{1}{12} = \frac{1}{4}$; and hence $2xy \times \frac{1}{4} = \frac{1}{2}xy =$ the area of the hyperbola when $x$ is infinite; and which is therefore equal to a triangle of the ſame baſe and altitude.

*Corol.* 3. Hence the ſpace included between the arc of an hyperbola, infinitely produced, and its chord, is of a finite magnitude, although it be of an infinite length. For this ſpace is the difference between the hyperbola and triangle, which, as above, is nothing, that is, nothing in compariſon with an infinite ſpace.

*Corol.*

RULE III.

Take BH $= \frac{3AB^2}{8AC}$ = to three-fourths of a third proportional to the tranſverſe axe 2AC, and the abſciſs AB; upon the diameter AI, = the parameter of the axe, deſcribe a circle meeting BD in K, and HL, parallel to BD, in L; and draw AK, AL.




Then will the ſegment DAF be nearly equal to $\frac{4AL + AD}{15} \times 4AB$, or $= \frac{4\sqrt{tx + \frac{3}{4}xx} + \sqrt{tx}}{15} \times \frac{4cx}{t}$; putting

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*Corol.* 4. By cor. 3 rule 2 for the ſegment of a circle, the area of a circle whoſe diameter is 1, is

$n = 2 \times (\frac{1}{3} + \frac{1}{1\cdot3\cdot5} - \frac{1}{3\cdot5\cdot7} + \&c)$, and by cor. 2 to this prob. $\frac{1}{2} = 2 \times (\frac{1}{3} - \frac{1}{1\cdot3\cdot5} - \frac{1}{3\cdot5\cdot7} - \&c)$; from hence are deduced the following equations:

I. $n = \frac{5}{6} - 4 \times (\frac{1}{3\cdot5\cdot7} + \frac{1}{7\cdot9\cdot11} + \frac{1}{11\cdot13\cdot15} + \frac{1}{15\cdot17\cdot19}$ &c)

II. $n = \frac{1}{2} + 4 \times (\frac{1}{1\cdot3\cdot5} + \frac{1}{5\cdot7\cdot9} + \frac{1}{9\cdot11\cdot13} + \frac{1}{13\cdot15\cdot17}$ &c)

III. $n = \frac{2}{3} + 12 \times (\frac{1}{1\cdot3\cdot5\cdot7} + \frac{1}{5\cdot7\cdot9\cdot11} + \frac{1}{9\cdot11\cdot13\cdot15}$ &c)

IV. $\frac{1}{96} = \frac{1}{1\cdot3\cdot5\cdot7} + \frac{2}{5\cdot7\cdot9\cdot11} + \frac{3}{9\cdot11\cdot13\cdot15}$ &c)

putting $t$ = the tranſverſe 2AC, $c$ = the conjugate, and $x$ = AB the abſciſs.*

EXAMPLE.

Taking the ſame example, in which $t = 30$, $c = 18$, and $x = 10$; we ſhall have $\frac{\sqrt{tx} + 4\sqrt{tx + \frac{4}{9}x^2}}{15} \times \frac{4cx}{t} = \frac{\sqrt{300} + 4\sqrt{375}}{450} \times 720 = \frac{10\sqrt{3} + 20\sqrt{15}}{5} \times 8 = (\sqrt{3} + 2\sqrt{15}) \times 16 = 151{\cdot}64828 =$ the area nearly.



* DEMONSTRATION.

By proceeding as in rule 2 for the circular ſegment, we obtain $\frac{cx\sqrt{tx}}{t} \times (\frac{2}{3} + \frac{x}{5t} - \frac{x^2}{28t^2}$ &c) for the true value of the ſemi-ſegment ADB. Then, after the method uſed in demonſtrating rule 6 for the circular ſegment, ſuppoſe ADB to be $\frac{cmx\sqrt{tx + \frac{1}{4}xx}}{t} + \frac{cnx\sqrt{tx}}{t} = \frac{cx\sqrt{tx}}{t} \times (m + n + \frac{3mx}{8t}$ &c): hence, comparing the like terms, $m + n = \frac{2}{3}$, and $\frac{3m}{8} = \frac{1}{5}$; and therefore $m = \frac{8}{15}$, and $n = \frac{2}{3} - \frac{8}{15} = \frac{2}{15}$. Which values being ſubſtituted for them, we have ADB $= \frac{2cx}{t} \times \frac{4\sqrt{tx + \frac{1}{4}xx} + \sqrt{tx}}{15} = 2x \times \frac{4\sqrt{\frac{c^2x}{t} + \frac{3c^2x^2}{4t^2}} + \sqrt{\frac{c^2x}{t}}}{15} = \frac{2}{15}$AB $\times [4\sqrt{(\text{AB} + \frac{3\text{AB}^2}{8\text{AC}}) \times \text{AI}} + \sqrt{\text{AB} \times \text{AI}}] =$ 2AB $\times \frac{4\sqrt{\text{AH} \times \text{AI}} + \sqrt{\text{AB} \times \text{AI}}}{15} =$ 2AB $\times \frac{4\text{AL} + \text{AD}}{15}$. *Q. E. D.*

And this rule was firſt given by Sir *I. Newton*; but without demonſtration.

RULE IV.

If BH be taken $= \frac{5xx}{7t} = \frac{5AB^2}{14AC}$ = to five-ſevenths of the ſaid third proportional to the axe 2AC, and the abſciſs AB; and the lines drawn as in the laſt rule.

Then $\frac{21AL+4AK}{75} \times 4AB$, or $\frac{21\sqrt{tx+\frac{1}{7}xx}+4\sqrt{tx}}{75} \times \frac{4cx}{t}$ will expreſs the area nearer than by the laſt rule.*

EXAMPLE.

Taking, ſtill, the ſame example, we have $\frac{21\sqrt{tx+\frac{1}{7}xx}+4\sqrt{tx}}{75} \times \frac{4cx}{t} = \frac{21\sqrt{37\frac{1}{7}}+4\sqrt{300}}{75} \times \frac{720}{30} = \frac{30\sqrt{182}+40\sqrt{3}}{75} \times 24 = \frac{3\sqrt{182}+4\sqrt{3}}{5} \times 16 = 151.681\,33$ the area nearly.

PRO-

* DEMONSTRATION.

Taking BH $= \frac{Axx}{t}$, let, as before, the area be expreſſed by $AB \times (m \times AL + n \times AK) = AB \times (m\sqrt{HA \times AI} + n\sqrt{BA \times AI}) = mx\sqrt{(x+\frac{Axx}{t}) \times \frac{cc}{t}} + nx\sqrt{x \times \frac{cc}{t}} = (m\sqrt{tx+Axx}+n\sqrt{tx}) \times \frac{cx}{t} = \frac{cx\sqrt{tx}}{t} \times (m+n+\frac{mAx}{2t} - \frac{mA^2x^2}{8tt}$ &c): which being compared with the true ſeries $\frac{cx\sqrt{tx}}{t} \times (\frac{2}{3}+\frac{x}{5t}-\frac{x^2}{28t^2}$ &c),

we

## PROBLEM VII.

*To find the Area of the Fruſtum of an Hyperbola, included between two Parallel Double Ordinates.*

Multiply the ſemi-tranſverſe by the one ordinate, and its diſtance from the center by the ſemi-conjugate, and take the ſum of the products; do the ſame by the other ordinate and its diſtance from the center, taking the ſum of the products alſo; divide the greater ſum by the leſs, and multiply the hyperbolic logarithm of the quotient by the product of the ſemi-axes; then ſubtract this laſt product from the difference of the products ariſing from the multiplication of each ordinate by its own diſtance from the center, and the remainder will be the area required.

That is, $\text{V}\Upsilon - vy - ac \times$ hyp. log. of $\frac{a\Upsilon + c\text{V}}{ay + cv} =$ the area included by $2\Upsilon$ and $2y$, V and $v$ being their reſpective diſtances from the center, and $a$ and $c$ the ſemi-axes. As is proved in cor. 8 to rule 1 to the laſt problem.

Or,

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we obtain theſe three equations, $m + n = \frac{2}{3}$, $\frac{m\text{A}}{2} = \frac{1}{5}$, and $\frac{m\text{A}^2}{8} = \frac{1}{28}$; hence $\text{A} = \frac{5}{7}$, $m = \frac{14}{25}$, and $n = \frac{8}{75}$.

Conſequently $\text{BH} = \frac{\text{A}xx}{t} = \frac{5xx}{7t}$, and the ſemi-ſegment $= 2\text{AB} \times \frac{21\text{AL} + 4\text{AK}}{75} = \frac{2cx}{t} \times \frac{21\sqrt{tx + \frac{5}{7}xx} + 4\sqrt{tx}}{75}$.

*Q. E. D.*

Which likewiſe is another rule given by Sir *I. Newton*, without demonſtration.

Or, if there be calculated the two ſegments whoſe baſes are $2Y$ and $2y$, their difference will be the area ſought.

EXAMPLE.

To find the area of the fruſtum whoſe altitude is 5; its leſs end being, as in the example to the laſt problem, 24, its diſtance from the center of the hyperbola 25, the tranſverſe axe 30, and the conjugate 18.

Here $a = 15$, $c = 9$, $v = 25$, $y = 12$, and $V = 30$; hence $Y = \frac{c\sqrt{Vv - aa}}{a} = \frac{9\sqrt{30^2 - 15^2}}{15} = 9\sqrt{2^2 - 1^2}$ $= 9\sqrt{3} = 15{\cdot}58845727$.

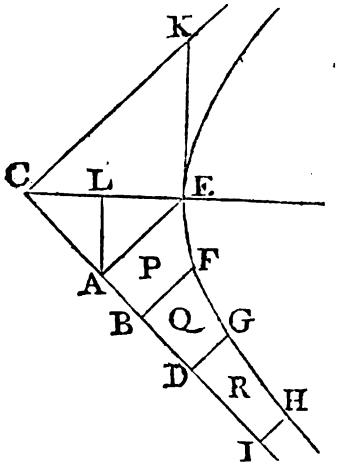
Then $\frac{aY + cV}{ay + cv} = \frac{15 \times 9\sqrt{3} + 9 \times 30}{15 \times 12 + 9 \times 25} = \frac{9\sqrt{3} + 3 \times 6}{12 + 3 \times 5} =$ $\frac{\sqrt{3} + 2}{3} = 1{\cdot}244016936$, whoſe hyp. log. is ·2183456, which multiplied by $ac = 15 \times 9$, gives 29·476656.

And $VY - vy = 30 \times 9\sqrt{3} - 25 \times 12 = 30 \times$ $(9\sqrt{3} - 10) = 30 \times 5{\cdot}58845727 = 167{\cdot}653718$.

Theref. $167{\cdot}653718 - 29{\cdot}476656 = 138{\cdot}177062$ is the area required.

PROBLEM VIII.

*To find the Area included between two Ordinates* BF, DG, *to an Aſymptote* CD *of an Hyperbola, drawn parallel to the other Aſymptote.*




From the vertex E of the curve, draw EA parallel to BF, or CK.

Then the rectangle of CA and AE multiplied by the hyp. log. of the quotient ariſing from the diviſion of CD by CB, or from the diviſion of BF by DG, will produce the area of BFGD.*

EX-

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* DEMONSTRATION.

Put CA $= a$, AE $= c$, AB $= x$, and BF $= y$; then, by the property of the curve, $ac =$ CB $\times$ BF $= (a + x) \times y$, and hence

$$y = \frac{ac}{a+x}, \text{ and } y\dot{x} = \frac{ac\dot{x}}{a+x};$$ whoſe fluent is

$ac \times$ h. l. of $\frac{a+x}{a} =$ CA $\times$ AE $\times$ h. l. of $\frac{CB}{CA} =$ the area ABFE.

In

EXAMPLE.

Required the area BDGF included by two ordinates BF, DG, whoſe lengths are 8 and 6; the ſemi-tranſverſe CE being 15, and the ſemi-conjugate EK 9.

Here $CK = \sqrt{CE^2 + EK^2} = \sqrt{15^2 + 9^2} = 3\sqrt{5^2 + 3^2} = 3\sqrt{34}$.

And in the ſimilar triangles CEK, CAE,

$$CK : CE :: CE : CA = \frac{75}{\sqrt{34}},$$

$$CK : CE :: EK : AE = \frac{45}{\sqrt{34}}.$$

Conſequently $CA \times AE = \frac{75 \times 45}{34} = \frac{3375}{34}$.

But the hyp. log. of $\frac{BF}{DG} = \frac{8}{6} = \frac{4}{3}$ is ·287682.

Theref.

---

In like manner ADGE is $= CA \times AE \times$ h. l. of $\frac{CD}{CA}$.

And, taking the difference, we have BDGF $= CA \times AE \times$ h. l. of $\frac{CD}{CA}$ — h. l. of $\frac{CB}{CA} = CA \times AE \times$ h. l. of $\frac{CD}{CB}$; or by writing $\frac{CA \times AE}{BF}$ for CB, and $\frac{CA \times AE}{DG}$ for CD, the ſame area will be $= CA \times AE \times$ h. l. of $\frac{BF}{DG}$. *Q. E. D.*

*Corol.* 1. If CB, CD, CI, &c, be in geometrical progreſſion; then $\frac{CD}{CB}$ will be $= \frac{CI}{CD}$, &c, and conſequently the ſpaces P, Q, R, &c, equal to one another; or the ſpaces ABFE, ADGE, AIKE, &c, a ſeries of arithmeticals, and are the logarithms of the geometricals $\frac{CB}{CA}$, $\frac{CD}{CA}$, $\frac{CI}{CA}$, &c, to the modulus CA; or the ſaid ſpaces will form a ſcale of hyperbolic logarithms, whoſe abſolute numbers are CB, CD, CI, &c, when CA is 1.

*Corol.*

Theref. $\cdot 287682 \times \frac{3375}{34} = 28 \cdot 55667$ is the area required.

## PROBLEM IX.

### *To find the Curve Surface of an Hyperboloid.*

Let $a$ and $c$ be the ſemi-axes of the generating hyperbola, and $v$ the diſtance of its baſe from the center. Alſo let $A = \frac{aa}{\sqrt{aa+cc}}$ be the ſemi-tranſverſe of another hyperbola, whoſe ſemi-conjugate is $c$, the ſame with that of the former. Then find, by problem 7, the area of the fruſtum of this latter hyperbola, whoſe two ends are diſtant from the center by $v$ and $a$; multiply this area by 3·14159, and the product will be the ſurface required.

That

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*Corol.* 2. The ſpace ABFE is alſo equal to

$$ac \times \left(\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \text{ \&c}\right).$$

For the fluxion of the area

$$\dot{y}\dot{x} = \frac{ac\dot{x}}{a+x} \text{ is } = ac\dot{x} \times \left(\frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4} \text{ \&c}\right)$$

whoſe fluent is $ac \times \left(\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \text{ \&c}\right)$

$$= CA \times AE \times \left(\frac{AB}{CA} - \frac{AB^2}{2CA^2} + \frac{AB^3}{3CA^3} \text{ \&c}\right).$$

After the ſame manner

$$ADGE \text{ is } = CA \times AE \times \left(\frac{AD}{CA} - \frac{AD^2}{2CA^2} + \frac{AD^3}{3CA^3} \text{ \&c}\right).$$

And conſequently, by taking the difference, we obtain EDGF

$$= CA \times AE \times \left(\frac{AD - AB}{CA} - \frac{AD^2 - AB^2}{2CA^2} + \frac{AD^3 - AB^3}{3CA^3} \text{ \&c}\right).$$

*Corol.* 3. Hence the hyperbolic logarithm of

$$\frac{a+x}{a} \text{ is } \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} \text{ \&c.}$$

That is, $p \times (v\,\text{Y} - ay - \text{A}c \times \text{hyp. log. of } \frac{\text{AY} + cv}{\text{A}y + ac})$ = the ſurface required; $p$ being = 3·14159, and Y, $y$, the ordinates, of the latter hyperbola, whoſe diſtances from the center are $v$, $a$.*

EX-

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* DEMONSTRATION.

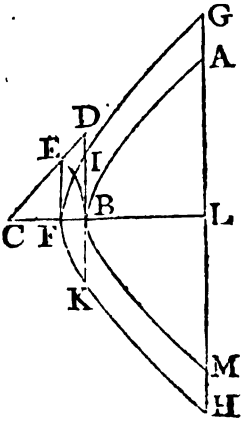
Put here, $w$ = the ordinate to the abſciſs $v$, and $z$ = the curve; then $a : c :: \sqrt{vv - aa} : w = \frac{c\sqrt{vv-aa}}{a}$, and $\dot{w} = \frac{cv\dot{v}}{a\sqrt{vv-aa}}$; hence the fluxion of the ſurface will be $2pvz = 2pv\sqrt{\dot{v}\dot{v} + \dot{w}\dot{w}}$

$$= \frac{2pc\sqrt{vv-aa}}{a}\sqrt{\dot{v}\dot{v} + \frac{ccvv\dot{v}\dot{v}}{aa(vv-aa)}} = \frac{2pcv\sqrt{aa(vv-aa)+ccvv}}{aa}\dot{v}$$

$$= \frac{2pcv\sqrt{aa+cc}}{aa}\sqrt{vv - \frac{aaaa}{aa+cc}}\dot{v} = \frac{2pcv\sqrt{vv - \text{AA}}}{\text{A}}\dot{v};$$

which is evidently equal to $p$ drawn into the fluxion of an hyperbolic ſegment, the ſemi-axes being A, $c$, and abſciſs $v$. And, by taking the correct fluent, we obtain the expreſſion in the rule above given; which is the difference of two ſegments whoſe baſes are diſtant from the center by $v$ and $a$; or the fruſtum whoſe two ends are diſtant from the center by the ſame quantities $v$ and $a$. *Q. E. D.*

*Corol.* 1. Hence if ABM be the generating hyperbola, BC its ſemi-tranſverſe, BD perpendicular to BC, and equal to the ſemi-conjugate.——Draw CD; make CE = CB, and on CB let fall the perpendicular EF; with the ſemi-tranſverſe CF, and ſemi-conjugate of ABM, deſcribe the hyperbola GFH.




Then, as the diameter of a circle is to its circumference, ſo is the fruſtum GIKH, included by the parallels EI, AM, produced, to the ſurface generated by AB.

For,

EXAMPLE.

Required the curve surface of an hyperboloid whose altitude is 10, the transverse and conjugate axes of the generating hyperbola being 30 and 18.

Here $a = 15$, $c = 9$, $A = \frac{aa}{\sqrt{aa+cc}} = \frac{15^2}{\sqrt{15^2+9^2}} = \frac{75}{\sqrt{34}}$,

$v = 25$, $Y = \frac{c\sqrt{vv - AA}}{A} = \frac{9\sqrt{25^2 - \frac{75^2}{34}}}{\frac{75}{\sqrt{34}}} = 15$,

and $y = \frac{c\sqrt{aa - AA}}{A} = \frac{9\sqrt{15^2 - \frac{75^2}{34}}}{\frac{75}{\sqrt{34}}} = \frac{27}{5} = 5\frac{2}{5}$.

Hence $vY - ay = 25 \times 15 - 15 \times 5\frac{2}{5} = 19\frac{3}{5} \times 15 = 294$; $Ac = \frac{75 \times 9}{\sqrt{34}} = \frac{75 \times 9\sqrt{34}}{34} = 115.761545$; and

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For, by similar triangles, CD : CB :: CB : CF $= A = \frac{CB^2}{CD} = \frac{CB^2}{\sqrt{CB^2 + BD^2}} = \frac{aa}{\sqrt{aa + cc}}$. And all the rest is evident.

*Corol.* 2. Putting $z = \frac{FL}{2CF + FL}$, and $z = \frac{FB}{2CF + FB}$; and proceeding as in rule 2 for the hyperbolic segment, we shall obtain the surface generated by BA, equal to 3.14159 drawn into the difference between the series

$$2FL \times GH \times \left(\frac{1}{3} - \frac{1}{1 \cdot 3 \cdot 5}z - \frac{1}{3 \cdot 5 \cdot 7}z^2 - \frac{1}{5 \cdot 7 \cdot 9}z^3 \text{ \&c}\right)$$

and the series

$$2FB \times BK \times \left(\frac{1}{3} - \frac{1}{1 \cdot 3 \cdot 5}z - \frac{1}{3 \cdot 5 \cdot 7}z^2 - \frac{1}{5 \cdot 7 \cdot 9}z^3 \text{ \&c}\right).$$

and

$$\frac{AY+cv}{Ay+ca} = \frac{\frac{75\times 15}{\sqrt{34}}+9\times 25}{\frac{75\times 27}{5\sqrt{34}}+9\times 15} = \frac{5}{3}\times\frac{5+\sqrt{34}}{3+\sqrt{34}} = \frac{19+2\sqrt{34}}{15}$$

$= 2{\cdot}0441269$; whose hyperbolic logarithm is $\cdot 7149706$; which multiplied by $AC = 115{\cdot}7615451$ produces $82{\cdot}7661049$; which, being taken from $vY - ay = 294$, leaves $211{\cdot}2338951$, which multiplied by $3{\cdot}14159$, produces $663{\cdot}610853$, the curve ſurface required.

## PROBLEM X.

### *To find the Solidity of an Hyperboloid.*

### RULE I.*

As the ſum of the tranſverſe axe of the generating hyperbola, and the height of the ſolid, is to the ſum of

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* DEMONSTRATION.

Let $t$ be the tranſverſe, and $c$ the conjugate axe of the generating hyperbola, $x$ its abſciſs, or the altitude of the ſolid, $y$ the ordinate, or radius of the baſe, and $p = 3{\cdot}14159$.

Then $yy = cc \times \frac{tx+xx}{tt}$, and the fluxion of the ſolid or $p\dot{y}y\dot{x}$ is $pcc\dot{x}x \times \frac{t+x}{tt}$, whoſe fluent is $pcc\dot{x}x \times \frac{\frac{1}{2}t+\frac{1}{3}x}{tt} = \frac{1}{2}pxyy \times \frac{t+\frac{2}{3}x}{t+x} = \frac{1}{2}$ baſe $\times$ altitude $\times \frac{t+\frac{2}{3}x}{t+x}$ the meaſure of the ſolid. *Q. E. D.*

*Corol.* 1. An hyperboloid is to a paraboloid of the ſame baſe and altitude, as $t+\frac{2}{3}x$ to $t+x$; and therefore the former is always leſs than the latter, by the quantity $\frac{1}{6}pxyy \times \frac{x}{t+x}$: which difference, when $x$ is infinitely little, or nothing is nothing; and the hyperboloid approaches nearer and nearer to the

of the ſaid tranſverſe and $\frac{2}{3}$ of the height; ſo is half the cylinder of the ſame baſe and altitude, to the ſolidity of the hyperboloid.

That is, the ſolidity is $= \frac{1}{2}$ baſe $\times$ altitude $\times \frac{t+\frac{2}{3}a}{t+a} = \frac{1}{2}parr \times \frac{t+\frac{2}{3}a}{t+a}$;

putting $a$ for the altitude, $r$ the radius of the baſe, $t$ the tranſverſe axe, and $p = 3{\cdot}14159$.

EXAMPLE.

Required the content of an hyperboloid whoſe altitude is 10, the radius of its baſe 12, and the ordinate to the middle of the abſciſs or height $3\sqrt{7}$.



the paraboloid, as the common altitude is diminiſhed; till at laſt they vaniſh in a ratio of equality. Alſo when the altitude is very little, the hyperboloid is equal to the paraboloid very nearly. But when $x$ is infinite, the ſame difference $\frac{1}{8}pxyy \times \frac{x}{t+x}$ becomes barely $\frac{1}{8}pxyy$, and the value of the infinitely long hyperboloid is $\frac{1}{2}pxyy - \frac{1}{8}pxyy = \frac{3}{8}pxyy =$ a cone of the ſame baſe and altitude, to which meaſure the hyperboloid continually approaches; and when the altitude is very great, it is equal to the cone very nearly.

*Corol.* 2. When $x$ is $= nt$, the general expreſſion becomes $\frac{1}{2}$ baſe $\times$ altitude $\times \frac{1+\frac{2}{3}n}{1+n}$; where $n$ may be any number, integral or fractional. When $n$ is infinitely little, this expreſſion becomes $\frac{1}{2}$ baſe $\times$ altitude or $=$ the paraboloid of the ſame baſe and altitude, as before.—If $n$ be infinitely great, it will become $\frac{1}{2}$ baſe $\times$ altitude $\times \frac{2}{3}$, that is, $\frac{2}{3}$ of the paraboloid, or $=$ the cone of the ſame baſe and altitude, as above. When $n$ is $= 1$, or $t = x$, the expreſſion becomes $\frac{1}{2}$ baſe $\times$ altitude $\times \frac{5}{8}$, or $\frac{5}{8}$ of the paraboloid. Moreover, when $n$ is between 1 and 0, the hyperboloid is between $\frac{5}{8}$ and $\frac{6}{8}$ of the paraboloid; and when

Here, by prob. 3, we ſhall have $\frac{10^2 \times 3^2 \times 7}{5 \times 12^2} \text{ to } \frac{5^2 \times 12^2}{10 \times 3^2 \times 7} = \frac{70-40}{8-7} = \frac{30}{1} = 30$ the tranſverſe.

Then $\frac{1}{2} parr \times \frac{t+\frac{2}{3}d}{t+a} = p \times 10 \times 144 \times \frac{36\frac{2}{3}}{80} = p \times 1440 \times \frac{11}{24} = 3{\cdot}14159 \times 660 = 2073{\cdot}45115369$, the content required.

RULE II.

To the ſquare of the radius of the baſe, add the ſquare of the diameter in the middle between the baſe and top; multiply the ſum by the altitude, the product by 3·14159, and $\frac{1}{6}$ of the laſt product will be the content of the ſegment.

That is, $\frac{rr+dd}{6} \times ap =$ the content of the ſegment; putting $r$ and $d$ for the radius of the baſe and diameter in the middle, $a$ for the altitude, and $p = 3{\cdot}14159$.

This is proved in cor. 1 to rule 2 for the next prob.

EX-

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when $n$ is between 1 and infinite, the hyperboloid is between $\frac{4}{9}$ and $\frac{4}{8}$ of the paraboloid of the ſame baſe and altitude.

*Corol.* 3. If the generating hyperbola be equilateral; ſince $t$ is then $= c$, $yy$ will be $= tx + xx$, and hence $t = \frac{yy-xx}{x}$; which being ſubſtituted inſtead of it, the general expreſſion for the hyperboloid will become $px \times \frac{yy-\frac{1}{4}xx}{2}$, which is the ſame expreſſion as that for the ſpherical ſegment, differing only in the ſign of the latter term, it being $-\frac{1}{4}xx$ here and $+\frac{1}{4}xx$ there.

*Corol.* 4. When $t$ is $= 0$, the rule becomes $\frac{1}{3}$ baſe $\times$ altitude, as it ought; for when $t$ is $= 0$, the hyperboloid becomes a cone.

EXAMPLE.

Taking here the laſt example, in which $r = 12$, $d = 6\sqrt{7}$, and $a = 10$;

We ſhall have $\frac{12^2 + 36 \times 7}{6} \times 10p = 11 \times 6 \times 10p = 660p$, the content the ſame as before.

PROBLEM XI.

*To find the Content of the Fruſtum of an Hyperboloid.*

RULE I.*

From the ſum of the ſquares of the ſemi-diameters of the two ends, ſubtract $\frac{1}{3}$ of a fourth proportional to the ſquare of the tranſverſe, the ſquare of the conjugate, and the ſquare of the altitude of the fruſtum; multiply the remainder by the ſaid altitude, and the product by 3·14159, and the half of the laſt product will be the content of the fruſtum.

That is, $(DD + dd - \frac{aacc}{3tt}) \times \frac{1}{2}pa =$ the content; putting D and $d$ for the ſemi-diameters of the ends,



* DEMONSTRATION.

Uſing here $y$, Y for the ordinates or ſemi-diameters of the ends, $x$ for the altitude; and putting A for the diſtance of the leſs ordinate $y$ from the vertex of the whole ſolid; ſince YY is $= \frac{(t + A + x) \times (A + x)}{tt} \times cc$, we ſhall have the fluxion of the ſolid $\dot{s} = p\text{YY}\dot{x} = pcc\dot{x} \times \frac{At + AA + 2Ax + tx + xx}{tt}$; and the fluents give $s = pcc\dot{x} \times \frac{At + AA + Ax + \frac{1}{2}tx + \frac{1}{3}xx}{tt}$; and this,

$a$ the altitude, $t$ the tranſverſe, $c$ the conjugate, and $p = 3{\cdot}14159$.

EXAMPLE.

If a caſk, in the form of two fruſtums of an hyperboloid, have its bung diameter 32 inches, its head diameter 24, and the diameter in the middle between the bung and head $\frac{8}{3}\sqrt{310}$; required the content in ale gallons, the length being 40 inches.

By problem 4 we ſhall have

$$\frac{10\sqrt{(-12^2+\frac{64}{25}\times 310-16^2)^2-4\times 12^2\times 16^2}}{12^2-\frac{32}{25}\times 310+16^2} = \frac{10\sqrt{(\frac{64\times 310}{25}-20^2)^2-2^2\times 12^2\times 16^2}}{20^2-\frac{32\times 310}{25}} = \frac{10\sqrt{(248-125)^2-120^2}}{125-124} = 30$$

this, by ſubſtituting $\frac{yy}{cc}$ for $\frac{At+AA}{tt}$, and $\frac{YY}{cc}$ for $\frac{At+AA}{tt}+\frac{2Ax+tx+xx}{tt}$, becomes $(YY+yy-\frac{ccxx}{3tt})\times\frac{1}{2}px$.

*Q. E. D.*

*Corol.* 1. When the generating hyperbola is equilateral, $t$ is $= c$, and the rule becomes $(YY+yy-\frac{1}{3}xx)\times\frac{1}{2}px$; which is the ſame with the rule for the fruſtum of the ſphere.

*Corol.* 2. When the axe is $= 0$, or the hyperboloid becomes a cone; ſince then $Y : y :: A+x : A$, or $Y-y : y :: x : A = \frac{xy}{Y-y}$, and $t : c :: A : y$, we ſhall have $\frac{cc}{tt} = \frac{yy}{AA} = \frac{(Y-y)^2}{xx}$, and therefore $\frac{ccxx}{3tt} = \frac{(Y-y)^2}{3}$; which being ſubſtituted for it in the rule $(YY+yy-\frac{ccxx}{3tt})\times\frac{1}{2}px$, produces $(YY+Yy+yy)\times\frac{1}{3}px$ for the fruſtum of the cone, as it ought.

$= 30\sqrt{41^2 - 40^2} = 30\sqrt{81} = 270 = t$ the tranſverſe axe.

And

$$\frac{270}{10}\sqrt{\frac{12^2}{2} - \frac{16 \times 310}{25} + \frac{16^2}{2}} = 27 \times 4\sqrt{\frac{3^2}{2} - \frac{62}{5} + \frac{4^2}{2}}$$

$$= 108\sqrt{\frac{5^2}{2} - \frac{62}{5}} = 108\sqrt{\frac{125 - 124}{10}} = 108\sqrt{\frac{1}{10}} =$$

the conjugate axe $c$.

Then

$$(DD + dd - \frac{aacc}{3tt}) \times \tfrac{1}{4}ap \times 2 = (16^2 + 12^2 - \frac{20^2 \times 108^2}{30 \times 270^2}) \times 20p = (20^2 - \frac{20^2 \times 2^2}{30 \times 5^2}) \times 20p = \frac{30 \times 25 - 4}{3} \times 4^2 \times 2p =$$

$$\frac{746 \times 32}{3} \times p = 7957\tfrac{1}{3} \times 3{\cdot}14159 = 24998{\cdot}69994216$$

inches $= 88{\cdot}64787213$ ale gallons.

* RULE II.

Add together the ſquares of the greateſt and leaſt ſemi-diameters and the ſquare of the whole diameter in



* DEMONSTRATION.

Put $x$ = the abſciſs, whoſe ordinate is $\delta$, the axes being $t$ and $c$; and, from the nature of the hyperbola, we ſhall have theſe three equations:

$$tt\delta\delta = cc(t+x) \times x = cc(tx + xx),$$

$$ttdd = cc(t + x - \tfrac{1}{2}a) \times (x - \tfrac{1}{2}a) = cc(tx + xx - ax - \tfrac{1}{2}at + \tfrac{1}{4}aa),$$

$$ttDD = cc(t + x + \tfrac{1}{2}a) \times (x + \tfrac{1}{2}a) = cc(tx + xx + ax + \tfrac{1}{2}at + \tfrac{1}{4}aa);$$

from the ſum of the two latter of which ſubtract double the former, and there will reſult $tt \times (DD - 2\delta\delta + dd) = \tfrac{1}{2}aacc$; and hence $\frac{aacc}{3tt}$ will be $= \frac{2DD - 4\delta\delta + 2dd}{3}$. Which being ſubſtituted inſtead of it, in the laſt rule, will give $(DD + 4\delta\delta + dd) \times \frac{1}{6}ap$ for the content of the fruſtum required. *Q. E. D.*

*Corol.*

in the middle, multiply the ſum by the altitude, and the product by 3·14159, and one-ſixth of the laſt product will be the content.

That is, $(DD + 4\delta\delta + dd) \times \frac{1}{6}ap =$ the content, putting D, $\delta$, and $d$ for the greateſt, middle, and leaſt ſemi-diameters, $a =$ the altitude, and $p = 3{\cdot}14159$.

EXAMPLE.

Taking here the ſame example as before, we ſhall have $\frac{16^2 + \frac{64 \times 310}{25} + 12^2}{6} \times 40p = \frac{20^2 + \frac{64 \times 62}{5}}{6} \times 40p = \frac{5 \times 20^2 + 64 \times 62}{3} \times 4p = 7957\frac{1}{3}p$, the content the ſame as before.

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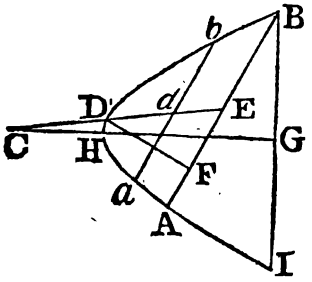
*Corol.* 1. If $d$ the leaſt diameter be ſuppoſed to become infinitely little, or nothing, the above rule will become $(DD + 4\delta\delta) \times \frac{1}{6}ap$ for the content of a ſegment.

*Corol.* 2. An hyperbolic fruſtum is equal to a cylinder of the ſame altitude, and whoſe diameter is $\sqrt{\frac{DD + 4\delta\delta + dd}{6}}$. And the ſegment is equal to the cylinder whoſe diameter is $\sqrt{\frac{DD + 4\delta\delta}{6}}$, the altitude being the ſame.

## PROBLEM XII.

*To find the Content of the Segment cut off an Hyperboloid by a Plane Oblique to the Axe of the Solid.*




As the ſum of DE and 2CD,
Is to the ſum of $\frac{3}{2}$DE and 2CD,
So is $\frac{1}{2}$ the cylindroid of the ſame baſe and alt.
To the ſolidity.

That is, $\frac{2CD + \frac{3}{2}DE}{2CD + DE} \times \frac{1}{2}DF \times$ baſe is the content of ADB.—Where 2CD is the diameter to the double ordinate AB, or tranſverſe diameter of the baſe, DE the continuation of the diameter to the middle of AB, and DF perpendicular to AB.*



* DEMONSTRATION.

Put $2CD = a$, $DE = b$, $EB = c$, $Dd = x$, $db = y$, $B =$ the elliptic baſe whoſe tranſverſe axe is AB, and $s =$ the ſine of the angle E to the radius 1.

By the property of the hyperbola, $ab + bb : ax + xx :: cc : yy = cc \times \frac{ax + xx}{ab + bb}$; and becauſe parallel ſections are ſimilar, and ſimilar figures are as the ſquares of their like diameters, we obtain $cc : yy :: B : B \times \frac{ax + xx}{ab + bb} =$ the ſection at $ab$.

Hence

*Note.* If the generating hyperbola be equilateral, the theorem will be barely $(\frac{LL + DD}{D} \times s - \frac{1}{3}L^3) \times \frac{1}{2}p$ for the content of the ſpindle; putting D = BL the greateſt diameter. As in corollary 3.

RULE

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*Corol.* 1. When $l$ becomes = L, the laſt theorem becomes $2pcs - \frac{pt^2L^3}{6cc}$ for the whole ſpindle ABCLA.

*Corol.* 2. If from the ſemi-ſpindle be ſubtracted the fruſtum, there will remain $2pc \times$ area AFE $- \frac{paatt}{cc} \times \frac{3L - 2a}{6}$ for the ſegment MAF; putting $a$ for AE the altitude.

*Corol.* 3. When the generating hyperbola is equilateral, we ſhall have $LL = 4cc - 4tt = 4cc - (2c - D)^2 = 4cD - DD$; and hence $c = \frac{LL + DD}{4D}$; which being ſubſtituted in the foregoing rules, we obtain $\frac{LL + DD}{2D} \times ps - \frac{1}{6}pL^3$ for the whole ſpindle, $\frac{LL + DD}{2D} \times p \times$ area EFBNO $- pl \times \frac{3LL - LL}{12}$ the middle zone FBNPLM, and $\frac{LL + DD}{2D} \times p \times$ area AFE $- paa \times \frac{3L - 2a}{6}$ for the ſegment MAF.

*Corol.* 4. Putting $d$ for MF the leaſt diam. of the fruſt. or zone, $n = \frac{1}{4}p = .785398$, and the reſt of the quantities as above; ſince $LL = ll \times \frac{DC - \frac{1}{4}DD}{DC - dC - \frac{1}{4}DD + \frac{1}{4}dd}$, $\frac{tt}{cc} = \frac{DC - dC - \frac{1}{4}DD + \frac{1}{4}dd}{\frac{1}{4}ll}$, and the area EFBNO $= \frac{1}{2}dl + s$, putting $s$ for the area FBN; if theſe values be ſubſtituted in the general forms, we ſhall obtain $\frac{1}{3}nl \times (2DD + dd + (-D + d + \frac{3s}{l})$ for the value of the middle zone FBNPLM,

and $\frac{1}{3}nL \times (2DD + (-D + \frac{3s}{L}) \times 8c)$ for that of the whole ſpindle.

*Corol.*

RULE II.

Divide 3 times the generating area, by the length of the ſpindle; from the quotient ſubtract the greateſt diameter of the ſpindle; multiply the remainder

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*Corol.* 5. And if here, again, the generating hyperbola be ſuppoſed to be equilateral, ſince in that caſe $ll$ is $=(2c-d)^2-4tt=(2c-d)^2-(2c-D)^2=-4cd+dd+4cD-DD$, and hence $c=\frac{ll+DD-dd}{4D-4d}=\frac{LL+DD}{4D}$; we ſhall then have $\frac{1}{3}nL\times[2DD+(-D+\frac{3s}{L})\times\frac{2LL+2DD}{D}]$ or $\frac{1}{6}pL\times(3s\times\frac{LL+DD}{LD}-LL)$ for the whole ſpindle, and $\frac{1}{3}nl\times[2DD+dd+(-D+d+\frac{3s}{l})\times\frac{2ll+2DD-2dd}{D-d}]$ or $\frac{1}{6}pl\times(\frac{3}{2}dd-ll+\frac{3s}{l}\times\frac{ll+DD-dd}{D-d})$ for the zone.

Hence it may be obſerved, that the content of an equilateral hyperbolic ſpindular caſk, may be found from having only its length, with the bung and head diameters given.

*Corol.* 6. The fluent of $\dot{p}yy\dot{x}$ or $\dot{p}\dot{x}\times(cc+tt+\frac{ttxx}{cc}-\frac{2ct\sqrt{cc+xx}}{c})$ is alſo $=px\times(cc+tt+\frac{ttxx}{3cc})-\frac{ctx\sqrt{cc+xx}}{c}-ctc\times$ hyp. log. of $\frac{x+\sqrt{cc+xx}}{c}$, which is another expreſſion for the content of the fruſtum FBLM.

*Corol.* 7. Putting $m$ for the diameter in the middle of the fruſtum or ſemi-ſpindle; then, by the nature of the hyperbola, we ſhall have $(c-\frac{1}{2}d)^2-(c-\frac{1}{2}D)^2:(c-\frac{1}{2}m)^2-(c-\frac{1}{2}D)^2::4:1$; hence $(c-\frac{1}{2}d)^2=4(c-\frac{1}{2}m)^2-3(c-\frac{1}{2}D)^2$, and $c=\frac{1}{4}\times\frac{4m^2-3D^2-d^2}{4m-3D-d}$ in the fruſtum, or $c=\frac{1}{4}\times\frac{4m^2-3D^2}{4m-3D}$ in

mainder by four times the central diſtance; and to the product add the ſquare of the greateſt diameter; then the ſum multiplied by the length, and the product by 3·14159, $\frac{1}{6}$ of the laſt product will be the content.

That is, $[DD + (\frac{3S}{L} - D) \times 4C] \frac{1}{6}pL =$ the content; the letters as before.——By corollary 4.

*Note.* If the generating hyperbola be ſuppoſed to be equilateral, 4 times the central diſtance will be $= \frac{LL + DD}{D}$, and then $(3S \times \frac{LL + DD}{LD} - LL) \times \frac{1}{6}pL$ is the content.——By corollary 5.

RULE III.

Divide the difference between 4 times the ſquare of the middle diameter and 3 times the ſquare of the greateſt, by the difference between 4 times the ſaid middle diameter and 3 times the greateſt, and $\frac{1}{4}$ of the quotient will be the central diſtance.

Then proceed as in the laſt rule.

That is, $[DD + (\frac{3S}{L} - D) \times \frac{4mm - 3DD}{4m - 3D}] \times \frac{1}{6}pL$ = the content.——By corollary 7.

*Note.*

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in the whole ſpindle; and hence the rules in corol. 4 will become $[DD + (\frac{3S}{L} - D) \times \frac{4m^2 - 3D^2}{4m - 3D}] \times \frac{1}{6}pL$ for the whole ſpindle; and $[DD + \frac{1}{2}dd + (\frac{3s}{l} + d - D) \times \frac{4m^2 - 3D^2 - d^2}{4m - 3D - d}] \times \frac{1}{6}pl$ for the zone.

*Corol.* 8. And if the generating hyp. be equilateral, then $\frac{1}{4}ll = (C - \frac{1}{2}d)^2 - (C - \frac{1}{2}D)^2 =$, by writing for C its value found in the laſt

*Note.* When the generating hyperbola is equilateral, the content may be found without having the length L given, for then L is =

$2\sqrt{\frac{(mm-\frac{3}{4}DD)^2-[(D-m)^2-\frac{1}{4}DD]^2}{(4m-3D)^2}}$, by corollary 8.

Another rule might be drawn from corollary 6.

## PROBLEM XIV.

### *To find the Content of a Zone or Double Fruſtum of an Hyperbolic Spindle.*

### RULE I.

Uſe here rule 1 for the zone of an elliptic ſpindle, and the content will be obtained.

That is, $2pc \times$ area EFBNO $- \frac{pttl}{cc} \times \frac{3LL-ll}{12} =$ the content of the zone FBNPLM; where $p = 3{\cdot}14159$, $c =$ GD the central diſtance, $t =$ BG the ſemi-tranſverſe axe, $c =$ GH the ſemi-conjugate, $l =$ EO the length of the zone, and $L =$ AC the length of the whole ſpindle.

By the general inveſtigation of the laſt problem.

*Note.*

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laſt corollary, $\frac{[(m-d)^2-\frac{3}{4}(D-d)^2]^2-[(D-m)^2-\frac{1}{4}(D-d)^2]^2}{(4m-3D-d)^2}$;

hence $2\sqrt{\frac{[(m-d)^2-\frac{3}{4}(D-d)^2]^2-[(D-m)^2-\frac{1}{4}(D-d)^2]^2}{(4m-3D-d)^2}} = l$

the length of the zone, and when $d$ vaniſhes, we have

$2\sqrt{\frac{(m^2-\frac{3}{4}D^2)^2-[(D-m)^2-\frac{1}{4}D^2]^2}{(4m-3D)^2}} =$ L the length of the whole ſpindle. So that when the hyperbola is equilateral, the length of the ſpindle need not be given, if the diameters D and $m$ be given.

*Note.* When the generating hyperbola is equilateral, the rule will be barely $p \times \frac{LL + DD}{2D} \times$ area EFBNO $- pl \times \frac{3LL - ll}{12}$.—By corollary 3 to the laſt problem. Putting D = BL the greateſt diameter.

RULE II.

Uſe the ſecond rule for the zone of an elliptic ſpindle, only here add the firſt general product inſtead of ſubtracting it, and the content will be got.

That is, $[2DD + dd + 8c \times (\frac{3s}{l} + d - D) \times 8c] \times \frac{1}{12}pl =$ FBNPLM. Where $d =$ FM the leaſt diameter, $s =$ the area FBN, and the other letters as before.——By corollary 4 to the laſt problem.

*Note.* If the generating hyperbola be equilateral, 4 times the central diſtance will be $= \frac{ll + DD + dd}{D - d}$, and then the content of the zone will be

$$[\tfrac{3}{2}dd - ll + \tfrac{3s}{l} \times (\frac{ll + DD - dd}{D - d}) \times \tfrac{3s}{l}] \times \tfrac{1}{6}pl.$$

By corollary 5 to the laſt problem.

RULE III.

From 4 times the ſquare of the diameter equidiſtant from the greateſt and leaſt, ſubtract the ſum of the ſquare of the leaſt and 3 times the ſquare of the greateſt diameter; and from 4 times the ſaid middle diameter, take the ſum of the leaſt and 3 times the greateſt; then divide the former difference by the latter, and $\frac{1}{4}$ of the quotient will be the central diſtance.

That

That is, $\frac{1}{4} \times \frac{4m^2 - 3D^2 - d^2}{4m - 3D - d} = c$, $m$ being the middle diameter.—By corollary 7 to the laſt problem.

Then proceed as in the laſt rule.

*Note.* When the generating hyperb. is equilateral, $l$ is $= 2\sqrt{\frac{[(m-d)^2 - \frac{3}{4}(D-d)^2]^2 - [(D-m)^2 - \frac{1}{4}(D-d)^2]^2}{(4m - 3D - d)^2}}$, by corollary 8 to the laſt problem; ſo that the content may then be found without having the length given.

Another rule might be drawn from corollary 6 to the laſt problem.

## PROBLEM XV.

*To find the Content of the Segment of an Hyperbolic Spindle.*

Uſe here the rule for the ſegment of an elliptic ſpindle; only inſtead of what is there called the leſs axe, take here the tranſverſe, and inſtead of the greater, the conjugate axe; and the content will be obtained.——By corollary 2 to problem 13.

That is, $2pc \times$ area AFE $- \frac{paatt}{cc} \times \frac{3L - 2a}{6} =$ the ſegment MAF. Where $a$ is $=$ AE its altitude, and the other letters as in the laſt problems.

Other rules might be found by ſubtracting the fruſtum from the ſemi-ſpindle, according to the correſponding rules of the laſt two problems.

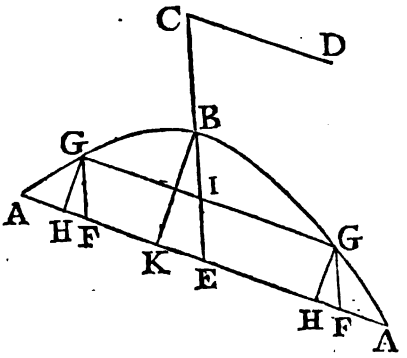
PRO-

## PROBLEM XVI.

*To find the Content of an Univerſal Hyperbolic Spindle; that is, of a Solid Generated from the Revolution of an Hyperbola about an Ordinate to Any Diameter.*

### INVESTIGATION.

Put $a$ = CB the ſemi-diameter to which belongs the double ordinate AA, about which the figure ABA revolves, $c$ = CD its ſemi-conjugate, $b$ = AE, c = CE, $z$ = EF, $y$ = GF parallel to CE, $m$ = ſine of the angle E or F, and $n$ = its coſine, to the radius 1, alſo $p = 3{\cdot}14159$; and let GH be perpendicular to AE.




Then, by the nature of the hyperbola, $c : a :: \sqrt{cc+zz} : \frac{a\sqrt{cc+zz}}{c} =$ CI; hence $y =$ c − CI $=$ $c - \frac{a\sqrt{cc+zz}}{c}$ : but GH is $= my$, and HF $= ny$; hence EH is $= z \pm ny$, and the fluxion of the ſolid is $p \times \text{GH}^2 \times \dot{\text{HK}} = pmmyy \times (\dot{z} \pm n\dot{y}) = pmm \times (\pm ny^2\dot{y} + c^2\dot{z} - \frac{2caz\dot{\sqrt{cc+zz}}}{c\sqrt{cc+zz}} + a^2\dot{z} + \frac{a^2z^2\dot{z}}{cc})$;

whoſe

whose correct fluent is $pmm \times [\mp \frac{1}{3}n(\text{BE}^3 - y^3) + c^2z + a^2z + \frac{a^2z^3}{3c^2} - \frac{caz\sqrt{cc+zz}}{c} - cac \times$ hyp. log. of $\frac{z+\sqrt{cc+zz}}{c} =$ the solid generated by KBGH.

And when $z$ becomes $= b$, the above fluent gives $pmm \times (c^2b + a^2b + \frac{a^2b^3}{3c^2} - \frac{cab\sqrt{cc+bb}}{c} - cac \times$ hyp. log. of $\frac{b+\sqrt{cc+bb}}{c}$) for half the content of the whole solid ABA.

*Corollary.* From the general investigation above it appears that $pm^2 \times (\mp \frac{1}{3}n\,\text{BE}^3 + c^2b + a^2b + \frac{a^2b^3}{3cc} - \frac{cab\sqrt{cc+bb}}{c} - cac \times$ hyp. log. of $\frac{b+\sqrt{cc+bb}}{c}$) expresses the solid generated by ABK; the sign of the first term $\frac{1}{3}n\text{BE}^3$ being $-$ or $+$, according as K falls between A and E, or without them: and consequently half the diff. of the solids generated by the spaces ABK, KBA, is $\frac{1}{3}pnm^2 \times \text{BE}^3 = \frac{1}{3}p \times \text{BK}^2 \times \text{KE} =$ the cone generated by the triangle BKE. And hence the solids generated by the two equal areas ABE, EBA, are also equal; the value of each being the half content of the whole solid mentioned above in the investigation.

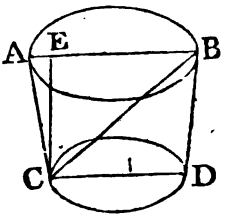
*Scholium.* The solution of this problem was never but once before attempted, viz. in a periodical performance; but the solution there given is erroneous, the fluxion and fluent of the general frustum being both falsely assigned.

## SECTION VIII.

### PROMISCUOUS QUESTIONS CONCERNING SOLIDS.

#### QUESTION I.

REQUIRED the content of a tub whoſe greater diameter AB is 60, diagonal BC 66, and the length of the ſtave AC 30 inches.




Here AB : BC + AC :: BC − AC : $\frac{36 \times 96}{60} = 6 \times 9{\cdot}6 = 57{\cdot}6$ = BE − EA = DC.

Hence AE = $\frac{1}{2}$AB − $\frac{1}{2}$CD = 30 − 28·8 = 1·2.

And EC = $\sqrt{AC^2 - AE^2} = \sqrt{30^2 - 1{\cdot}2^2} = 6\sqrt{5^2 - {\cdot}2^2}$ = $6\sqrt{25 - {\cdot}04} = 6\sqrt{24{\cdot}96}$.

Then $(AB^2 + AB \times DC + DC^2) \times \frac{1}{3}CE \times {\cdot}785398$ = $(60^2 + 60 \times 57{\cdot}6 + 57{\cdot}6^2) \times 2\sqrt{24{\cdot}96} \times {\cdot}785398$ = 10373·76 × 4√·39 × 3·14159 = 81410·112 cubic inches = 288·688 ale gallons, the content required.

#### QUESTION II.

Three perſons having bought a ſugar loaf, would divide it equally among them by ſections parallel to the baſe; it is required to find the altitude of each perſon's ſhare, ſuppoſing the loaf to be a cone whoſe height is 20 inches.

Similar cones being as the cubes of their altitudes, we ſhall have as $\sqrt[3]{3} : \sqrt[3]{1} :: 20 : 20\sqrt[3]{\frac{1}{3}} = 13{\cdot}8672247 =$ the height of the upper part; and as $\sqrt[3]{3} : \sqrt[3]{2} :: 20 : 20\sqrt[3]{\frac{2}{3}} = 17{\cdot}4716107 =$ that of the upper and middle part together; conſequently $17{\cdot}4716107 - 13{\cdot}8672247 = 3{\cdot}604386$ is the height of the middle part, and $20 - 17{\cdot}4716107 = 2{\cdot}5283893$ is that of the lower part.

### QUESTION III.

A ſilver cup, in form of the fruſtum of a cone, whoſe top diameter is 3 inches, its bottom diameter 4, and its altitude 6 inches, being filled with liquor, a perſon drank out of it till he could ſee the middle of the bottom; it is required to find how much he drank?

By problem 14 of ſection 1 we have $D = 4$, $d = 3$, $b = 6$, and $BD = \frac{1}{2}D = 2$; hence $D^3 - d^3 = 64 - 27 = 37$, $P =$ the tabular area whoſe verſed ſine is $\frac{BD}{D}$ or $\frac{1}{2}$, $= \frac{1}{2}n = \frac{1}{2}$ of $\cdot 78539816$; $Q =$ the tabular area whoſe verſed ſine is $\frac{BD - D + d}{d}$ or $\frac{1}{3}$, $= \cdot 22945528$; $\frac{BD}{BD - D + d} = 2$; and $\frac{\frac{1}{3}b}{D - d} = 2$: conſequently $(37n - 32n + 54Q\sqrt{2}) \times 2 = (5n + 54Q\sqrt{2}) \times 2 = 7{\cdot}8539816 + 35{\cdot}0458624 = 42{\cdot}899844$ cubic inches $= \cdot 152127$ ale gallons, or 1 gill and $\frac{3}{5}$ nearly, the quantity required.

### QUESTION IV.

I have a right cone which coſt me £5 13 7, at 10s a cubic foot, the diameter of its baſe being to its altitude as 5 to 8; and would have its convex

ſurface divided in the ſame ratio by a plane parallel to the baſe, the upper part to be the greater: required the ſlant height of each part.

Here $\frac{£5\ 13\ 7}{108} = \frac{1363}{120}$ is the ſolidity of the cone in feet; and $\cdot 785398 \times 5^2 \times \frac{8}{3} = 200 \times \cdot 785398 \times \frac{1}{3} = \frac{200\pi}{3}$ is that of a cone ſimilar to it, whoſe altitude is 8.

Now, the ſurfaces of ſimilar ſolids being as the ſquares of their like dimenſions, we have $\sqrt{5+8} : \sqrt{8} :: \sqrt{2\cdot 5^2 + 8^2}$ the ſide of the ſaid ſimilar cone $\sqrt{\frac{562}{13}}$ the ſlant height of the upper part of this cone, when its ſurface is divided in the ratio propoſed; and conſequently $\sqrt{70\frac{1}{4}} - \sqrt{\frac{562}{13}} = \sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}}$ is the ſlant height of the under part of it.

Then, becauſe ſimilar ſolids are as the cubes of their like dimenſions, we ſhall have $\sqrt[3]{\frac{200\pi}{3}} : \sqrt[3]{\frac{1363}{120}} :: \sqrt{\frac{562}{13}} : \frac{1}{120}\sqrt[3]{\frac{1363}{\pi}} \times \sqrt{\frac{562}{13}} = 3\cdot 9506486$ the ſlant height of the upper part required; and $\sqrt[3]{\frac{200\pi}{3}} : \sqrt[3]{\frac{1363}{120}} :: \sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}} : \frac{1}{120}\sqrt[3]{\frac{1363}{\pi}} \times (\sqrt{\frac{562}{8}} - \sqrt{\frac{562}{13}}) = \frac{1}{120}\sqrt[3]{\frac{1363}{\pi}} \times \sqrt{\frac{562}{8}} - 3\cdot 9506486 = 5\cdot 0361098 - 3\cdot 9506486 = 1\cdot 0854612$ the ſlant height of the under part.

QUES-

QUESTION V.

There is a mill-hopper in the form of a ſquare pyramid, whoſe ſolid content is $13\frac{1}{2}$ feet; but one foot is cut off its perpendicular altitude, to make a paſſage for the grain from the fruſtum or hopper to the mill-ſtone: The ſides of its greater and leſs end are in proportion as $4\frac{1}{2}$ to 1. Required the content of the fruſtum, in corn meaſure, in which 268·8025 cubic inches make a gallon, or 2150·42 a buſhel.

Since ſimilar ſolids are as the cubes of their like dimenſions, we ſhall have $4{\cdot}5^3 : 1^3 :: 13\frac{1}{2} : \frac{13\frac{1}{2}}{4{\cdot}5^3} = \frac{3}{4{\cdot}5^2} = \frac{12}{81} = \frac{4}{27}$ the content of the part cut off.

Therefore $13\frac{1}{2} - \frac{4}{27} = \frac{27}{2} - \frac{4}{27} = \frac{27^2-8}{54} = \frac{729-8}{54}$ $= \frac{721}{54}$ cubic feet $= \frac{721}{54} \times 1728$ or $\frac{721 \times 288}{9}$ or $721 \times 32$ or 23072 cubic inches, the content of the hopper. Which being divided by 2150·42, gives 10·729067 buſhels for the content in corn meaſure.

QUESTION VI.

A piece of round tapering timber, whoſe top and bottom diameters are 40 and 50 inches, and its height 6 feet, is to be cut through the extremity of the leſs diameter, and parallel to the tapering direction; required the content of the two parts or hoofs into which it is cut.

By prob. 14 ſect. 1, $D = 50$, $d = 40$, $b = 6$ feet $= 72$ inches; then $A = 50^2 \times$ tabular ſegment whoſe height is $\frac{50-40}{50}$ or $\frac{10}{50}$ or $\cdot 2 = 50^2 \times \cdot 1118238$ $= 279{\cdot}5595$.

Hence $[\frac{A \times D}{D-d} - \frac{4}{3}d\sqrt{(D-d)d}] \times \frac{1}{3}b =$ $[\frac{279{\cdot}5595 \times 50}{50-40} - \frac{160}{3}\sqrt{(50-40)40}] \times 24 =$ $(279{\cdot}5595 \times 5 - \frac{160-20}{3}) \times 24 = (279{\cdot}5595 \times 3 - 160 \times 4) \times 40 = 198{\cdot}6785 \times 40 = 7947{\cdot}14$, the content of the hoof cut off.

But, by prob. 8 ſect. 1, $\frac{50^3-40^3}{50-40} \times \frac{72}{3} \times {\cdot}785398 = (5^3 - 4^3) \times 100 \times 24 \times {\cdot}785398 = 166400 \times {\cdot}785398 = 250690{\cdot}2543893$, the content of the whole piece.

Conſequently $130690{\cdot}2543893 - 7947{\cdot}14 = 122743{\cdot}1143893$ is the ſolidity of the complemental hoof.

## QUESTION VII.

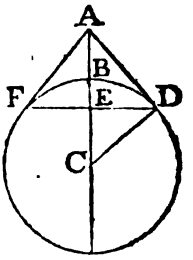
"Two Oxonians meeting at an inn, encountered with a tankard of negus; the one, being pot-valiant, gave it a black-eye, as it is called, that is, he drank till he could ſee the center of the bottom of the tankard; the other drank the reſt: Now, if the liquor coſt 1s 6d, and the tankard meaſure 4 inches diameter at the top and bottom, and 6 inches in depth, what muſt each perſon pay, proportionable to the liquor he drank?"

By prob. 4 ſect. 1, we have $\frac{1}{6}ddb$ for the quantity left by the firſt perſon; and, by prob. 2 ſect. 1, $nddb$ is the content of the whole tankard; $d$ being the diameter, $b$ the height, and $n = {\cdot}785398$; that is, the whole is to the leſs ſhare as $n$ to $\frac{1}{6}$; conſequently $n : \frac{1}{6} :: 18d : \frac{3}{n} = \frac{3}{{\cdot}785398} =$ $3{\cdot}8197185$

3·8197185 pence, which is the ſum the latter perſon muſt pay; and hence 18d − 3·8197185d = 14·1802815 pence is the ſum the firſt drinker muſt pay.

## QUESTION VIII.

How many acres of the earth's ſurface may be ſeen from the top of a ſteeple whoſe height is 400 feet; the earth being ſuppoſed a perfect ſphere whoſe circumference is 25000 miles?




Having drawn from A, the top of the ſteeple, two lines AD, AF, to touch the earth, whoſe center is C, and the other lines as in the figure; FBD will be the ſurface required.

By ſimilar triangles, CA : CD or CB :: CD or CB : CE; hence CA : CA − CB = AB :: CB : CB − CE =

$$BE = \frac{CB \times BA}{AC} = \frac{CB \times BA}{CB + BA} = \frac{\frac{12500}{3{\cdot}14159} \times \frac{400}{5280}}{\frac{12500}{3{\cdot}14159} + \frac{400}{5280}} =$$

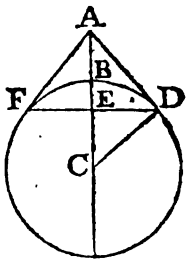
$$\frac{12500 \times 4}{1250 \times 528 + 3{\cdot}14159 \times 4} = \frac{50000}{660012{\cdot}5663706144} =$$

·07575613336416945.

Then, by problem 19 ſection 1, we have ·07575613336416945 × 250000 × 640 (the acres in a ſquare mile) = 12120981·338267112, the number of acres required.

### QUESTION IX.

How high above the ſurface of the earth muſt a perſon be raiſed, that he may ſee one third of its ſurface?



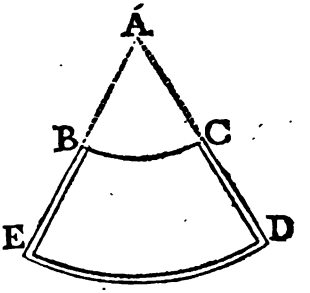

The ſurfaces of ſegments being, by prob. 19 ſect. 1, as their altitudes, the altitude of the ſegment in the queſtion muſt be one-third of the diameter, or two-thirds of the radius of the ſphere; that is, BE $=$ $\frac{2}{3}$BC; and conſequently CE $=$ $\frac{1}{3}$CB.

But, by ſimilar triangles, as CE : CD or CB :: CD or CB : CA, and hence CE : CB $-$ CE $=$ BE :: CB : CA $-$ CB $=$ AB $=$ $\frac{\text{CB} \times \text{BE}}{\text{CE}}$, which is a general expreſſion for the height above the ſurface, and which when, as above, BE is $=$ 2EC, becomes 2BC $=$ the diameter of the earth; which is the height required.

### QUESTION X.

If from a piece of tin AED, in the form of the ſector of a circle, whoſe radius AE or AD is 30 inches, and the length of its arc ED 36 inches, be cut another ſector ABC whoſe radius AB or AC is 20 inches; and if then the remaining fruſtum BGDE be rolled up ſo as to form the fruſtum of a cone; it is required to find its content, ſuppoſing one-eighth of an inch to be allowed off its ſlant height BE for the bottom, and the ſame allowance off the circumference, of both top and bottom, for what the ſides BE, CD, fold over each other, in order to their being ſoldered together.

By



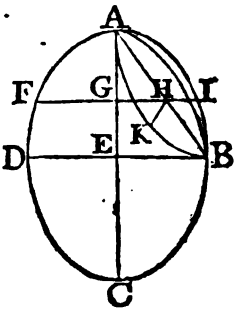

By similar figures, as AE : AB :: ED : BC $= \frac{36 \times 20}{30}$ $= 24$. Then the circumferences of the top and bottom are $36 - \frac{1}{8}$ $= 35\frac{7}{8} = 35{\cdot}875$, and $23{\cdot}875$; and the side of the vessel is $30 - 20 - \frac{1}{8}$ $= 9\frac{7}{8} = 9{\cdot}875$.

Hence the diameters of the ends will be $D = \frac{35{\cdot}875}{3{\cdot}14159}$ and $d = \frac{23{\cdot}875}{3{\cdot}14159}$; half their difference is $\frac{6}{3{\cdot}14159}$; consequently the perpendicular altitude will be found, by the property of right-angled triangles, to be $\sqrt{9{\cdot}875^2 - \frac{6^2}{3{\cdot}14159^2}} = \sqrt{\frac{9{\cdot}875^2 \times 3{\cdot}14159^2 - 6^2}{3{\cdot}14159^2}}$ $= \frac{30{\cdot}43749}{3{\cdot}14159}$.

Then, by problem 8 section 1, we shall have $\frac{35{\cdot}875^2 + 35{\cdot}875 \times 23{\cdot}875 + 23{\cdot}875^2}{3{\cdot}14159} \times \frac{30{\cdot}43749}{3{\cdot}14159} \times \frac{3{\cdot}14159}{12}$ $= \frac{287^2 + 287 \times 191 + 191^2}{8^2 \times 3{\cdot}14159^2} \times \frac{30{\cdot}43749}{12} = \frac{170667}{768} \times \frac{30{\cdot}43749}{3{\cdot}14159^2}$ $= 685{\cdot}3263$ cubic inches, the content required.

## QUESTION XI.

It is required to find the area of the section of any spheroid, formed by a plane passing through the extremities of the two axes, the one axe being 80, and the other 60.




By prop. 1 sect. 4, the section AKB will be an ellipse, whose axes are AB and 2HK, H being the middle of AB.

But

But AB $= \sqrt{AE^2 + EB^2} = \sqrt{40^2 + 30^2} = 50$.

And, having drawn FGHI parallel to DB, by the nature of the ellipſe, $AE^2 : EB^2 :: AE^2 - GE^2 = AE^2 - \frac{1}{4}AE^2 = \frac{3}{4}AE^2 : GI^2 = \frac{3}{4}EB^2$; and hence, by the nature of the circle, $2\sqrt{FH \times HI} = 2\sqrt{GI^2 - GH^2} = 2\sqrt{\frac{3}{4}EB^2 - \frac{1}{4}EB^2} = EB\sqrt{2} =$ the other axe 2HK.

Conſequently AB × 2HK × ·785398 = 50 × 30√2 × ·785398 = 1666·081101807, the area of the ſection, when the ſpheroid is oblong. And 50 × 40√2 × ·785398 = 2221·441469076, the area when it is oblate.

## QUESTION XII.

There is a punch bowl in form of the ſegment of an oblong ſpheroid, whoſe axes are to each other in the proportion of 3 to 4, the depth of the bowl being one-fourth of the whole tranſverſe axe, and the diameter of its top 20 inches: it is required to determine what number of rounds a company of 10 men may drink out of it, when filled with liquor, uſing a conical glaſs, whoſe depth is 2 inches, and the diameter of its top an inch and a half.

The ſegment, whoſe altitude is 1, of the ſpheroid, whoſe axes are 3 and 4, is ſimilar to the propoſed one; and, by the nature of the ellipſe, $4 : 3 :: 2\sqrt{3 \times 1} : \frac{3\sqrt{3}}{2}$ the diameter of its top; but by prob. 14 ſect. 5, its content is $\frac{6-1}{3 \times 4^2} \times 3^2 \times 3{\cdot}14159 = \frac{15 \times {\cdot}785398}{4}$; and ſimilar ſolids are as the cubes of their like dimenſions; therefore $(\frac{3\sqrt{3}}{2})^3 : 20^3 ::$

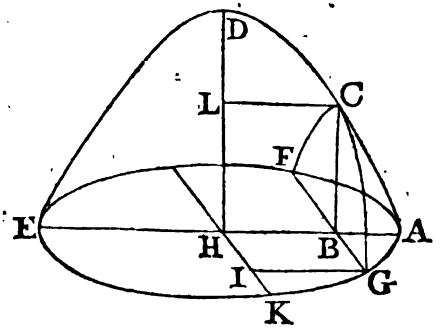
$\frac{15 \times .785398}{4} : \frac{40^3}{3^4 \times \sqrt{3}} \times \frac{15 \times .785398}{4} = \frac{80000\sqrt{3}}{81} \times .785398$, the content of the bowl.

But $1.5^2 \times 2 \times \frac{.785398}{3} = \frac{3}{2} \times .785398$, the content of the glass, and $15 \times .785398 =$ each round.

Therefore $\frac{80000\sqrt{3} \times .785398}{81 \times 15 \times .785398} = \frac{16000\sqrt{3}}{243} =$ 114.0444976 is the number of rounds required.

### QUESTION XIII.

Two persons would divide between them, by a plane perpendicular to the base, a hay-rick, in the form of a paraboloid, whose altitude is 40, and the diameter of its base 30 feet; it is required to find the difference between the solidities of the parts, supposing the altitude of the section to be 28 feet.




Here DH = 40, EA = 30, and BC = HL = 28.

Hence DL = DH − HL = 40 − 28 = 12; and, by prob. 3 sect. 6, as $\sqrt{}$DH : $\sqrt{}$DL :: HA : LC = HB $= \frac{15\sqrt{12}}{\sqrt{40}} = \frac{3\sqrt{30}}{2}$; hence $\sqrt{HA^2 - HB^2} = \sqrt{15^2 - \frac{9 \times 30}{4}} = \sqrt{\frac{315}{2}} =$ BG, and $\frac{BA}{EA} = \frac{15 - \frac{3}{2}\sqrt{30}}{30} = \frac{10 - \sqrt{30}}{20} =$ .2261387 the tabular versed sine similar to AB; whose tabular area is .13322474; which taken from .39269908, the tabular semi-circle, leaves .25947434 the

the tabular area ſimilar to double the area HBGK; conſequently $\frac{1}{2} \times 30^2 \times .25947434 = 116.763453$ is the area HBGK.

Then, by cor. 4 to prob. 14 ſect. 6, we ſhall have $116.763453 \times 40 \times 2 + \frac{315\sqrt{28 \times 12}}{3} = 9341.07624 + 420\sqrt{21} = 9341.07624 + 1924.681794 = 11265.75803$, the difference required.

## QUESTION XIV.

The curve ſurface of a paraboloid being $(109\sqrt{109} - 27) \times 2 \times 3.14159 = 6980.57746$, it is required to find its altitude, and the diameter of its baſe, ſuppoſing them in proportion as 5 to 6.

By prob. 7 ſect. 6, $\frac{(3^2+10^2)^{\frac{3}{2}}-3^3}{3^2+10^2-3^2} \times 3 \times \frac{2}{3} \times 3.14159 = \frac{109\sqrt{109}-27}{100} \times 2 \times 3.14159$ is the ſurface of a paraboloid whoſe altitude is 5 and baſe diameter 6, which is ſimilar to the propoſed paraboloid; but the ſurfaces of ſimilar bodies are as the ſquares of their lineal dimenſions, or the dimenſions are as the roots of the ſurfaces; therefore as $\sqrt{\frac{109\sqrt{109}-27}{100}} \times 2 \times 3.14159$ is to $\sqrt{\frac{109\sqrt{109}-27}{1}} \times 2 \times 3.14159$, or as 1 : 10 :: 5 : 50 the altitude, and :: 6 : 60, the baſe-diameter required.

## QUESTION XV.

If a cubical foot of braſs were to be drawn into wire, of one-ſortieth of an inch in diameter, it is required to determine the length of the ſaid wire, allowing no loſs in the metal.

Here $1 \times 12^3 = 1728$ is the ſolidity of the wire in inches; and $(\frac{1^2}{40^2}) \times .785398 = \frac{1}{1600} \times .785398$ the

the area of its end; consequently

$$\frac{1728}{\frac{1}{1600} \times .785398} = \frac{1728 \times 1600}{.785398} = 3520252.69329$$

inches $= 55\frac{5}{9}$ miles, is the length required.

QUESTION XVI.

A gentleman having a bowling green, 300 feet long and 200 feet broad, which he would raise one foot higher by means of the earth to be dug out of a ditch around it; it is required to find to what depth the ditch must be dug, its breadth being every where 8 feet.

Here the ditch is a kind of ring whose breadth is 8, and length $= 300 \times 2 + 200 \times 2 + 8 \times 4 = 516 \times 2 = 1032$, and consequently the area of its bottom $= 8 \times 1032 = 8256$; but $300 \times 200 \times 1 = 60000$ is its content; consequently $\frac{60000}{8256} = 7\frac{288}{516}$ feet is the depth required.

QUESTION XVII.

Required the weight of a bomb-shell, or hollow sphere of cast iron, whose outside diameter is 15, and the thickness of the metal $2\frac{1}{2}$ inches; the gravity of water being to that of cast iron, as 1 to 7, and a cubic inch of water weighing .5787 ounces avoirdupois.

Here $15^3 \times 3.14159 \times \frac{1}{6} = 15^3 \times .5236$ is the solidity of the whole sphere, supposing it all solid; and, since $15 - 2\frac{1}{2} \times 2 = 15 - 5 = 10$ is the diameter of the cavity, $10^3 \times .5236$ will be the content of the cavity; consequently their difference $(15^3 - 10^3) \times .5236 = (3^3 - 2^3) \times 5^3 \times .5236 = 19 \times 125 \times .5236$ is the solidity of the metal in inches.

But

But 1 : 7 :: ·5785 : 4·0509 ounces, the weight of a cubic inch of the metal.

Conſequently 4·0509 × 19 × 12·5 × ·5236 = 5037·4849152 ounces = 314·8428072 pounds avoirdupois is the weight required.

### QUESTION XVIII.

Of what diameter muſt the bore of a cannon be caſt for a ball of 24 pound weight, ſo that the diameter of the bore may be one tenth of an inch more than that of the ball?

By the laſt, the weight of a cubic inch of caſt iron is 4·0509 ounces, conſequently as 4·0509 ounces : 1 inch :: 24 lb. or 24 × 16 oz. : $\frac{24 \times 16}{4{\cdot}0509} = \frac{128}{1{\cdot}3503}$ inches, the ſolidity of the ball; but the ſolidity is equal to the cube of the diam. multiplied by ·5236; therefore $\sqrt[3]{\frac{128}{1{\cdot}3503 \times {\cdot}5236}} = 5{\cdot}657098$ inches, the diameter of the ball; to which adding ·1 makes 5·757098 inches, the diameter required.

PART

# PART IV.

Having given the menſuration of all figures which are uſually thought to be met with in real practice, viz. the figures formed by right or circular lines, or by the conic ſections; I ſhall, under this part, give a very brief treatiſe of other things relating to menſuration in general; ſuch as the quadrature and cubature of figures, from general equations expreſſing their nature or property; the method of equidiſtant ordinates; and the relation between the areas or ſolidities of figures, and their centers of gravity.

## SECTION I.

### OF THE TRUE QUADRATURE AND CUBATURE OF CURVES IN GENERAL.

#### PROPOSITION I.

*IF z be the abſciſs of any curve, and the ordinate y be equal to* $z^{p-1} \times (\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n} + \&c.)^{q-1} \times (a + b z^n + c z^{2n} + d z^{3n} + e z^{4n} \&c)$; *and if* $\frac{p}{n}$ *be put equal to* $r$, $r + q = s$, $s + q = t$, $t + q = v$, $v + q = w$, *&c. I ſay the area will be equal to*

$z^p \times (\alpha + \beta z^n + \gamma z^{2n} + \delta z^{3n} + \varepsilon z^{4n} + \&c)^q$

*drawn into the ſeries*

+

$$+\frac{\frac{a}{n}}{\alpha r}$$

$$+\frac{\frac{b}{n}-A\beta s}{\alpha(r+1)}z^{n}$$

$$+\frac{\frac{c}{n}-A\gamma t-B\beta(s+1)}{\alpha(r+2)}z^{2n}$$

$$+\frac{\frac{d}{n}-A\delta v-B\gamma(t+1)-C\beta(s+2)}{\alpha(r+3)}z^{3n}$$

$$+\frac{\frac{e}{n}-A\varepsilon w-B\delta(v+1)-C\gamma(t+2)-D\beta(s+3)}{\alpha(r+4)}z^{4n}$$

+ &c.

Where $A$, $B$, $C$, &c, are the whole coefficients of the preceding terms, with their ſigns + or −; viz. $A=\frac{\frac{a}{n}}{\alpha r}$, $B=\frac{\frac{b}{n}-A\beta s}{\alpha(r+1)}$, &c.

DEMONSTRATION.

The fluxion of the area is $y\dot{z}=z^{r-1}\dot{z}\times(\alpha+\beta z^{n}+\gamma z^{2n}\,\&c)^{s-1}\times(a+bz^{n}+cz^{2n}+\&c)$.

Let the fluent of this expreſſion, or the area required, be repreſented by $z^{r}(\alpha+\beta z^{n}+\gamma z^{2n}+\&c)^{s}\times(A+Bz^{n}+Cz^{2n}+\&c)$; where A, B, C, &c, are not ſuppoſed to repreſent the ſame quantities as in the propoſition, but other quantities yet to be determined.

Then

Then let the fluxion of this laſt aſſumed area or fluent be taken, and compared with the given fluxion, ſo ſhall we have equations for determining the values of the aſſumed letters A, B, C, &c.

Thus the fluxion of the aſſumed area being $z^{p-1}\dot{z} \times (\alpha + \beta z^n + \gamma z^{2n} + \&c)^q \times (Ap + (p+n)Bz^n + (p+2n)Cz^{2n} + \&c, + nqz^{p-1}z \times (\alpha + \beta z^n + \gamma z^{2n} + \&c)^{q-1} \times (\beta z^n + 2\gamma z^{2n} + 3\delta z^{3n} + \&c) \times (A + Bz^n + Cz^{2n} + \&c)$.

Or, by proper multiplication, &c,

$z^{p-1}\dot{z} \times (\alpha + \beta z^n + \gamma z^{2n} + \&c)^{q-1}$ drawn into

$$\begin{aligned} &p\alpha A + (p+qn)\beta A z^n + (p+2qn)\gamma A z^{2n} + (p+3qn)\delta A z^{3n} + \&c, \\ &+ (p+n)\alpha B z^n + (p+qn+n)\beta B z^{2n} + (p+2qn+n)\gamma B z^{3n} \\ &+ (p+2n)\alpha C z^{2n} + (p+qn+2n)\beta C z^{3n} \\ &+ (p+3n)\alpha D z^{3n} \\ &\&c. \end{aligned}$$

If the ſeveral terms of this ſeries be compared with the correſponding terms of the ſeries in the given fluxion, we ſhall obtain theſe equations, viz.

$$\begin{aligned} a &= p\alpha A, \\ b &= (p+qn)\beta A + (p+n)\alpha B, \\ c &= (p+2qn)\gamma A + (p+qn+n)\beta B + (p+2n)\alpha C, \\ d &= (p+3qn)\delta A + (p+2qn+n)\gamma B + (p+qn+2n)\beta C \\ &\quad + (p+3n)\alpha D, \\ e &= p+4qn)\varepsilon A + (p+3qn+n)\delta B + (p+2qn+2n)\gamma C \\ &\quad + (p+qn+3n)\beta D + (p+4n)\alpha E, \\ \&c. &\quad \&c. \end{aligned}$$

From whence we obtain

$$A = \frac{a}{\alpha p},$$

$$B = \frac{b - (p+qn)\beta A}{\alpha(p+n)},$$

$$C = \frac{c - (p+2qn)\gamma A - (p+qn+n)\beta B}{\alpha(p+2n)},$$

$$D = \frac{d - (p+3qn)\delta A - (p+2qn+n)\gamma B - (p+qn+2n)\beta C}{\alpha(p+3n)},$$

$$E = \frac{e - (p+4qn)\varepsilon A - (p+3qn+n)\delta B - (p+2qn+2n)\gamma C - (p+qn+3n)\beta D}{\alpha(p+4n)},$$

&c. &c.

Where now the letters A, B, C, &c, in the terms on the right hand ſide of theſe equations, denote the preceding terms, as ſpecified in the propoſition.

And if $\frac{p}{n}$ be put $= r$, $r + q = s$, $s + q = t$, $t + q = v$, &c, theſe laſt equations will become

$$A = \frac{\frac{a}{n}}{\alpha r},$$

$$B = \frac{\frac{b}{n} - A\beta s}{\alpha(r+1)},$$

$$C = \frac{\frac{c}{n} - A\gamma t - B\beta(s+1)}{\alpha(r+2)},$$

$$D = \frac{\frac{d}{n} - A\delta v - B\gamma(t+1) - C\beta(s+2)}{\alpha(r+3)},$$

$$E = \frac{\frac{e}{n} - A\varepsilon w - B\delta(v+1) - C\gamma(t+2) - Db(s+3)}{\alpha(r+4)},$$

&c. &c.

Which

Which values of A, B, C, &c, being ſubſtituted for them in the aſſumed fluent or area, will give the area as in the propoſition. *Q. E. D.*

And much after the ſame manner we may find the area of the curve whoſe abſciſs is $z$, and ordinate $(\alpha+\beta z^n+\gamma z^{2n}+\&c)\times z^{p-1}\times(\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}+\&c)^{q-1}$ $\times\,(A+Bz^n+Cz^{2n}+Dz^{3n}+\&c)^{r-1}\times\,\&c$, whatever be the number of the ſeries.

When, after ſome of the firſt terms, the numerators of each of the following terms of the ſeries $\frac{\frac{a}{n}}{\alpha r}+\frac{\frac{b}{n}-A\beta s}{\alpha(r+1)}z^n+\&c$, become equal to nothing, the ſeries will break off, and terminate; and then the curve is ſaid to be quadrable. If otherwiſe, it is ſaid to be non-quadrable.——If $r$ be either nothing, or a negative integer number, it is evident that the denominator of one of the terms of the above ſeries will become equal to nothing, and then that term will be infinite; and if this happen before the ſeries terminate, by means of the numerators becoming equal to nothing, the value of the area will come out infinite; in which caſe the ſeries is ſaid to fail.

The curve is denominated from the number of terms contained in the quantity $(\alpha+\beta z^n+\gamma z^{2n}+\delta z^{3n}+\&c)^q$. So, if it contain only one term $\alpha$, it is a ſimple nomial; if it contain two terms $(\alpha+\beta z^n)^q$, it is a binomial; if three $(\alpha+\beta z^n+\gamma z^{2n})^q$, it is a trinomial; and ſo on.

But in what follows I ſhall conſider none beyond the trinomial.

The area might be expreſſed by a deſcending ſeries, and from thence other caſes of the termination of the ſeries might be pointed out; but this I ſhall do in the particular forms, as in them it will be done with greater eaſe.

Nor ſhould it be wondered at that the area admits of two different values; for when an ordinate flows, the area on one ſide of it will increaſe as faſt as that on the other decreaſes; conſequently the fluxions of thoſe two areas will be equal to each other, that is, the fluxions of the areas are both expreſſed by the ſame quantity; and it is therefore but right that the fluxion ſhould admit of two different fluents, anſwering to the two areas on the oppoſite ſides of the ordinate. When the expreſſion comes out affirmative, it denotes the area lying on that ſide of the ordinate from which it is ſuppoſed to move; but when it comes out negative, it denotes the area on the other ſide of the ordinate. When it comes out infinite, it denotes the area lying along the abſciſs infinitely produced.

If all the terms of the ſeries $a + bz^n + cz^{2n} +$ &c, after the firſt, vaniſh, and by that means the expreſſion for the ordinate become only $az^{p-1} \times (\alpha + \beta z^n + \gamma z^{2n} +$ &c$)^{q-1}$, the curve or area is ſaid to be ſimple; but if there be more terms than one, it is ſaid to be compound.

In what follows I ſhall conſider chiefly thoſe caſes in which the curve is ſaid to be ſimple; not only becauſe thoſe are the caſes that commonly happen

happen in practice, but becauſe every compound caſe may be reſolved into as many ſimple ones, as there are terms in the ſeries $a + bz^n + cz^{2n} +$ &c); and then, by finding the area for every ſimple caſe, the aggregate of thoſe will be the area for the whole compound one.

In the quantity expreſſing any area, write that particular value of the abſciſs which it is ſuppoſed to have where the area commences, or when it is equal to nothing, and the value of the area reſulting from that ſubſtitution, will be equal to nothing, if the firſt area be rightly aſſigned, and then it needs no correction; but if the area, by this ſubſtitution, come out of ſome value, then by juſt ſo much will the firſt area differ from the truth, and it muſt be corrected by ſubtracting the ſaid value from it.

It may alſo be obſerved, that when the ordinate is oblique to the abſciſs, the area, as found by the ſeries, muſt be drawn into the ſine of the angle of inclination of the ordinate to the abſciſs.

EXAMPLE.

If an ordinate $y$ be $= \frac{3a - bzz}{zz\sqrt{az - bz^3 + cz^4}}$.

To have this in the ſame form with the general ſeries, we muſt expreſs it thus

$$y = z^{-\frac{3}{2}-1} \times (3a - bzz) \times (a - bz^2 + cz^3)^{\frac{1}{2}-1};$$

or it may be expreſſed thus

$$y = z^{-1-1} \times (-b + 3az^{-2}) \times (c - bz^{-1} + az^{-3})^{\frac{1}{2}-1}.$$

Now by comparing the firſt of theſe forms with the general expreſſion of the area, we obtain

$a = 3a$, $b = 0$, $c = -b$; $a = \alpha$, $\beta = 0$, $\gamma = -b$, $\delta = c$; $p = -\frac{3}{2}$, $n = 1$, $q = \frac{1}{2}$.

Hence $r = \frac{p}{n} = -\frac{3}{2}$, $s = r + q = -1$, $t = s + q = -\frac{1}{2}$, $v = t + q = 0$, $w = v + q = \frac{1}{2}$, &c.

Then, by ſubſtituting theſe values in the general ſeries, we have

$z^{-\frac{3}{2}} \times (a - b z^2 + c z^3)^{\frac{1}{2}} \times -2 - 2\sqrt{\frac{a - bz^2 + cz^3}{z^3}}$,

all the reſt of the terms after the firſt vaniſhing. And becauſe this expreſſion is negative, it denotes the area on the other ſide of the ordinate.

Again, by comparing the latter form with the general ſeries, we obtain $a = -b$, $b = 0$, $c = 3a$; $\alpha = c$, $\beta = -b$, $\gamma = 0$, $\delta = a$; $p = -1$, $n = -1$, $q = \frac{1}{2}$.

Hence $r = \frac{p}{n} = 1$, $s = r + q = \frac{3}{2}$, $t = s + q = 2$, &c.

Then ſubſtituting theſe values in the general ſeries, we obtain $\sqrt{\frac{a - bz^2 + cz^3}{z^5}} \times$ $(\frac{b}{c} + \frac{3bA}{4cz} + \frac{5bB - 6a}{6cz^2} + \frac{7bC - 5aA}{8cz^3} + \frac{9bD - 7aB}{10cz^4} + \&c)$ for the area in this caſe; where the law of the progreſſion is manifeſt, and where A, B, C, &c, denote the whole coefficients of the firſt, ſecond, third, &c, terms,

### COROLLARY I.

When $b$, $c$, $d$, &c, are each equal to nothing, the curve will be ſimple, and the general expreſſion for ſimple

ple areas becomes $\frac{az^{p}}{an} \times (\alpha + \beta z^{n} + \gamma z^{2n} + \delta z^{3n} + \&c)^{q} \times$

$+ \frac{1}{r}$

$- \frac{s\beta A}{(r+1)\alpha} z^{n}$

$- \frac{t\gamma A + (s+1)\beta B}{(r+2)\alpha} z^{2n}$

$- \frac{v\delta A + (t+1)\gamma B + (s+2)\beta C}{(r+3)\alpha} z^{3n}$

$- \frac{w\varepsilon A + (v+1)\delta B + (t+2)\gamma C + (s+3)\beta D}{(r+4)\alpha} z^{4n}$

&c.

COROLLARY II.

If the curve be only a trinomial, as $y = az^{p-1} \times (\alpha + \beta z^{n} + \gamma z^{2n})^{q-1}$, that is, $\delta$, $\varepsilon$, &c, each equal to nothing, the area in the laſt corollary will become $\frac{az^{p}}{an} \times (\alpha + \beta z^{n} + \gamma z^{2n})^{q} \times (\frac{1}{r} - \frac{s\beta A}{(r+1)\alpha} z^{n} - \frac{(s+1)\beta B + t\gamma A}{(r+2)\alpha} z^{2n}$

$- \frac{(s+2)\beta C + (t+1)\gamma B}{(r+3)\alpha} z^{3n} - \frac{(s+3)\beta D + (t+2)\gamma C}{(r+4)\alpha} z^{4n} - \&c)$.

But to find another expreſſion of the area in a deſcending ſeries, we have the ordinate $y$ or $az^{p-1} \times (\alpha + \beta z^{n} + \gamma z^{2n})^{q-1} = az^{p-1+2nq-2n} \times (\gamma + \beta z^{-n} + \alpha z^{-2n})^{q-1}$; then, by comparing this with the original ſeries, the area will come out $\frac{az^{p-2n}}{\gamma n} \times (\alpha + \beta z^{n} + \gamma z^{2n})^{q} \times (\frac{1}{t-2}$

$- \frac{(s-2)\beta A}{(t-3)\gamma z^{n}} - \frac{(s-3)\beta B + (r-2)\alpha A}{(t-4)\gamma z^{2n}} - \frac{(s-4)\beta C + (r-3)\alpha B}{(t-5)\gamma z^{3n}}$

$-$ &c).

When any particular example is propoſed, if, after having ſubſtituted in the former or aſcending form, it come out an infinite ſeries, let the latter or deſcending form be tried; for ſometimes an area is quadrable by the one ſeries, when it is not ſo by the other.

EXAMPLE I.

Let the ordinate $y$ be $= \frac{a}{\sqrt{z^6 - 2z^7 + 3z^8}}$; which reduced to form is $az^{-2\cdot1} \times (1 - 2z + 3z^2)^{\frac{1}{2}-1}$.

Here then we have $a = a$; $\alpha = 1, \beta = -2, \gamma = 3$; $p = -2$, $n = 1$, $q = \frac{1}{2}$.

Whence $r = \frac{p}{n} = -2$, $s = r + q = -\frac{3}{2}$, $t = s + q = -1$, $v = t + q = -\frac{1}{2}$, $w = v + q = 0$, &c.

And by ſubſtituting theſe values in the aſcending ſeries, we obtain $az^{-2} \times (1 - 2z + 3z^2)^{\frac{1}{2}} \times (-\frac{1}{2} - \frac{3}{2}z)$ all the reſt of the terms vaniſhing, $= \frac{1 + 3z}{2zz} \times -a\sqrt{1 - 2z + 3zz}$ for the area required; and which therefore is quadrable.

But if the ſame values be ſubſtituted in the deſcending ſeries, we get the area expreſſed by the infinite ſeries

$$\frac{a\sqrt{1 - 2z + 3z^2}}{9z^4}\left(-1 - \frac{7A}{4z} - \frac{9B - 4A}{5z^2} - \frac{11C - 5B}{6z^3} \text{ \&c}\right) =$$

$$\frac{a\sqrt{1 - 2z + 3z^2}}{9z^4}\left(-1 + \frac{7}{4z} - \frac{9\cdot7 + 4\cdot4}{4\cdot5z^2} + \frac{11(9\cdot7 + 4\cdot4) + 5\cdot5}{4\cdot5\cdot6z^3} \text{ \&c}\right) =$$

$$\frac{a\sqrt{1 - 2z + 3z^2}}{9z^4} \times \left(-1 + \frac{7}{4z} - \frac{79}{4\cdot5z^2} + \frac{894}{4\cdot5\cdot6z^3} - \text{ \&c}\right).$$

EX-

EXAMPLE II.

Suppose $y = \frac{azz}{2+3z+3z^2|^{\frac{2}{3}}}$; which reduced to the proper form is $az^{3\cdot 1} \times (2+3z+3z^2)^{\frac{1}{3}\cdot 1}$.

Here $a=a$; $\alpha=2, \beta=3, \gamma=3, p=3, n=1, q=\frac{1}{3}$.

Then $r=\frac{p}{n}=3$, $s=3\frac{1}{3}$, $t=3\frac{2}{3}$, $v=4$, &c.

And, by the ascending series, the area will be expressed by the infinite series $\frac{1}{4}az^3\sqrt[3]{2+3z+3z^2} \times (\frac{1}{3}-\frac{10A}{8}z-\frac{13B+11A}{10}z^2-\frac{16C+14B}{12}z^3-\frac{19D+17C}{14}z^4$ &c) $=az^3\sqrt[3]{2+3z+3z^2} \times (\frac{1}{6}-\frac{10}{6.8}+\frac{42}{6.8.10}+\frac{728}{6.8.10.12}$ &c); and therefore it is not quadrable by this method.

But by the descending series the area is quadrable, and comes out $\frac{z-2}{5} \times a\sqrt{2+3z+3z^2}$, all the terms after the second vanishing.

So that sometimes the area is quadrable by the one series, and sometimes by the other; but it is also sometimes quadrable by neither, and sometimes by both of them.

COROLLARY III.

In the case of a binomial $y = az^{p-1} \times \overline{\alpha+\beta z^n}|^{q-1}$, beside the letters which in the last corollary were equal to nothing, $\gamma$ also will be nothing; and by supposing it to vanish in the ascending series in the last corollary, there will result $\frac{az^p}{an} \times (\alpha+\beta z^n)^q \times (\frac{1}{r}-\frac{s\beta}{(r+1)\alpha}Az^n-\frac{(s+1)\beta}{(r+2)\alpha}Bz^{2n}-\frac{(s+2)\beta}{(r+3)\alpha}Cz^{3n}-\frac{(s+3)\beta}{(r+4)\alpha}Dz^{4n}$ &c. for the ascending series to be used in this case; where

where the ſeveral letters have the ſame value as before. And this ſeries, it is evident, will always terminate when $s$ is either nothing or a negative integer, and the number of terms will be one more than the number of units in $s$.

But the ſame ordinate $y = az^{p-1} \times (\alpha + \beta z^n)^{q-1}$ is $= az^{p-1+qn-n} \times (\beta + \alpha z^{-n})^{q-1}$. Then by writing, in the above aſcending ſeries, $\alpha$ for $\beta$, $\beta$ for $\alpha$, $p + qn - n$ for $p$, and $-n$ for $+n$, we ſhall have $\frac{az^{p-n}}{\beta n} \times (\alpha + \beta z^n)^q \times (\frac{1}{s-1} - \frac{(r-1)\alpha A}{(s-2)\beta z^n} - \frac{(r-2)\alpha B}{(s-3)\beta z^{2n}} - \frac{(r-3)\alpha C}{(s-4)\beta z^{3n}}$ &c) for the area expreſſed by a deſcending ſeries; and which, it is evident, will terminate, and be quadrable, when $r$ is any affirmative integer number.

EXAMPLE I.

If the ordinate be $y = \frac{azz}{bb + 2bcz^3 + c^2z^6} = \frac{azz}{(b + cz^3)^2} = azz\,(b + cz^3)^{-2} = az^{3-1}(b + cz^3)^{-1-1}$.

We ſhall have $a = a$; $\alpha = b$, $\beta = c$; $p = 3$, $n = 3$, and $q = -1$. Hence $r = \frac{p}{n} = 1$, and $s = r + q = 1 - 1 = 0$; ſo that the curve is quadrable by both forms of the ſeries. Thus,

By ſubſtituting in the aſcending ſeries, we ſhall have $\frac{az^3}{3c}(b + cz^3)^{-1} \times \frac{1}{1} = \frac{az^3}{3b(b + cz^3)}$ for the area on the one ſide of the ordinate.

And by uſing the deſcending ſeries, we obtain $\frac{az^0}{3b}(b + cz^3)^{-1} \times -\frac{1}{1} = -\frac{a}{3c(b + cz^3)}$ for the area on the other ſide of the ordinate.

EX-

EXAMPLE II.

Suppoſe the ordinate be $y = \frac{az^3}{\sqrt{b+cz^2}} = az^3 \times (b+cz^2)^{-\frac{1}{2}} = az^{4-1} \times (b+cz^2)^{\frac{1}{2}-1}$.

Then $a = a$; $\alpha = b, \beta = c$; $p = 4, n = 2$, and $q = \frac{1}{2}$. Hence $r = \frac{p}{n} = 2$, and $s = r + q = 2\frac{1}{2}$. So that it appears that the deſcending ſeries will bring out a terminate, but the aſcending one an interminate, area. Thus,

By ſubſtituting in the aſcending ſeries, we have $\frac{az^4\sqrt{b+cz^2}}{2b} \times (\frac{1}{2} - \frac{5cz^2}{2.3.2b} - \frac{5.7c^2z^4}{2.3.4.2^2b^2} - \frac{5.7.9c^3z^6}{2.3.4.5.2^3b^3} - \&c)$ for the area. Or by writing $d$ for $2b$, it will be $\frac{az^4\sqrt{b+cz^2}}{d} \times (\frac{1}{2} - \frac{5cz^2}{2.3d} - \frac{5.7c^2z^4}{2.3.4d^2} - \frac{5.7.9c^3z^6}{2.3.4.5d^3} - \&c)$.

But by ſubſtituting in the deſcending ſeries, the area is quadrable, and comes out $\frac{az^2\sqrt{b+czz}}{2c} \times (\frac{2}{3} - \frac{4b}{3czz}) = a\sqrt{b+czz} \times \frac{czz-2b}{3cc}$.

EXAMPLE III.

If the ordinate $y$ be $= \sqrt{a(b+x)} = \sqrt{ab+ax} = a^{\frac{1}{2}}x^0(b+x)^{\frac{1}{2}} = a^{\frac{1}{2}}x^{1-1}(b+x)^{\frac{2}{2}-1}$; which expreſſes the common parabola, $a$ being the parameter, and $b+x$ the whole abſciſs, or diſtance of the ordinate from the vertex of the curve.

Here $a = a^{\frac{1}{2}}$; $\alpha = b, \beta = 1, z = x$; $p = 1, n = 1$, and $q = \frac{1}{2}$. Hence $r = \frac{p}{n} = 1$, and $s = r + q = \frac{3}{2}$. So that the curve is quadrable by the deſcending ſeries,

ſeries, but not by the aſcending one. Thus,

By ſubſtituting in the deſcending ſeries, we obtain $a^{\frac{1}{2}}x^{\circ}(b+x)^{\frac{3}{2}} \times \frac{2}{3} = \frac{2}{3}\sqrt{a}(b+x)^{\frac{3}{2}}$ for the area.——And ſince this area vaniſhes only when $b+x$ or the whole abſciſs is equal to nothing, it denotes the area of the whole parabola. But when $x$ is $= 0$, this area becomes $\frac{2}{3}\sqrt{a} \times b^{\frac{3}{2}}$ for the area to the abſciſs $b$; hence $2\sqrt{a} \times \frac{(b+x)^{\frac{3}{2}} - b^{\frac{3}{2}}}{3}$ will be the area of the fruſtum included by the two ordinates anſwering to the two abſciſſes $b$ and $b+x$. And, after the ſame manner, $2\sqrt{a} \times \frac{(b+x)^{\frac{3}{2}} - (b-x)^{\frac{3}{2}}}{3}$ will be found to denote the fruſtum contained by the two ordinates whoſe abſciſſes are $b+x$ and $b-x$.

But, by ſubſtituting in the aſcending ſeries, the area will come out the infinite ſeries

$$\frac{x\sqrt{a}}{b} \times (b+x)^{\frac{1}{2}} \times \left(1 - \frac{5x}{4b} + \frac{5 \cdot 7x^2}{4.6b^2} - \frac{5 \cdot 7 \cdot 9x^3}{4.6.8b^3} + \&c\right).$$

As this ſeries vaniſhes when $x$ is put equal to nothing, it expreſſes the area beginning where $x$ begins, viz. the fruſtum included by the two ordinates whoſe abſciſſes are $b$ and $b+x$; and therefore this ſeries is equal to $2\sqrt{a} \times \frac{(b+x)^{\frac{3}{2}} - b^{\frac{3}{2}}}{3}$, which was found above to expreſs the ſame area. Then, by making this quantity equal to the ſaid ſeries, and reducing, we obtain $\frac{2b}{3x} \times \left[1 - \left(\frac{b}{b+x}\right)^{\frac{3}{2}}\right]$ for the value of the infinite ſeries $1 - \frac{5x}{4b} + \frac{5 \cdot 7x^2}{4.6b^2} - \frac{5 \cdot 7 \cdot 9x^3}{4.6.8b^3} + \&c$. And by ſuppoſing $b$ and $x$ in theſe laſt expreſſions

to

to be equal to each other, there will result $\frac{2\sqrt{2}-1}{3\sqrt{2}} = \frac{4-\sqrt{2}}{6}$ for the sum of the infinite series

$$1 - \frac{5}{4} + \frac{5\cdot7}{4.6} - \frac{5\cdot7\cdot9}{4.6.8} + \frac{5\cdot7\cdot9\cdot11}{4.6.8.10} - \&c;$$

$$\text{or } \frac{1+\sqrt{2}}{3\sqrt{2}} = \frac{2+\sqrt{2}}{6} = \frac{5}{4} - \frac{5\cdot7}{4.6} + \frac{5\cdot7\cdot9}{4.6.8} - \frac{5\cdot7\cdot9\cdot11}{4.6.8.10} + \&c.$$

Now by the binomial theorem the quantity

$$\frac{(b+x)^{\frac{3}{2}}}{b} = b^{\frac{1}{2}} \times \left(1 + \frac{3x}{2b} + \frac{3x^2}{2.4b^2} - \frac{1\cdot3x^3}{2.4.6b^3} + \frac{1\cdot3\cdot5x^4}{2.4.6.8b^4} - \&c\right),$$

therefore the above quantity

$$\frac{(b+x)^{\frac{3}{2}}}{b} \times \left(1 - \frac{5x}{4b} + \frac{5\cdot7x^2}{4.6b^2} - \&c\right), \text{ is}$$

$$= b^{\frac{1}{2}} \times \left(1 + \frac{x}{4b} - \frac{1x^2}{4.6b^2} + \frac{1\cdot3x^3}{4.6.8b^3} - \frac{1\cdot3\cdot5x^4}{4.6.8.10b^4} + \&c\right),$$

and consequently the area becomes

$$a^{\frac{1}{2}}b^{\frac{1}{2}}x \times \left(1 + \frac{x}{4b} - \frac{1x^2}{4.6b^2} + \frac{1\cdot3x^3}{4.6.8b^3} - \frac{1\cdot3\cdot5x^4}{4.6.8.10b^4} + \&c\right),$$

which must be $= \frac{2}{3}a^{\frac{1}{2}} \times \left[(b+x)^{\frac{3}{2}} - b^{\frac{3}{2}}\right]$;

so that, in general,

$$\frac{2}{3} \times \frac{(b+x)^{\frac{3}{2}} - b^{\frac{3}{2}}}{b^{\frac{1}{2}}x} = 1 + \frac{x}{4b} - \frac{1x^2}{4.6b^2} + \frac{1\cdot3x^3}{4.6.8b^3} - \frac{1\cdot3\cdot5x^4}{4.6.8.10b^4} \&c).$$

If, in this equation, we take $x = +b$ and $-b$, we shall have

$$\frac{2}{3}(2\sqrt{2}-1) = 1 + \frac{1}{4} - \frac{1}{4.6} + \frac{1\cdot3}{4.6.8} - \frac{1\cdot3\cdot5}{4.6.8.10} + \&c.$$

$$\text{and } \frac{2}{3} = 1 - \frac{1}{4} - \frac{1}{4.6} - \frac{1\cdot3}{4.6.8} - \frac{1\cdot3\cdot5}{4.6.8.10} + \&c.$$

And by adding and subtracting these two, we have

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{4.6} + \frac{1\cdot3}{4.6.8} + \frac{1\cdot3\cdot5}{4.6.8.10} + \frac{1\cdot3\cdot5\cdot7}{4.6.8.10.12} + \&c,$$

$$\frac{2}{3}\sqrt{2} = 1 - \frac{1}{4.6} - \frac{1\cdot3\cdot5}{4.6.8.10} - \frac{1\cdot3\cdot5\cdot7\cdot9}{4.6.8.10.12.14} - \&c,$$

$$\frac{2}{3}\sqrt{2} - 1 = \frac{1}{4} + \frac{1\cdot3}{4.6.8} + \frac{1\cdot3\cdot5\cdot7}{4.6.8.10.12} + \frac{1\cdot3\cdot5\cdot7\cdot9\cdot11}{4.6.8.10.12.14.16} + \&c.$$

And,

And, further, if $x$ be taken $= nb$, then, in general,

$$\frac{3}{2} \times \frac{(n+1)^{\frac{3}{2}} - 1}{n} = 1 + \frac{n}{4} - \frac{n^2}{4.6} + \frac{1.3n^3}{4.6.8} - \frac{1.3.5n^4}{4.6.8.10} + \&c.$$

COROLLARY IV.

In the case of a simple nomial $y = az^{p-1} \times \alpha^{q-1} = a\alpha^{q-1}z^{p-1}$, supposing $\beta$ in the last corollary to become nothing, the area will be $\frac{a\alpha^{q-1}z^p}{p} = \frac{mz^p}{p}$, putting $m = a\alpha^{q-1}$.

Or if instead of $mz^{p-1}$ be written its value $y$, the area will be $\frac{yz}{p}$.

And the curve in this case is always quadrable; except when $p$ is $= 0$; for then the area $\frac{mz^p}{p}$ becomes $\frac{mz^0}{0} = \frac{m}{0}$, viz. infinitely great.

To this case of areas belong all simple parabolic spaces, and the spaces lying between the curves and asymptotes of all hyperbolas. And all such are, evidently, to the parallelograms of the same base and altitude, as 1 to $p$.

EXAMPLE I.

The general equation to parabolas is $p^m z^n = y^{m+n}$, or $y = p^{\frac{m}{m+n}} z^{\frac{n}{m+n}}$; where $y$ is the ordinate, $z$ the abscis, and $p$ a given quantity.

Here then $m = p^{\frac{m}{m+n}}$, and $p = \frac{n}{m+n} + 1 = \frac{m+2n}{m+n}$; which value of $p$ being of some magnitude, we conclude that all parabolas are quadrable, and the area

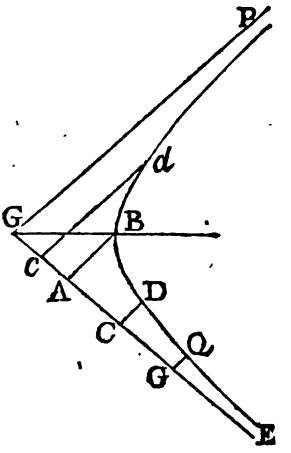
will be, by ſubſtituting in the two forms above, either $\frac{m+n}{m+2n} \times p^{\frac{m}{m+n}} z^{\frac{m+2n}{m+n}}$, or $\frac{m+n}{m+2n} \times yz$.

So that the area of any parabola is to that of its circumſcribing parallelogram, as $m+n$ is to $m+2n$.

In the common parabola, $m$ and $n$ are each equal to 1, or $pz = yy$, and the general area becomes $\frac{2}{3}yz$ or $\frac{2}{3}$ of the circumſcribing parallelogram.

EXAMPLE II.

The general equation to hyperbolas is $z^m y^n = p^{m+n}$, or $y = p^{\frac{m+n}{n}} z^{-\frac{m}{n}}$; where $z$ is one aſymptote GC, and $y$ an ordinate CD drawn parallel to the other GP.




Here then $m = p^{\frac{m+n}{n}}$, and $p = 1 - \frac{m}{n} = \frac{n-m}{n}$; ſo that the ſpace will always be quadrable, except when $n$ and $m$ are equal to each other, as in the caſe of the common or conic hyperbola; and by ſubſtituting in the two forms above, we obtain either $\frac{np^{\frac{n+m}{n}} z^{\frac{n+m}{n}}}{n-m}$ or $\frac{nyz}{n-m}$ for the area GP$d$DC.

So that hyperbolic ſpaces are to their inſcribed parallelograms, as $\frac{n}{n-m}$ to 1, or as $n$ to $n-m$; and ſince when $n$ is $= m$, as in the conic hyperbola, this ratio becomes that of $n$ to 0; it appears that the ſaid hyperbolic ſpace GP$d$DC is in this caſe infinitely great;

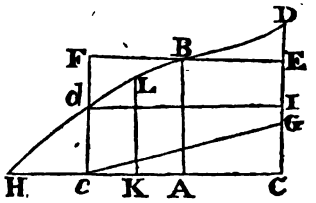
great; but in any other, of a finite magnitude, notwithſtanding it is of an infinite length towards P.

### COROLLARY V.

If in the original ſeries, in the propoſition, only $\beta$, $\gamma$, $\delta$, &c, vaniſh, by becoming equal to nothing, the ſeries for the area will become

$$\alpha^{q-1} z^{p} \times \left(\frac{a}{p} + \frac{bz^{n}}{p+n} + \frac{cz^{2n}}{p+2n} + \frac{cz^{3n}}{p+3n} + \&c\right).$$

And if here again $p$ and $n$ be ſuppoſed each equal to 1, and $\alpha^{q-1}$ be actually drawn into the ſeries, and then the literal coefficients of the ſeveral terms be repreſented by A, B, C, &c, we ſhall obtain $Ax + \frac{1}{2}Bx^{2} + \frac{1}{3}Cx^{3} + \frac{1}{4}Dx^{4} + \&c$, for the area correſponding to the ordinate $y = A + Bx + Cx^{2} + Dx^{3} + \&c$, putting $x$ for $z$.




And by this ſeries we might find a great variety of areas. But to reduce it to another ſeries, converging quicker; if $x$ be $=$ AC, $y =$ CD, as the above ſeries will repreſent the area ABDC of the curve $dBD$; ſo likewiſe if $y$ flow back from A to $cd$, making the ordinate $y = A - Bx + Cx^{2} - Dx^{3} + \&c$, $x$ being now negative, the area AB$dc$ will be $Ax - \frac{1}{2}Bx^{2} + \frac{1}{3}Cx^{3} - \frac{1}{4}Dx^{4} + \&c$.

Then, ſuppoſing the former $x$ or AC, to be equal to this latter $x$ or A$c$, and adding the two ſeries together, there will reſult $2Ax + \frac{2}{3}Cx^{3} + \frac{2}{5}Ex^{5} + \&c$, for the area $cd$DC. Or if now $z$ be put $= cc = 2x$, this area will be equal to $Az + \frac{Cz^{3}}{3\cdot 4} + \frac{Ez^{5}}{5\cdot 4^{2}} + \frac{Gz^{7}}{7\cdot 4^{3}} + \&c$, where $c$ may either be at the vertex of the curve, or not.

When

When $x$ is $= o$, then $y = A = AB$; so that AB is always the value of A, or A denotes the middle ordinate of the area $cd$DC.

If FBE be drawn parallel to $cc$, and meet CD and $cd$ in E and F; the rectangle $c$FEC will be equal to $cc \times AB = Az$, and consequently the difference between $cd$DC and $c$FEC will be $\frac{Cz^3}{3 \cdot 4} + \frac{Ez^5}{3 \cdot 4^2} +$ &c.

EXAMPLE I.

If $d$BD be a right line; its equation will be $y = A + Bx$, B here denoting the ratio of the sine of the angle $d$BA to its cosine. Then, since the value of each of the letters C, D, E, &c, in the general formula, is nothing, the general area $Az + \frac{Cz^3}{3 \cdot 4} +$ &c, will be only $Az =$ the rectangle CF $= AB \times cc = \frac{dc + DC}{2} \times cc$ for the area $d$DC$c$, as in rule 6 prob. 3 sect. 1 part 2.

EXAMPLE II.

If $d$BD be a parabola, its equation will be $y = \sqrt{a} \times (b + x) = \sqrt{ab} \times (1 + \frac{x}{2b} - \frac{x^2}{2.4b^2} + \frac{1.3x^3}{2.4.6b^3} - \frac{1.3.5x^5}{2.4.6.8b^4} +$ &c, by extracting the root.

Here $A = \sqrt{ab}$, $C = -\frac{A}{2.4b^2}$, $E = -\frac{1.3.5A}{2.4.6.8b^4}$, &c, and, by substituting, the area $cd$DC will be $= 2\sqrt{ab} \times (x - \frac{x^3}{4.6b^2} - \frac{1.3.5x^5}{4.6.8.10b^4} - \frac{1.3.5.7.9x^7}{4.6.8.10.12.14b^6} -$ &c),

$= Az \times (1 - \frac{z^2}{4.6.2^2b^2} - \frac{1.3.5x^4}{4.6.8.10.2\ b} -$ &c).

And therefore the rectangle $Az$ or $BA \times cc$, exceeds the parabolic area, by all the terms of the series after the first.

But by example 3 to corollary 3, the ſame area $cd$DC is $= 2\sqrt{a} \times \frac{(b+x)^{\frac{3}{2}} - (b-x)^{\frac{3}{2}}}{3}$; conſeq. $\frac{(b+x)^{\frac{3}{2}} - (b-x)^{\frac{3}{2}}}{3} = \sqrt{b} \times (x - \frac{x^3}{4.6b^2} - \frac{1.3.5x^5}{4.6.8.10b^4} - \frac{1.3.5.7.9x^7}{4.6.8.10.12.14b^6}$ &c). And when $b$ and $x$ are equal to each other in this equation, we obtain $\frac{2}{3}\sqrt{2} = 1 - \frac{1}{4.6} - \frac{1.3.5}{4.6.8.10} - \frac{1.3.5.7.9}{4.6.8.10.12.14} -$ &c; the ſame as was found, by a different method, in ex. 3 cor. 3.

When $c$ is the vertex of the curve, then $b$ is $= x$, and the ſeries for the area $c$DC becomes A$z \times (1 - \frac{1}{4.6} - \frac{1.3.5}{4.6.8.10} -$ &c); which alſo, by ſubſtituting for this laſt ſeries its value $\frac{2}{3}\sqrt{2}$, as found above, will be $= \frac{2}{3}\sqrt{2}$A$z = \frac{2}{3}\sqrt{2} \times$ the rectangle $c$FEC.

### EXAMPLE III.

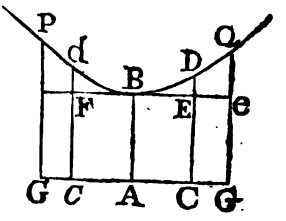
If the curve be a circle, and A were any point between the circumference and the center, the ſeries would be very complex; but if A be the center, and $r$ the radius, then will $y$ be $= \sqrt{rr - xx} = r \times (1 - \frac{x^2}{2r^2} - \frac{x^4}{2.4r^4} - \frac{1.3x^6}{2.4.6r^6} -$ &c, and conſequently the area will be $2rx \times (1 - \frac{x^2}{2.3r^2} - \frac{x^4}{2.4.5r^4} - \frac{1.3x^6}{2.4.6.7r^6} - \frac{1.3.5x^8}{2.4.6.8.9r^8} -$ &c); which ſeries converges very faſt when $x$ is ſmall in reſpect of $r$.

When $x$ becomes equal to $r$, we obtain $4rr \times (1 - \frac{1}{2.3} - \frac{1}{2.4.5} - \frac{1.3}{2.4.6.7} - \frac{1.3.5}{2.4.6.8.9} -$ &c, for the whole circle.

Hence $1 - .785398$ &c, $= \frac{1}{2.3} + \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7}$ + &c, as in pa. 104.

EXAMPLE. IV.

If $dBD$ be an hyperbola; put its ſemi-tranſverſe AB $= t$, ſemi-conjugate AG $= c$, abſciſs AC $= x$, and ordinate CD $= y$; then its equation will be $yy = tt \times \frac{cc + xx}{cc}$, or $y = t \times$

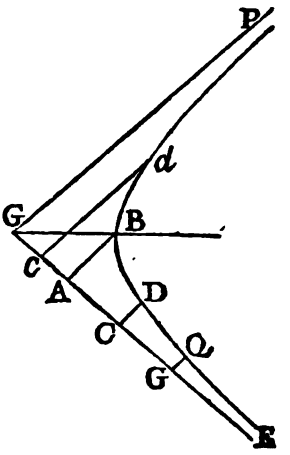



$\sqrt{1 + \frac{xx}{cc}} = t \times (1 + \frac{x^2}{2c^2} - \frac{x^4}{2.4c^4} + \frac{1.3x^6}{2.4.6c^6} - \frac{1.3.5x^8}{2.4.6.8c^8}$ &c); and by ſubſtituting in the general ſeries, the area $cd$DC will be $2tx \times (1 + \frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4} + \frac{1.3x^6}{2.4.6.7c^6} -$ &c), the ſame with the ſeries in the laſt example for the area of a circle, differing only in the ſigns of the ſecond, fourth, ſixth, &c, terms.

When $x$ is $= c$, the ſeries becomes $2tc \times (1 + \frac{1}{2.3} - \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7} -$ &c) for the area GPQG.

The rectangle ABEC is $= tx$, which taken from the area ABDC, will leave AB $\times$ BE $\times$ $(\frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4} + \frac{1.3x^6}{2.4.6.7c^6} -$ &c) for the area BDE. So that the rectangle ABEC is to the trilineal BDE, as 1 is to the ſeries $\frac{x^2}{2.3c^2} - \frac{x^4}{2.4.5c^4} + \frac{1.3x^6}{2.4.6.7c^6} -$ &c. And when $x$ or AC is $= c$ or AG, the rectangle ABEG will be to the trilineal BEQ, as 1 is to $\frac{1}{2.3} - \frac{1}{2.4.5} + \frac{1.3}{2.4.6.7} -$ &c.

Parallel to one aſymptote GP of an hyperbola, draw ordinates BA, CD, to the other GG; B being the vertex of the curve: And put GA $= a$, AB $= b$, AC $= x$, and CD $= y$.




Then $a + x : a :: b : y = \frac{ab}{a+x} = b \times (1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{x^3}{a^3}$ &c.) $=$ the ordinate.

And by comparing this with the general ſeries, we obtain $2bx \times (1 + \frac{x^2}{3a^2} + \frac{x^4}{5a^4} + \frac{x^6}{7a^6}$ + &c), for the area *cd*DC, taking A*c* $=$ AC.

And when $x = a$, or AC $=$ AG, the above expreſſion will become
2GA × AB × (1 + $\frac{1}{3}$ + $\frac{1}{5}$ + $\frac{1}{7}$ + $\frac{1}{9}$ &c) for the area GPBQG.

## PROPOSITION II.

*If there be Any Solid, in which all the Parallel Sections, either Right or Oblique to the Axe, are Like and Similar Figures; and if, when the Abſciſs, or Part of the Axe drawn through the Centers of the Parallel Sections, is repreſented by* z, *the Value of the Section be expreſſed by Any Series of Terms involving* z, *and known quantities, after the ſame manner as the Ordinate is expreſſed in the laſt Propoſition: Then will the Solidity be expreſſed by the ſame Quantity as the Area in the laſt Propoſition: That is, ſuppoſing the relation between the Abſciſs and Section to be expreſſed by Any Equation of which one ſide is the Section, after the manner of the relation between the Abſciſs and Ordinate in the laſt Propoſition; whenever the value of*

*of the Ordinate agrees with that of the Section, then will the value of the Solidity be the ſame with that of the Area as found by the ſaid Propoſition.*

DEMONSTRATION.

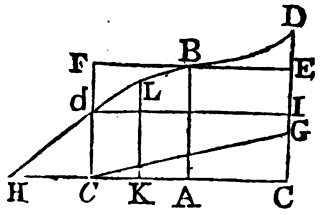
For, the fluxion of the ſolid being equal to the ſection drawn into the fluxion of the abſciſs, and the fluxion of the area equal to the ordinate drawn into the ſame fluxion of the abſciſs, whenever the ordinate and ſection agree, the area and ſolidity muſt likewiſe agree, ſince equal fluxions have equal fluents. *Q. E. D.*

SCHOLIUM.

All the examples that have been given to the corollaries of the laſt propoſition, may be conſidered as examples to this, the quadrature in thoſe being conſidered as the cubature here; and it is evident that curves will be cubable not only whenever they are quadrable, but they will often be cubable when the area cannot be expreſſed except in an infinite ſeries, becauſe the ſection of a ſolid, uſing the area of a given circle as a given number, is often a terminate expreſſion, when the ordinate is denoted by an infinite ſeries; and this is the caſe in the conic ſections; for thoſe curves, the parabola excepted, are not quadrable, yet all the ſolids generated by their revolution have finite expreſſions, as we have already found in the foregoing part of this work, and as will more generally appear in what follows; which I have ſet down not merely to ſhew that the ſame concluſions may be brought out by different means, but chiefly for the ſake of ſome eaſy, curious, and general rules, with which this method of inveſtigation ſo readily ſupplies us.

COROLLARY.

Thus, by applying here what is done in the laſt corollary of the laſt propoſition, to the curves of the ſecond kind, we ſhall obtain general and terminate expreſſions for their ſolidities. For, by prop. 2 ſect. 4 part 3, the flowing ſection being always as $A + Bx + Cxx$, if $p$ denote a given quantity, $p \times (a + Bx + Cxx)$ will denote the ſection itſelf; and conſequently $px \times (A + \frac{1}{2}Bx + \frac{1}{3}Cx^2)$ or $p \times (Az + \frac{1}{1\frac{1}{2}}Cz^3)$ will denote the ſolidity $cdDC$, generated by the ſection of a ſolid, in flowing from the place $cd$ to the place $CD$, $z$ being $= cc$.




Since $pA$ repreſents the middle ſection $AB$, the firſt term $pAz$ in the above expreſſion will be equal to the cylinder $cFBEC$, whoſe baſe is equal to the middle ſection $AB$, and altitude $cc = z$; and conſequently $\frac{1}{1\frac{1}{2}}pcz^3$ will always be the difference between the ſolid $cdBDC$ and the cylinder $cFBEC$.

Or if there be taken $CG : Cc :: \sqrt{C} : 1$, and the three areas, viz. the curve $cdBDC$, the rectangle $cFBEC$, and the right-angled triangle $cGC$, be ſuppoſed to revolve together about the common axe $cC$; then will the difference between the ſolid generated by $cdBDC$, and the cylinder generated by $cFBEC$, be conſtantly equal to one-fourth of the cone generated by $cGC$.

The ſruſt. generated by $cdBDC$ is { greater than / equal to / or leſs than } the cylinder generated by $cFBEC$ according as the value of $c$ is { affirmative / nothing / or negative } that is, according as the curve $dBD$ is { an hyperbola / a parabola / or an ellipſe.

In

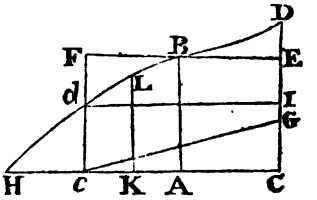
In the parabola c is $= o$, the equation to the curve being $yy = A + Bx$; and therefore the parabolic conoid, or any fruſtum of it, is equal to a cylinder of the ſame height with it, and whoſe end is equal to the middle ſection of the conoid, viz. the ſection parallel to and equally diſtant from its two ends.

In the hyperbola $\sqrt{c}$ is $= \frac{n}{m}$, putting $m$ for the whole axe of which $cc$ is the continuation, and $n$ for its conjugate axe; and therefore the hyperbolic conoid, or its fruſtum exceeds the cylinder by one-fourth of a cone of which the radius of the baſe is to the common altitude $cc$ of all the three ſolids, as $n$ is to $m$.

But in the ellipſe the value of c is $-\frac{nn}{mm}$, $m$ and $n$ being the axes as above; and therefore the ſemi-ſpheroid, or its fruſtum, is leſs than the cylinder by one-fourth of the ſaid cone, of which the radius of the baſe is to the common altitude, as $n$ is to $m$.

When the ellipſe becomes a circle, $m$ will be equal to $n$; and conſequently the ſemi-ſphere, or its fruſtum, will be leſs than the cylinder by one-fourth of the cone, of which the radius of the baſe is equal to the common altitude of all the ſolids.

When $d$BD is a right line, or the ſolid a cone, then the value of $\sqrt{c}$ or $\frac{CG}{Cc}$ will be equal to $\frac{DC}{CH}$, producing the right lines D$d$, $cc$ to meet in H; conſequently the triangles DHC, G$cc$ are ſimilar, and the conic fruſtum exceeds the cylinder by one-fourth of a ſimilar cone; the ſolids being ſtill all of the ſame altitude. The radius CG of the baſe of the ſaid ſimi-



lar cone, will evidently be $= \frac{DC \times cc}{CH}$, a fourth proportional to CH, CD and $cc$: Or, if $di$ be drawn parallel to $cc$, the triangle $di$D will be equal to the triangle $cc$G in all reſpects, or CG $=$ ID $=$ DC $-$ $cd$, that is, the radius CG of the baſe of the ſimilar cone, is equal to the difference between the greateſt CD and leaſt $cd$ radius of the fruſtum.——When the fruſtum becomes a complete cone, it is evident that it will exceed the cylinder by one-fourth of itſelf, or the cone will be to the cylinder as 4 to 3.

From the general manner of conſidering the generation of the fruſtum, in the beginning of this corollary, by the parallel motion of a flowing ſection, it is evident that the above properties will obtain in the ſame ſolids, whether the ends are perpendicular or oblique to the axe; and alſo that the general method will include the fruſtum of any pyramid, whether right or oblique; and ſuch a fruſtum will exceed the priſm, of the ſame altitude and whoſe baſe is equal to the middle ſection of the fruſtum, by one-fourth of a cone of the ſame altitude alſo, and of which the radius of the baſe is $\frac{DC \times cc}{CH}$.

It may farther be obſerved, in general, that in the ſame or in ſimilar ſolids, when the altitude and the inclination of the ends to the axe are the ſame, the cone, or the difference between the fruſtum of the ſolid and the cylinder or priſm, will be conſtantly the ſame quantity, whatever the magnitude of the ends may be. When the altitude is conſtant, and the inclinations of the ends vary, the ſaid difference is reciprocally as the cube of the diameter of the generating plane which is conjugate to $cc$ the axe of the fruſtum, or diameter connecting the centers of its ends. When both the altitude and inclination vary, the difference is as the cube of the altitude directly, and cube of the

ſaid

ſaid conjugate diameter reciprocally; but when they vary ſo as that the altitude is always reciprocally as that diameter, then the difference is a conſtant quantity.

### PROPOSITION III.

*In the Fruſtum of any Solid, generated by the Revolution of Any Conic Section about its Axe, if to the Sum of the two Ends be added four times the Middle Section, or ſection parallel to and equally diſtant from the two Ends, one-ſixth of the Laſt Sum will be a Mean Area, and being drawn into the Altitude of the Solid, will produce the Content.*

### DEMONSTRATION.

By the corollary to the laſt prop. the content is $px \times (A + \frac{1}{2}Bx + \frac{1}{3}Cx^2) = px \times \frac{6A + 3Bx + 2Cx^2}{6}$; but by prop. 2 ſect. 2 part 3, the one end $pD^2$ is $= p \times (A + Bx + Cx^2)$; then by writing $\frac{1}{2}x$ for $x$, the middle ſection will be $p\delta^2 = p \times (A + \frac{1}{2}Bx + \frac{1}{4}Cx^2)$; and by writing o for $x$, the other end will be $p\,d^2 = pA$; now it is evident that the ſum of the two ends with four times the middle ſection, is equal to the numerator of the quantity expreſſing the content; conſequently $px \times \frac{D^2 + 4\delta^2 + d^2}{6} = px \times \frac{6A + 3Bx + 2Cx^2}{6}$ will be the ſolidity of the fruſtum. *Q. E. D.*

### COROLLARY I.

When the fruſtum becomes the complete ſolid, the leſs end vaniſhes, and the content is barely $px \times \frac{D^2 + 4\delta^2}{6}$.

COROL-

COROLLARY II.

This propoſition, it is evident, includes all fruſtums, as well as the complete ſolids, whether right or oblique, not only of the ſolids generated from the revolution of the conic ſections, but alſo of all pyramids, cones, and in ſhort of any ſolid whoſe parallel ſections are ſimilar figures. The ſame theorem may alſo be applied to the areas of all curves, whoſe equation is of this form $y = A + Bx + Cx^2$, calling D and $d$ the extreme ordinates, and $\delta$ the middle one.

And of this form is the equation to the parabola $d$BD (figure to corollary 5 to prop. 1) CD being the axe of the curve, and putting $cA = x$, and $AB = y$; and conſequently any parabolic portion $cd$BA bounded by the curve $d$B, the two lines AB, $cd$ being parallel to the axe CD, and the line A$c$ perpendicular to the ſame, is truly expreſſed by $\frac{AB + 4KL + cd}{6} \times Ac$, KL being the middle ordinate.

## SECTION II.

### OF THE APPROXIMATE QUADRATURE AND CUBATURE OF CURVES BY MEANS OF EQUIDISTANT ORDINATES OR SECTIONS.

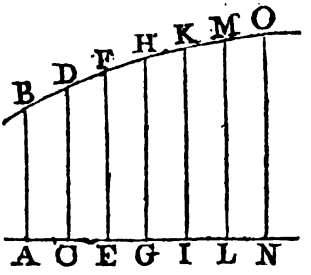
PROPOSITION I.

*IF a Right Line* AN *be divided into any Even Number of Equal Parts* AC, CE, EG, *&c; and at the points of diviſion be erected Perpendicular Ordinates* AB, CD, EF, *&c, terminated by Any Curve* BDF *&c; and if*

*if* A *be put for the ſum of the Extreme or Firſt and Laſt Ordinates* AB, NO; B *for the ſum of the Even Ordinates* CD, GH, LM, *&c, viz. the ſecond, fourth, ſixth, &c; and* C *for the ſum of all the reſt* EF, IK, *&c, viz. the third, fifth, &c, or the Odd Ordinates, wanting the firſt and laſt: Then I ſay that the Common Diſtance* AC *or* CE, *&c, of the Ordinates, being drawn into the ſum ariſing from the addition of* A, 4 *times* B, *and* 2 *times* C, *one-third of the Product will be the Area* ABON *very nearly.*

*That is* $\frac{A + 4B + 2C}{3} \times D =$ *the Area, putting* $D =$ AC *the Common Diſtance of the Ordinates.*

DEMONSTRATION.




If through the firſt three points B, D, F, of the propoſed curve, a parabola be conceived to be drawn, having its axe parallel to the ordinates; the parabolic area, by the ſecond corollary of the laſt propoſition, will be (AB + 4CD + EF) × $\frac{1}{6}$ AE = (AB + 4CD + EF) × $\frac{1}{3}$AC; but when the points B, D, F, are at no great diſtance from each other, the parabolic curve will nearly coincide with the curve propoſed, and conſequently the area of the one will be equal to the area of the other, very nearly; hence (AB + 4CD + EF) × $\frac{1}{3}$AC will be equal to the area ABFE, very nearly. After the ſame manner (EF + 4GH + IK) × $\frac{1}{3}$ AC, or EG, will be equal to the area EFKI; and (IK + 4LM + NO) × $\frac{1}{3}$ AC = the area IKON; and ſo on.

Wherefore, by taking the ſum of theſe areas, we ſhall have (AB + 4CD + 2EF + 4GH + 2IK + 4LM + NO) × $\frac{1}{3}$AC = (A + 4B + 2C) × $\frac{1}{3}$AC for the whole area ABON, very nearly. *Q. E. D.*

COROL-

COROLLARY.

The ſame theorem will alſo obtain for the contents of all ſolids, by uſing the ſections perpendicular to the axe, inſtead of the ordinates, as will appear from the laſt propoſition of the laſt ſection; for the fruſtum of the ſolid there may be ſuppoſed to coincide in the extremes and middle with any other fruſtum, after the ſame manner as the parabola with any other curve. The propoſition is accurately true for all parabolic and right-lined areas, as alſo for all ſolids generated from the revolution of conic ſections or right lines, with all kinds of pyramids; and for all other kinds of areas, and ſolidities, it is a very good approximation.

SCHOLIUM.

It is evident that the greater the number of ordinates or ſections are uſed, the more accurately will the area or ſolidity be determined; but in a caſe of real practice, ſuch as caſk-gauging, three ſections are ſufficient; and becauſe of the ſimplicity and accuracy of this method, it is by far the moſt proper for practical gaugers of any thing that can well be deviſed; for by only taking the bung and head diameters, and a diameter in the middle between them, the ſum of the bung, head, and 4 times the middle circle, drawn into half the length of the caſk, will be ſix times the content, very nearly. And the ſame method may be uſed to good purpoſe in all caſes of ullaging either ſtanding or lying caſks, by taking the area at the top, bottom, and middle of the liquor.

EXAMPLE I.

Let it be required to find the area of the quadrant of a circle whoſe radius is 1.

Let

Let the radius be divided into 10 equal parts by 11 ordinates, the firſt being nothing, and the laſt equal to the radius, and the intermediate 9 ordinates the ſines of the arcs whoſe verſed ſines are reſpectively $\frac{1}{10}$, $\frac{2}{10}$, $\frac{3}{10}$, $\frac{4}{10}$, $\frac{5}{10}$, $\frac{6}{10}$, $\frac{7}{10}$, $\frac{8}{10}$, and $\frac{9}{10}$.

Now, by the property of the circle, we ſhall have

$\sqrt{1\cdot9 \times \cdot1} = \sqrt{\cdot19} = \cdot4358899$ the ſecond ordinate,
$\sqrt{1\cdot8 \times \cdot2} = \sqrt{\cdot36} = \cdot6$ the 3d ordinate,
$\sqrt{1\cdot7 \times \cdot3} = \sqrt{\cdot51} = \cdot7141428$ the 4th,
$\sqrt{1\cdot6 \times \cdot4} = \sqrt{\cdot64} = \cdot8000000$ the 5th,
$\sqrt{1\cdot5 \times \cdot5} = \sqrt{\cdot75} = \cdot8660254$ the 6th,
$\sqrt{1\cdot4 \times \cdot6} = \sqrt{\cdot84} = \cdot9165151$ the 7th,
$\sqrt{1\cdot3 \times \cdot7} = \sqrt{\cdot91} = \cdot9539392$ the 8th,
$\sqrt{1\cdot2 \times \cdot8} = \sqrt{\cdot96} = \cdot9797959$ the 9th,
$\sqrt{1\cdot1 \times \cdot9} = \sqrt{\cdot99} = \cdot9949874$ the tenth,

the 11th, being $= 1$ the radius, as was before obſerved.

Hence $A = 0 + 1 = 1$,
$B =$ the 2d $+$ 4th $+$ 6th $+$ 8th $+$ 10th $= 3\cdot9649847$,
$C =$ the 3d $+$ 5th $+$ 7th $+$ 9th $= 3\cdot296311$,
and $D = \cdot1$.

Conſequently $(A + 4B + 2C) \times \frac{1}{3}D = 23\cdot45256 \times \frac{1}{30} = \cdot78175 =$ the area, pretty near the truth.

This area differs from the truth by more than in curves in general, on account of the obliquity of the curve to the ordinates near the beginning of the arc; but if by the ſame rule we find the arc belonging to the 9 greateſt ordinates, which will be $\cdot70363$, and to it add the area of the ſemi-ſegment whoſe altitude is $\cdot2$ and baſe $\cdot6$, conſidering it as a parabola, viz. $\cdot6 \times \cdot2 \times \frac{2}{3} = \cdot08$, the ſum $\cdot7836$ will be nearer the true area; and if $\cdot08175$ the true area of the ſegment had been added, we ſhould have had $\cdot78538$ the area very exact.

EXAMPLE II.

Taking example 1 to prob. 5 of the parabola, in which the abſciſs of a parabola is 2, and the baſe or double ordinate 12, to find the area of the parabola.

Here, by taking three ordinates, of which the firſt and laſt are each nothing, and the middle one $= 2$, their common diſtance being 6; we ſhall have $A = 0$, $B = 2$, $C = 0$, and $D = 6$.

Conſequently $(A + 4B + 2C) \times \frac{1}{3}D = 8 \times 2 = 16$, the true area as before.

EXAMPLE III.

Given the lengths of five equidiſtant ordinates of an area, or the ſections of a ſolid, 10, 11, 14, 16, 16, and the length of the whole baſe 20; to find the area or the ſolidity.

Here $A = 10 + 16 = 26$, $B = 11 + 16 = 27$, $C = 14$, and $D = \frac{20}{4} = 5$.

Therefore $(A + 4B + 2C) \times \frac{1}{3}D = (26 + 108 + 28) \times \frac{5}{3} = \frac{162}{3} \times 5 = 54 \times 5 = 270$, the area or ſolidity required.

EXAMPLE IV.

Given eleven ordinates to an hyperbola between the aſymptotes, to find the area. Thus, taking the equation at the end of example 4 to corollary 5 to prop. 1 ſect. 1; in which any ordinate $y$ is $= \frac{ab}{a+x} = \frac{1}{1+x}$; then ſuppoſing the firſt value of $x$ to be nothing, and the laſt value equal to 1, and conſequently

quently the common diſtance of the ordinates ·1 or $\frac{1}{10}$; we ſhall have for the ordinates

$\frac{1}{1+0}$, $\frac{1}{1\times\cdot1}$, $\frac{1}{1+\cdot2}$, &c,

or $\frac{10}{10}$, $\frac{10}{11}$, $\frac{10}{12}$, $\frac{10}{13}$, $\frac{10}{14}$, $\frac{10}{15}$, $\frac{10}{16}$, $\frac{10}{17}$, $\frac{10}{18}$, $\frac{10}{19}$, and $\frac{10}{20}$.

Here then $A = \frac{10}{10} + \frac{10}{20} = 1 + \frac{1}{2} = 1\cdot5$,
$B = \frac{10}{11} + \frac{10}{13} + \frac{10}{15} + \frac{10}{17} + \frac{10}{19} = 3\cdot4595393$,
$C = \frac{10}{12} + \frac{10}{14} + \frac{10}{16} + \frac{10}{18} = 2\cdot7281745\frac{5}{6}$,
and $D = \cdot1$

Therefore $(A + 4B + 2C) \times \frac{1}{3}D = \cdot69315021$ is the area required.

## PROPOSITION II.

*Given Any Equidiſtant Ordinates of a Curve, or Sections of a Solid, as a, b, c, d, e, f, &c; to find nearly a General Expreſſion for the Area or Solidity by the Terms a, b, c, &c, and the Baſe upon which they inſiſt.*

## INVESTIGATION.

The firſt differences of the terms $a, b, c, d, e, f$, &c, are

$$b-a,\ c-b,\ d-c,\ e-d,\ f-e,\ \&c,$$

The dif. of theſe are

$$c-2b+a,\ d-2c+b,\ e-2d+c,\ f-2e+d,\ \&c,$$

The dif. of theſe are

$$d-3c+3b-a,\ e-3d+3c-b,\ f-3e-3d-c,\ \&c,$$

The dif. of theſe are

$$e-4d+6c-4b+a,\ f-4e+6d-4c+b,\ \&c,$$

The dif. of theſe are

$$f-5e+10d-10c+5b-a,\ \&c,$$

Then by putting $\alpha, \beta, \gamma, \delta, \varepsilon$, &c, for the firſt term of theſe differences reſpectively, and tranſpoſing, we ſhall have

$b =$

$$\begin{aligned}
b &= +a + \alpha \\
c &= -a + 2b + \beta \\
d &= +a - 3b + 3c + \gamma \\
e &= -a + 4b - 6c + 4d + \delta \\
f &= +a - 5b + 10c - 10d + 5e + \varepsilon \\
&\&c. \qquad \&c.
\end{aligned}$$

Hence, by ſubſtituting the value of $b$ in that of $c$; and the values of $b$ and $c$ in that of $d$; and thoſe of $b$, $c$, and $d$ in that of $e$; and thoſe of $b$, $c$, $d$, and $e$ in that of $f$; and ſo on; we ſhall obtain

$$\begin{aligned}
a &= a \\
b &= a + \alpha \\
c &= a + 2\alpha + \beta \\
d &= a + 3\alpha + 3\beta + \gamma \\
e &= a + 4\alpha + 6\beta + 4\gamma + \delta \\
f &= a + 5\alpha + 10\beta + 10\gamma + 5\delta + \varepsilon \\
&\&c. \qquad \&c.
\end{aligned}$$

Where it is evident that the coefficients in the value of any term, are the ſame with thoſe of a binomial raiſed to the power denoted by the place or number of the term, and conſequently the $x$ term $z$ will be $a + x\alpha + x.\frac{x-1}{3}\beta + x.\frac{x-1}{2}.\frac{x-2}{3}\gamma + x.\frac{x-1}{2}.\frac{x-2}{3}.\frac{x-3}{4}\delta +$ &c. Which, by actually multiplying the factors together, &c, becomes $z = a$

$$\begin{aligned}
&+ \quad \alpha x + \quad \tfrac{1}{2}\beta x^2 + \quad \tfrac{1}{2.3}\gamma x^3 + \&c, \\
&- \quad \tfrac{1}{2}\beta x - \quad \tfrac{1+2}{2.3}\gamma x^2 - \quad \tfrac{1+2+3}{2.3.4}\delta x^3 - \&c, \\
&+ \quad \tfrac{1.2}{2.3}\gamma x + \quad \tfrac{1.2+3.1+2}{2.3.4}\delta x^2 + \tfrac{1.2+3.1+2+4.1+2+3}{2.3.4.5}\varepsilon x^3 + \&c, \\
&- \quad \tfrac{1.2.3}{2.3.4}\delta x - \tfrac{1.2.3+4\times 1.2+3.1+2}{2.3.4.5}\varepsilon x^2 \qquad \&c \\
&+ \quad \tfrac{1.2.3.4}{2.3.4.5}\varepsilon x \qquad \&c \\
&\quad \&c
\end{aligned}$$

$\alpha$

$$\text{or } z = a + \alpha x + \tfrac{1}{2}\beta x^2 + \tfrac{1}{6}\gamma x^3 + \tfrac{1}{24}\delta x^4 + \tfrac{1}{120}\epsilon x^5 \&c$$
$$- \tfrac{1}{2}\beta x - \tfrac{1}{2}\gamma x^2 - \tfrac{1}{4}\delta x^3 - \tfrac{1}{12}\epsilon x^4 \quad \&c$$
$$+ \tfrac{1}{3}\gamma x + \tfrac{11}{24}\delta x^2 + \tfrac{7}{24}\epsilon x^3 \quad \&c$$
$$- \tfrac{1}{4}\delta x - \tfrac{5}{12}\epsilon x^2 \quad \&c$$
$$+ \tfrac{1}{5}\epsilon x \quad \&c$$
$$\&c$$

And by cor. 5 prop. 1 sect. 1, the area will be

$$L \times : a + \tfrac{1}{2}\alpha x + \tfrac{1}{6}\beta x^2 + \tfrac{1}{24}\gamma x^3 + \tfrac{1}{120}\delta x^4 + \tfrac{1}{720}\epsilon x^5 \&c$$
$$- \tfrac{1}{4}\beta x - \tfrac{1}{6}\gamma x^2 - \tfrac{1}{10}\delta x^3 - \tfrac{1}{60}\epsilon x^4 \quad \&c$$
$$+ \tfrac{1}{6}\gamma x + \tfrac{11}{72}\delta x^2 + \tfrac{7}{90}\epsilon x^3 \quad \&c$$
$$- \tfrac{1}{8}\delta x - \tfrac{5}{30}\epsilon x^2 \quad \&c$$
$$+ \tfrac{1}{10}\epsilon x \quad \&c$$
$$\&c$$

Where L represents the length of the base.

Which area will be had accurately true when the differences are continued till one order of them become equal to nothing, for then the series will break off and terminate; that is, an area is quadrable whenever one of the orders of the differences of its equidistant ordinates consists of a series of nothings, or any other equals, or of a series of arithmeticals; but if an order of the differences never become equal to nothing, the area will be expressed by an infinite series.

When the terms or ordinates are taken near to one another, the differences will the sooner become equal to nothing, or nearly so; and if any order of differences, and consequently its succeeding ones, be rejected as inconsiderable, we shall have an approximate value of the area, and that the nearer to the truth as the more of the differences $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$, &c, are used.

Thus, if there be only one ordinate, or if $\alpha$, $\beta$, $\gamma$, &c, be rejected, the area will be $aL$.

If there be two ordinates, or $\beta$, $\gamma$, $\delta$, &c, be rejected, $x$ will be $= 1$, and the area will be

$$(a + \tfrac{1}{2}\alpha) \times L = \frac{a+b}{2} \times L.$$

If there be three ordinates, or $\gamma$, $\delta$, $\varepsilon$, &c, be rejected, $x$ will be $= 2$, and the area will be $(a + \alpha + \frac{2}{3}\beta - \frac{1}{3}\beta) \times L = (a + \alpha + \frac{1}{6}\beta) \times L = (a + 4b + c) \times \frac{1}{6}L$.

If there be four ordinates, or $\delta$, $\varepsilon$, &c, be rejected, $x$ will be $= 3$, and the area will be

$$\left.\begin{array}{r} a + \frac{3}{2}\alpha + \frac{2}{2}\beta + \frac{9}{8}\gamma \\ -\frac{3}{4}\beta - \frac{3}{2}\gamma \\ +\frac{1}{2}\gamma \end{array}\right\} \times L = (a + \tfrac{3}{2}\alpha + \tfrac{3}{4}\beta + \tfrac{1}{8}\gamma) \times L = (a + 3b + 3c + d) \times \tfrac{1}{8}L.$$

And thus by putting $x$ equal to every number ſucceſſively, we ſhall have the following table of areas, anſwering to the reſpective number of ordinates ſet oppoſite to them; of which every expreſſion is more accurate than the preceding ones, and in which A repreſents the ſum of the firſt and laſt terms, B the ſum of the ſecond and laſt but one, C the ſum of the third and laſt but two, &c, and the laſt of the letters denotes the double of the middle term when the number of terms is odd, alſo L denotes the length of the whole baſe, or the diſtance between the firſt and laſt terms.

| N° of ordin. | Areas. |
|---|---|
| 1 | $AL$ |
| 2 | $\frac{A}{2} \times L$ |
| 3 | $\frac{A+2B}{6} \times L$ |
| 4 | $\frac{A+3B}{8} \times L$ |
| 5 | $\frac{7A+32B+6C}{90} \times L$ |
| 6 | $\frac{19A+75B+50C}{288} \times L$ |
| 7 | $\frac{41A+216B+27C+136D}{840} \times L$ |
| 8 | $\frac{751A+3577B+1323C+2989D}{17280} \times L$ |
| 9 | $\frac{989A+5888B-928C+10496D-2270E}{28350} \times L$ |
| &c | &c |

EXAMPLE.

Taking the third example to the laſt propoſition, in which are given the five perpendiculars 10, 11, 14, 16, 16, and the diſtance between the firſt and laſt $= 20$; we ſhall have $A = 10 + 16 = 26$, $B = 11 + 16 = 27$, $C = 14 \times 2 = 28$, and $L = 20$.

Hence by the rule for five ordinates $\frac{7A+32B+6C}{90} \times L = \frac{182+864+168}{90} \times 20 = \frac{1214}{9} \times 2 = 269\frac{7}{9}$ the area as before, nearly.

SCHOLIUM.

When there are many ordinates given, the caſe may be reduced to fewer, by adding together the firſt and laſt ordinates, the ſecond and laſt but one, the third and laſt but two, and ſo on, and conſidering the ſums as a new ſet of terms upon a baſe equal to

half that of the former. Or there may be added together the two firſt and two laſt terms into one ſum, then the four terms next to theſe, and ſo on; or we may add three at the beginning to three at the end, then the next ſix, and ſo on; always diminiſhing the baſe in proportion to the number of terms that are added into each ſum. And the content will be nearly the ſame in each caſe.

### EXAMPLE.

Taking the fourth example to the laſt propoſition, in which are given the eleven ordinates $\frac{10}{10}$, $\frac{10}{11}$, $\frac{10}{12}$, $\frac{10}{13}$, $\frac{10}{14}$, $\frac{10}{15}$, $\frac{10}{16}$, $\frac{10}{17}$, $\frac{10}{18}$, $\frac{10}{19}$, $\frac{10}{20}$, and the diſtance of the firſt and laſt $= 1$; we ſhall have $\frac{10}{10} + \frac{10}{20} = 1{\cdot}5$, $\frac{10}{11} + \frac{10}{19} = 1{\cdot}4354069$, $\frac{10}{12} + \frac{10}{18} = 1{\cdot}3888889$, $\frac{10}{13} + \frac{10}{17} = 1{\cdot}357466$, $\frac{10}{14} + \frac{10}{16} = 1{\cdot}3392857$, and $\frac{10}{15} \times 2 = 1\frac{1}{3}$.

Hence $A = 1{\cdot}5 + 1\frac{1}{3} = \frac{5}{12}$,
$B = 1{\cdot}4354069 + 1{\cdot}3392857 = 2{\cdot}7746926$,
$C = 1{\cdot}3888889 + 1{\cdot}357466 = 2{\cdot}7463549$,
and $L = \frac{1}{2}$.

Then, by the rule for ſix ordinates, we ſhall have $\frac{19A + 75B + 50C}{288} \times L = {\cdot}6931476$, which is much nearer to the truth than the number found by the former rule, the true number being ·6931471 8, and is the hyperbolic logarithm of 2.

SEC.

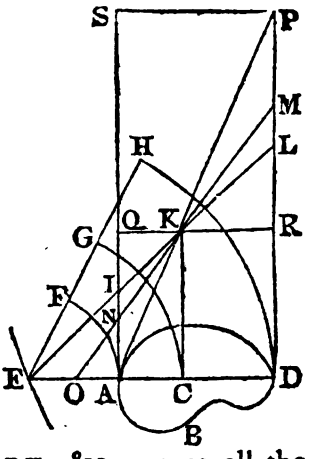
# SECTION III.

## OF THE RELATION BETWEEN THE AREAS AND SOLIDITIES OF FIGURES AND THE CENTERS OF GRAVITY OF THEIR GENERATING LINES AND PLANES.

### PROPOSITION I.

*IF any Line, Right or Curved, or Any Plane Figure, whether it be bounded by Right Lines or Curves, revolve about an Axe in the Plane of the Figure; the Surface or Solid generated will be reſpectively equal to the Surface or Solid whoſe baſe is the Given Line or Figure, and its Height equal to the Arc deſcribed by the Center of Gravity of the ſaid Generating Line or Figure; and conſequently the Content will be found by drawing the Generating Line or Figure into the Arc deſcribed by its Center of Gravity.*

### DEMONSTRATION.

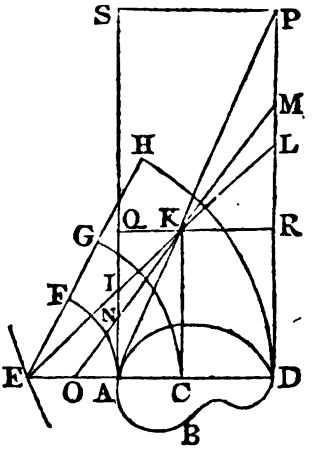



Let AFHD be the figure generated by the given line or plane ABD; through C the center of gravity of which draw DCAE perpendicular to the axe of revolution, and meeting HGFE in E; and let every point of the baſe be reduced to AD by means of perpendiculars to it.

The figure AFHD generated, is equal to all the AF, CG, DH, &c. But, by ſimilar figures, all the AF, CG, DH, &c, are as all the

EA, EC, ED, &c; and, by mechanics, the sum of all the EA, EC, ED, &c, is equal to as many times EC; therefore the sum of all the AF, CG, DH, &c, is equal to as many times CG, or equal to AD × CG; that is, the figure AFHD is equal to ABD × CG, the base drawn into the line described by its center of gravity. *Q. E. D.*

COROLLARY I.




From E draw EIKL cutting the upright prismatic figure erected upon the given base ABD, so as that any perpendicular AI may be equal to its corresponding arc AF. Then will the figure AILD be equal to the figure AFHD.

For, by similar figures, all the AF, CG, DH, &c, are as all the AI, CK, DL, &c, each to each; and as one of each are equal, therefore they are all equal, each to each; viz. all the AI, CK, DL, &c, equal to all the AF, CG, DH, &c; that is, the figure AILD equal to the figure AFHD.

COROLLARY II.

Through K draw MKNO; and the figure ANMD will be equal to the figure AIKLD, or equal to the figure AFHD.

For, by the last corollary, ANMD is equal to the figure described by the base AD revolving about O, till the arc described by C be equal to CK; which, by the proposition, is equal to AD × CK or AD × CG.

COROL-

COROLLARY III.

Hence all the upright figures AQKRD, AIKLD, ANKMD, AKPD, &c, of the ſame baſe, and bounded at the top by lines or planes cutting the upright ſides, and paſſing through the extremity K of the line CK erected upon the center of gravity of the baſe, are equal to one another; and the value of each will be equal to the baſe drawn into the line CK.

Hence alſo all figures, deſcribed by the revolution of the ſame line or plane about different centers or axes, will be equal to one another, when the arcs deſcribed by the center of gravity are equal. But if thoſe arcs be not equal, the figures generated will be as the arcs. And in general, the figures generated, will be to one another, as the revolving lines or planes drawn into the arcs deſcribed by their reſpective centers of gravity.

COROLLARY IV.

Moreover, the oppoſite parts NIK, MLK, of any two of theſe figures, are equal to each other.

COROLLARY V.

The figure ASPD is to the figure APD, as AS to CK; or, by ſimilar triangles, they will be as AD to AC.

For ASPD is equal to AD × AS, and APD is equal to AD × CK.

COROLLARY VI.

If the line or plane be ſuppoſed to be at an infinite diſtance from the center about which it revolves, the figure generated will be an upright ſurface or priſm, the altitude being the line deſcribed by the center of gravity; ſo that the baſe drawn into the ſaid line will

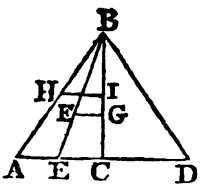
be equal to the baſe drawn into the altitude, as it ought for all upright figures, whoſe ſections parallel to the baſe are all equal to each other.

EXAMPLE I.

If a right line, or a parallelogram, revolve about a line perpendicular to the length, there will be deſcribed a ring either ſuperficial or ſolid; and as the center of gravity of the deſcribing line, or parallelogram, is in the middle of them, the general rule will become the ſame with rule 3 ſect. 1 part 3, and the rule at prob. 2 ſect. 3 part 3.

When the center of revolution is in the end of the line, the line will deſcribe a circle whoſe radius is the ſaid deſcribing line, and whoſe circumference is double the circumference deſcribed by the center of gravity; conſequently the radius drawn into half the circumference, will be the area of the circle.

EXAMPLE II.




If the right-angled triangle ABC revolve about the perpendicular BC, and deſcribe the cone ABD.

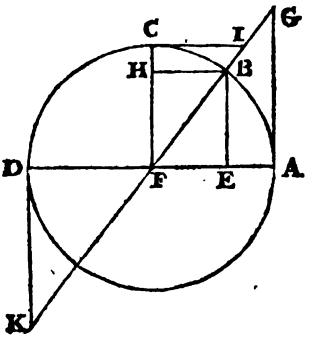
Draw BE to biſect AC, and take EF equal to one-third of BE; and F will be the center of gravity of the triangle ABC, as is well known. Draw FG parallel to AC.——Then the line deſcribed by F will be the circumference of a circle whoſe radius is FG; and, by the general rule, the cone will be equal to the triangle ABC × GF × $8n$, putting $n = .785398$ &c, or $=$ AC × $\frac{1}{2}$CB × $\frac{1}{3}$CE × $8n$ $=$ AC × $\frac{1}{2}$CB × $\frac{1}{3}$CA × $8n$ $=$ AC$^2$ × $\frac{1}{3}$CB × $4n$ $=$ AD$^2$ × $\frac{1}{3}$CB × $n$ $=$ the baſe drawn into one-third of the altitude, as it ought.

Again,

Again, from H, the middle of AB, draw HI parallel to AC; then is H the center of gravity of AB, and consequently the surface described by AB will be AB × circumference whose radius is HI = AB × half the circumference whose radius is AC = the side drawn into half the circumference of the base = the surface of the cone, as it ought.

EXAMPLE III.

Let the semi-circle DCA revolve about the diameter AD, and describe the surface of a sphere.




If there be taken DC : FC :: FC : FH = $\frac{FC^2}{DC} = \frac{4r}{c}$, putting $r$ for the radius, and $c$ for the whole circumference; H will be the center of gravity of the arc DCA, and consequently $r : c ::$ FH : $4r$ = the line or circumference described by H the center of gravity; and, by the general rule, DCA × $4r = \frac{1}{2}c \times 4r = 2rc$ = the surface of the sphere = the circumference into the diameter, as it ought.

And for the solidity of the sphere, we shall have first $\frac{1}{2}c : 2r :: \frac{2}{3}r : \frac{8rr}{3c}$ = the distance FH of the center of gravity of the semi-circle DCAD from the diameter AD, which is two-thirds of the distance of the

the center of gravity of the arc DCA from the ſame diameter DA, in the former caſe; conſequently the line deſcribed by the center of gravity in this caſe will be two-thirds of that in the former; but the deſcribing line in the former caſe is to the deſcribing ſpace in this, as 1 is to $\frac{1}{2}r$, therefore $1 : \frac{2}{3} \times \frac{1}{2}r ::$ ſurface of the ſphere : ſolidity $= \frac{1}{3}r \times$ ſurface.

COROLLARY.

The circumference of the circle, whoſe radius is the diſtance of the center of gravity of the ſemi-circumference of any circle from its center, is equal to four times the radius of that circle.

EXAMPLE IV.

For the ſolidity of the parabolic ſpindle, putting $b =$ the baſe and $a =$ the altitude or axe of the generating parabola, and $n = \cdot 785398$, as before.

It is known that $\frac{2}{3}a$ is the diſtance of the center of gravity from the baſe, and conſequently $\frac{16}{15}a =$ the line deſcribed by the center of gravity; but $\frac{2}{3}ab$ is $=$ the revolving area; therefore $\frac{16}{15}a \times \frac{2}{3}ab = \frac{32}{45}aab$ will be the content, which is $\frac{8}{15}$ of the circumſcribed cylinder.

EXAMPLE V.

For the paraboloid. Making the notation as in the laſt example, $\frac{3}{8}b$ will be the diſtance of the center of gravity of the ſemi-parabola from the axe, conſequently $\frac{3}{8}b \times 8n \times \frac{2}{3}ab = 2abbn =$ the ſolidity $=$ half the circumſcribed cylinder.

PART

# PART V.

## SECTION I.

### OF LAND SURVEYING.

IT is ſuppoſed that land-meaſuring firſt gave riſe to geometry, which has ſince been gradually riſing to the height at which we at preſent view it. Since the diviſion of common grounds has become ſo frequent in England, ſurveying has been univerſally taught and practiſed throughout the nation. I ſhall here give a ſhort account of two or three of the moſt uſeful inſtruments, before we enter upon the meaſurements themſelves.

### CHAPTER I.

### *The Deſcription and Uſe of the moſt uſeful Inſtruments for Surveying.*

#### I. OF THE CHAIN.

Land is meaſured with a chain, called Gunter's chain, of 4 poles or 22 yards in length, which conſiſts of 100 equal links, each link being $\frac{22}{100}$ of a yard, or $\frac{66}{100}$ of a foot, or 7·92 inches long, that is nearly 8 inches or $\frac{1}{3}$ of a foot.

An acre of land, is equal to 10 ſquare chains, that is, 10 chains in length and 1 chain in breadth. Or it is 220 × 22 or 4840 ſquare yards. Or it is 40 × 4 or 160 ſquare poles. Or it is 1000 × 100 or 100000 ſquare links. Theſe being all the ſame quantity.

Alſo, an acre is divided into 4 parts called roods, and

and a rood into 40 parts called perches, which are ſquare poles, or the ſquare of a pole of $5\frac{1}{2}$ yards long, or the ſquare of $\frac{1}{4}$ of a chain, or of 25 links, which is 625 ſquare links. So that the diviſions of land meaſure will be thus:

| | | |
|---|---|---|
| 625 ſq. links | = | 1 pole or perch |
| 40 perches | = | 1 rood |
| 4 roods | = | 1 acre |

The length of lines, meaſured with a chain, are ſet down in links as integers, every chain in length being 100 links; and not in chains and decimals. Therefore after the content is found, it will be in ſquare links; then cut off five of the figures on the right-hand for decimals, and the reſt will be acres. Thoſe decimals are then multiplied by 4 for roods, and the decimals of theſe again by 40 for perches.

EXAMPLES.

Suppoſe the length of a rectangular piece of ground be 792 links, and its breadth 385; to find the area in acres, roods, and perches.

```
     792
     385
    ----
    3960
   6336
  2376
  -------          ac  ro  p
  3·04920  Anſ.    3   0   7
        4
  -------
   ·19680
       40
  -------
  7·87200
  -------
```

2. OF

## 2. OF THE PLAIN TABLE.

This inſtrument conſiſts of a plain rectangular board of any convenient ſize, the center of which, when uſed, is fixed by means of ſcrews to a three-legged ſtand, having a ball and ſocket, or joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction.

To the table belongs,

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a ſheet of paper upon the table. The one ſide of this frame is uſually divided into equal parts, for drawing lines acroſs the table, parallel or perpendicular to the ſides; and the other ſide of the frame is divided into 360 degrees from a center which is in the middle of the table; by means of which the table is to be uſed as a theodolite, &c.

2. A needle and compaſs ſcrewed into the ſide of the table, to point out the directions, and to be a check upon the ſights.

3. An index, which is a braſs two-foot ſcale, with either a ſmall teleſcope, or open ſights erected perpendicularly upon the ends. Theſe ſights, and one edge of the index are in the ſame plane, and that edge is called the fiducial edge of the index.

Before you uſe this inſtrument, take a ſheet of paper which will cover it, and wet it to make it expand; then ſpread it flat upon the table, preſſing down the frame upon the edges, to ſtretch it and keep it fixed there; and when the paper is become dry, it will by contracting again, ſtretch itſelf ſmooth and flat from any cramps or unevenneſs. Upon this paper is to be drawn the plan or form of the thing meaſured.

In uſing this inſtrument, begin at any part of the ground you think the moſt proper, and make a point upon a convenient part of the paper or table, to repreſent that point of the ground; then fix in that point one

one leg of the compaſſes, or a fine ſteel pin, and apply to it the fiducial edge of the index, moving it round till through the ſights you perceive ſome remarkable object, as the corner of a field, &c, and from the ſtation point draw a line with the point of the compaſſes along the fiducial edge of the index; then ſet another object or corner, and draw its line; do the ſame by another, and ſo on till as many objects are ſet as may be thought neceſſary. Then meaſure from your ſtation towards as many of the objects as may be neceſſary, and no more, taking the requiſite offsets to corners or crooks in the hedges, &c, and lay the meaſures down upon their reſpective lines upon the table. Then, at any convenient place, meaſured to, fix the table in the ſame poſition, and ſet the objects which appear from thence, &c, as before; and thus continue till your work is finiſhed, meaſuring ſuch lines as are neceſſary, and determining as many as you can by interſecting lines of direction drawn from different ſtations.

And in theſe operations obſerve the following particular cautions and directions.

1. Let the lines upon which you make ſtations be directed towards objects as far diſtant as poſſible; and when you have ſet any ſuch object, go round the table and look through the ſights from the other end of the index, to ſee if any other remarkable object lie directly oppoſite; if there be not ſuch an one, endeavour to find another forward object, ſuch as ſhall have a remarkable backward oppoſite one, and make uſe of it rather than the other; becauſe the back object will be of uſe in fixing the table in the original poſition either when you have meaſured too near to the forward object, or when it may be hid from your ſight at any neceſſary ſtation by intervening hedges, &c.

2. Let the ſaid lines upon which the ſtations are taken, be purſued as far as you conveniently can; for that

that will be the means of preſerving more accuracy in the work.

3. At each ſtation it will be neceſſary to prove the truth of it; that is, whether the table be ſtraight in the line towards the object, and alſo whether the diſtance be rightly meaſured and laid down on the paper. —To know if the table be ſet down ſtraight in the line; lay the index upon the table in any manner, and move the table about till through the ſights you perceive either the fore or back object; then, without moving the table, go round it and look through the ſights by the other end of the index, to ſee if the other object can be perceived; if it be, the table is in the line; if not, it muſt be ſhifted to one ſide, according to your judgment, till through the ſights both objects can be ſeen.—The aforeſaid operation only informs you if the ſtation be ſtraight in the line; but to know if it be in the right part of the line, that is, if the diſtance has been rightly laid down; fix the table in the original poſition, by laying the index along the ſtation line, and turning the table about till the fore and back objects appear through the ſights, and then alſo will the needle point at the ſame degree as at firſt; then lay the index over the ſtation point and any other point on the paper repreſenting an object which can be ſeen from the ſtation; and if the ſaid object appear ſtraight through the ſights, the ſtation may be depended on as right; if not, the diſtance ſhould be examined and corrected till the object can be ſo ſeen. And for this very uſeful purpoſe, it is adviſable to have ſome high object or two, which can be ſeen from the moſt part of the ground, accurately laid down on the paper from the beginning of the ſurvey, to ſerve continually as proof objects.

When, from any ſtation, the fore and back objects cannot both be ſeen, the agreement of the needle with one of them may be depended on for placing the table ſtraight

ſtraight on the line, and for fixing it in the original poſition.

### *Of ſhifting the Paper on the Plain Table.*

When one paper is full, and you have occaſion for more; draw a line in any manner through the fartheſt point of the laſt ſtation line, to which the work can be conveniently laid down; then take the ſheet off the table, and fix another on, drawing a line upon it, in a part the moſt convenient for the reſt of the work; then fold or cut the old ſheet by the line drawn on it, apply the edge to the line on the new ſheet, and, as they lie in that poſition, continue the laſt ſtation line upon the new paper, placing upon it the reſt of the meaſure, beginning at where the old ſheet left off. And ſo on from ſheet to ſheet.

When the work is done, and you would faſten all the ſheets together into one piece, or rough plan, the aforeſaid lines are to be accurately joined together, as when the lines were transferred from the old ſheets to the new ones.

But it is to be noted, that if the ſaid joining lines, upon the old and new ſheet, have not the ſame inclination to the ſide of the table, the needle will not point to the original degree when the table is rectified; and if the needle be required to reſpect ſtill the ſame degree of the compaſs, the eaſieſt way of drawing the lines in the ſame poſition, is to draw them both parallel to the ſame ſides of the table, by means of the equal diviſions marked on the other two ſides.

## 3. OF THE THEODOLITE.

The theodolite is a brazen circular ring, divided into 360 degrees, and having an index with ſights, or a teleſcope, placed upon the center, about which the index is moveable; alſo a compaſs fixed to the center, to point out courſes and check the ſights; the whole

being fixed by the center upon a ſtand of a convenient height for uſe.

In uſing this inſtrument, an exact account, or field-book, of all meaſures and things neceſſary to be remarked in the plan, muſt be kept, from which to make out the plan upon your return home from the ground.

Begin at ſuch part of the ground, and meaſure in ſuch directions, as you judge moſt convenient; taking angles or directions to objects, and meaſuring ſuch diſtances as appear neceſſary, under the ſame reſtrictions as in the uſe of the plain table. And it is ſafeſt to fix the theodolite in the original poſition at every ſtation by means of fore and back objects, and the compaſs, exactly as in uſing the plain table; regiſtering the number of degrees cut off by the index when directed to each object; and, at any ſtation, placing the index at the ſame degree as when the direction towards that ſtation was taken from the laſt preceding one, to fix the theodolite there in the original poſition, after the ſame manner as the plain table is fixed in the original poſition, by laying its index along the line of the laſt direction.

The beſt method of laying down the aforeſaid lines of direction, is to deſcribe a pretty large circle, quarter it, and lay upon it the ſeveral numbers of degrees cut off by the index in each direction; then, by means of a parallel ruler, draw, from ſtation to ſtation, lines parallel to lines drawn from the center to the reſpective points in the circumference.

### 4. OF THE CROSS.

The croſs conſiſts of two pair of ſights ſet at right angles to each other, upon a ſtaff having a ſharp point at the bottom to ſtick in the ground.

The croſs is very uſeful to meaſure ſmall and crooked pieces of ground. The method is to meaſure a baſe or chief line, uſually in the longeſt direction of

the piece from corner to corner; and while meaſuring it, finding the places where perpendiculars would fall upon this line from the ſeveral corners and bends in the boundary of the piece, with the croſs, by fixing it, by trials, upon ſuch parts of the line as that through one pair of the ſights both ends of the line may appear, and through the other pair you can perceive the correſponding bends or corners; and then meaſuring the lengths of the ſaid perpendiculars.

### REMARKS.

Beſides the fore-mentioned inſtruments, which are moſt commonly uſed, there are ſome others; as the circumferentor, which reſembles the theodolite in ſhape and uſe; and the ſemi-circle, for taking angles, &c. But of all the inſtruments for meaſuring, the plain table is certainly the beſt; not only becauſe it may be uſed as a theodolite or ſemi-circle, by turning uppermoſt that ſide of the frame which has the 360 degrees upon it; but becauſe it is, in its own proper uſe, by much the eaſieſt, ſafeſt, and moſt accurate for the purpoſe; for by planning every part immediately upon the ſpot, as ſoon as meaſured, there is not only ſaved a great deal of writing in the field-book, but every thing can alſo be planned more eaſily and accurately while it is in view, than it can be afterwards from a field-book, in which many little things muſt be either neglected or miſtaken; and beſides, the opportunities which the plain table afford of correcting your work, or proving if it be right, at every ſtation, are ſuch advantages as can never be balanced by any other method. But although the plain table be the moſt generally uſeful inſtrument, it is not *always* ſo; there being many caſes in which ſometimes one inſtrument is the propereſt, and ſometimes another; nor is that ſurveyor maſter of his buſineſs who cannot in any caſe diſtinguiſh which is the fitteſt inſtrument or method, and

and use it accordingly: nay, sometimes no instrument at all, but barely the chain itself, is the best method, particularly in regular open fields lying together; and even when you are using the plain table, it is often of advantage to measure such large open parts with the chain only, and from those measures lay them down upon the table.

The perambulator is used for measuring roads, and other great distances on level ground, and by the sides of rivers. It has a wheel of $8\frac{1}{4}$ feet, or half a pole, in circumference, upon which the machine turns; and the distance measured is pointed out by an index, which is moved round by clock-work.

Levels, with telescopic or other sights, are used to find the level between place and place, or how much one place is higher or lower than another.

An offset-staff is a very useful and necessary instrument, for measuring the offsets and other short distances. It is 10 links in length, being divided and marked at each of the 10 links.

Ten small arrows, or rods of iron or wood, are used to mark the end of every chain length, in measuring lines. And sometimes pickets, or staves with flags, are set up as marks or objects of direction.

Various scales are also used in protracting and measuring on the plan or paper; such as plane scales, line of chords, protractor, compasses, reducing scales, parallel and perpendicular rulers, &c. Of plane scales, there should be several sizes, as a chain in 1 inch, a chain in $\frac{1}{2}$ of an inch, a chain in $\frac{1}{4}$ an inch, &c. And of these, the best for use, are those that are laid on the very edges of the ivory scale, to prick off distances by, without compasses.

## THE FIELD BOOK.

In surveying with the plain table, a field-book is not used, as every thing is drawn on the table immediately

when it is meaſured. But in ſurveying with the theodolite, or any other inſtrument, ſome ſort of a field-book muſt be uſed, to write down in it a regiſter or account of all that is done and occurs relative to the ſurvey in hand.

This book every one contrives and rules as he thinks fitteſt for himſelf. The following is a ſpecimen of a form very generally uſed. It is ruled into 3 columns: the middle, or principal column, is for the ſtations, angles, bearings, diſtances meaſured, &c; and thoſe on the right and left are for the offsets on the right and left, which are ſet againſt their correſponding diſtances in the middle column; as alſo for ſuch remarks as may occur, and be proper to note in drawing the plan, &c.

Here ⊙ 1 is the firſt ſtation, where the angle or bearing is 105° 25′. On the left, at 73 links in the diſtance or principal line, is an offset of 92; and at 610 an offset of 24 to a croſs hedge. On the right, at 0, or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the firſt line, the o denotes its terminating in the hedge. And ſo on for the other ſtations.

Draw a line under the work, at the end of every ſtation line, to prevent confuſion.

*Form*

*Form of the Field-book.*

| Offsets and Remarks on the left. | Stations, Bearings, and Diſtances. | Offsets and Remarks on the right. |
|---|---|---|
| | ⊙ 1 | |
| | 105° 25′ | |
| | 00 | 25 corner |
| 92 | 73 | |
| | 248 | Brown's hedge |
| croſs a hedge 24 | 610 | 35 |
| | 954 | 00 |
| | ⊙ 2 | |
| | 53° 10′ | |
| | 00 | 00 |
| houſe corner 51 | 25 | 21 |
| | 120 | 29 a tree |
| 34 | 734 | 40 a ſtile |
| | ⊙ 3 | |
| | 67° 20′ | |
| | 61 | 35 |
| a brook 30 | 248 | |
| | 639 | 16 a ſpring |
| foot path 16 | 810 | |
| croſs hedge 18 | 973 | 20 a pond |

But in ſmaller ſurveys and meaſurements, a very good way of ſetting down the work, is, to draw, by the eye on a piece of paper, a figure reſembling that

which is to be meaſured; and ſo write the dimenſions, as they are found, againſt the correſponding parts of the figure. And this method may be practiſed to a conſiderable extent, even in the larger ſurveys.

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## CHAPTER II.

## THE

# PRACTICE OF SURVEYING.

THIS part contains the ſeveral works proper to be done in the field, or the ways of meaſuring by all the inſtruments, and in all ſituations.

### PROBLEM I.

### *To Meaſure a Line or Diſtance.*

To meaſure a line on the ground with the chain: Having provided a chain, with 10 ſmall arrows, or rods, to ſtick one into the ground, as a mark, at the end of every chain; two perſons take hold of the chain, one at each end of it, and all the 10 arrows are taken by one of them, who is to go foremoſt, and is called the leader; the other being called the follower, for diſtinction's ſake.

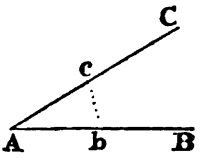
A picket, or ſtation ſtaff, being ſet up in the direction of the line to be meaſured, if there do not appear ſome marks naturally in that direction; the follower ſtands at the beginning of the line, holding the ring at the end of the chain in his hand, while the leader drags forward the chain by the other end of it, till

till it is ſtretched ſtraight, and laid or held level, and the leader directed, by the follower waving his hand, to the right or left, till the follower ſee him exactly in a line with the mark or direction to be meaſured to; there both of them ſtretching the chain ſtraight, and ſtooping and holding it level, the leader having the head of one of his arrows in the ſame hand by which he holds the end of the chain, let him there ſtick one of them down with it while he holds the chain ſtretched. This done, he leaves the arrow in the ground, as a mark for the follower to come to, and advances another chain forward, being directed in his poſition by the follower ſtanding at the arrow, as before; as alſo by himſelf now, and at every ſucceeding chain's length, by moving himſelf from ſide to ſide, till he brings the follower and the back mark into a line. Having then ſtretched the chain, and ſtuck down an arrow, as before, the follower takes up his arrow, and they advance again in the ſame manner another chain-length. And thus they proceed till all the 10 arrows are employed, and are in the hands of the follower; and the leader, without an arrow, is arrived at the end of the 11th chain-length. The follower then ſends or brings the 10 arrows to the leader, who puts one of them down at the end of his chain, and advances with the chain as before. And thus the arrows are changed from the one to the other at every 10 chains length, till the whole line is finiſhed; when the number of changes of the arrows ſhews the number of tens, to which the follower adds the arrows he holds in his hand, and the number of links of another chain over to the mark or end of the line. So if there have been 3 changes of the arrows, and the follower hold 6 arrows, and the end of the line cut off 45 links more, the whole length of the line is ſet down in links thus 3645.

## PROBLEM II.

### *To take Angles and Bearings.*

Let B and C be two objects, or two pickets set up perpendicular; and let it be required to take their bearings, or the angle formed between them at any station A.




#### 1. *With the Plain Table.*

The table being covered with a paper, and fixed on its stand; plant it at the station A, and fix a fine pin, or a point of the compasses, in a proper part of the paper, to represent the point A: Close by the side of this pin lay the fiducial edge of the index, and turn it about, still touching the pin, till one object B can be seen through the sights: then by the fiducial edge of the index draw a line. In the very same manner draw another line in the direction of the other object C. And it is done.

#### 2. *With the Theodolite, &c.*

Direct the fixed sights along one of the lines, as AB, by turning the instrument about till you see the mark B through these sights; and there screw the instrument fast. Then turn the moveable index about till, through its sights, you see the other mark C. Then the degrees cut by the index, upon the graduated limb or ring of the instrument, shews the quantity of the angle.

#### 3. *With the Magnetic Needle and Compass.*

Turn the instrument, or compass, so, that the north end of the needle point to the flower-de-luce. Then direct the sights to one mark, as B, and note the degrees cut

cut by the needle. Then direct the ſights to the other mark c, and note again the degrees cut by the needle. Then their ſum or difference, as the caſe is, will give the quantity of the angle BAC.

### 4. *By Meaſurement with the Chain, &c.*

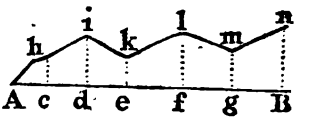
Meaſure one chain length, or any other length, along both directions, as to b and c. Then meaſure the diſtance b c, and it is done.—This is eaſily transferred to paper, by making a triangle A b c with theſe three lengths, and then meaſuring the angle A as in Practical Geometry, prob. XI.

## PROBLEM III.

### *To Meaſure the Offsets.*

A h i k l m n being a crooked hedge, or river, &c. From A meaſure in a ſtraight direction along the ſide of it to B. And in meaſuring along this line AB obſerve when you are directly oppoſite any bends or corners of the hedge, as at c d, e, &c; and from thence meaſure the perpendicular offsets c, h, d i, &c, with the offset-ſtaff, if they are not very large, otherwiſe with the chain itſelf; and the work is done. And the regiſter, or field-book, may be as follows:

| Offs. left. | Baſe line AB. |
|---|---|
| 0 | ⊙ A |
| ch 62 | 45 AC |
| di 84 | 220 Ad |
| ek 70 | 340 Ae |
| fl 98 | 510 Af |
| gm 57 | 634 Ag |
| Bn 91 | 785 AB |




*Note.* When the offsets are not very large, their places

places c, d, e, &c, on the baſe line, can be very well determined by the eye, eſpecially when aſſiſted by laying down the offset-ſtaff in a croſs or perpendicular direction. But when theſe perpendiculars are very large, find their poſitions by the croſs, or by the inſtrument which you happen to be uſing, in this manner: As you meaſure along AB, when you come about c, where you judge a perpendicular will ſtand, plant your inſtrument in the line, and turn the index till the marks A and B can be ſeen through both the ſights, looking both backward and forward; then look along the croſs ſights, or the croſs line on the index; and if it point directly to the corner or bend h, the place of c is right. Otherwiſe, move the inſtrument backward or forward on the line AB, till the croſs line points ſtraight to h. This being found, ſet down the diſtance meaſured from A to c; then meaſure the offset c h, and ſet it down oppoſite the former, and on the left hand ſide.

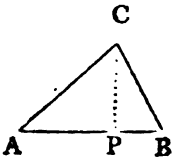
Then proceed forward in the line AB, till you arrive oppoſite another corner, and determine the place d of the perpendicular as before. And ſo on throughout the whole length.

### PROBLEM IV.

### *To Survey a Triangular Field* ABC.

#### 1. *By the Chain.*

AP 794
AB 1321
PC 826




Having ſet up marks at the corners, which is to be done in all caſes where there are not marks naturally; meaſure

measure with the chain from A to P, where a perpendicular would fall from the angle C, and set up a mark at P, noting down the distance AP. Then complete the distance AB by measuring from P to B. Having set down this measure, return to P, and measure the perpendicular PC. And thus, having the base and perpendicular, the area from them is easily found. Or having the place P of the perpendicular, the triangle is easily constructed.

Or, measure all the three sides with the chain, and note them down. From which the content is easily found, or the figure constructed.

### 2. *By taking one or more of the Angles.*

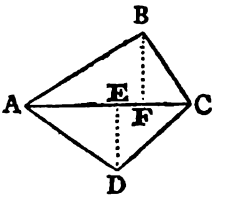
Measure two sides AB, AC, and the angle A between them. Or measure one side AB, and the two adjacent angles A and B. From either of these ways the figure is easily planned; then by measuring the perpendicular CP on the plan, and multiplying it by half AB, you have the content.

## PROBLEM V.

### *To measure a Four-sided Field.*

### 1. *By the Chain.*

| | |
|---|---|
| AE 214 | 210 DE |
| AF 362 | 306 BF |
| AC 592 | |

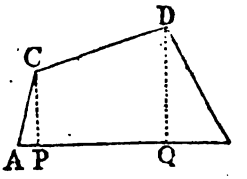



Measure along either of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the last problem; or else the sides AB, BC, CD, DA. From either of which the figure may be planned and computed as before directed.

*Other-*

*Otherwiſe by the Chain.*

| | |
|---|---|
| AP 110 | 352 PC |
| AQ 745 | 595 QD |
| AB 1110 | |




Meaſure on the longeſt ſide, the diſtances AP, AQ, AB; and the perpendiculars PC, QD.

2. *By taking one or more of the Angles.*

Meaſure the diagonal AC (ſee the firſt fig. above), and the angles CAB, CAD, ACB, ACD.—Or meaſure the four ſides, and any one of the angles as BAD.

| Thus | Or thus |
|---|---|
| AC 591 | AB 486 |
| CAB 37°20′ | BC 394 |
| CAD 41 15 | CD 410 |
| ACB 72 25 | DA 462 |
| ACD 54 40 | BAD 78°35′ |

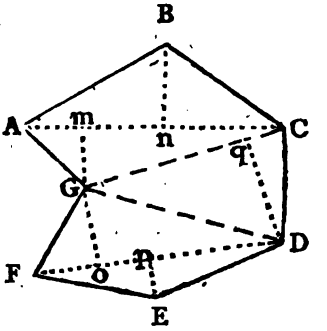
PROBLEM VI.

*To Survey any Field by the Chain only.*

Having ſet up marks at the corners, where neceſſary, of the propoſed field ABCDEFG. Walk over the ground, and conſider how it can beſt be divided into triangles and trapeziums; and meaſure them ſeparately as in the laſt two problems. And in this way it will be proper to divide it into as few ſeparate triangles, and as many trapeziums as may be, by drawing diagonals from corner to corner; and ſo as that all the perpendiculars may fall within the figure. Thus, the following figure is divided into the two trapeziums ABCG, GDEF, and the triangle GCD. Then, in the firſt, beginning at A, meaſure the diagonal AC, and

and the two perpendiculars GM, BN. Then the baſe GC, and the perpendicular DQ. Laſtly the diagonal DF, and the two perpendiculars PE, OG. All which meaſures write againſt the correſponding parts of a rough figure drawn to reſemble the figure to be ſurveyed, or ſet them down in any other form you chooſe.

| | |
|---|---|
| AM 135 | 130 MG |
| AN 410 | 180 NB |
| AC 550 | |
| CQ 152 | 230 QD |
| CG 440 | |
| FO 206 | 120 OG |
| FP 288 | 80 PE |
| FD 520 | |




*Or thus.*

Meaſure all the ſides AB, BC, CD, DE, EF, FG, and GA; and the diagonals AC, CG, GD, DF.

*Otherwiſe.*

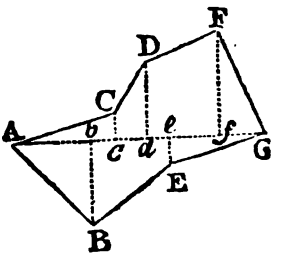
Many pieces of land may be very well ſurveyed, by meaſuring any baſe line, either within or without them, together with the perpendiculars let fall upon it from every corner of them. For they are by thoſe means divided into ſeveral triangles and trapezoids, all whoſe parallel ſides are perpendicular to the baſe line; and the ſum of theſe triangles and trapeziums will be equal to the figure propoſed if the baſe line fall within it; if not, the ſum of the parts which are without being taken from the ſum of the whole which are both within and without, will leave the area of the figure propoſed.

In pieces that are not very large, it will be ſufficiently exact to find the points, in the baſe line, where the

the ſeveral perpendiculars will fall, by means of the *croſs*, and from thence meaſuring to the corners for the lengths of the perpendiculars.—And it will be moſt convenient to draw the line ſo as that all the perpendiculars may fall within the figure.

Thus, in the following figure, beginning at A, and meaſuring along the line AG the diſtances and perpendiculars, on the right and left, are as below.

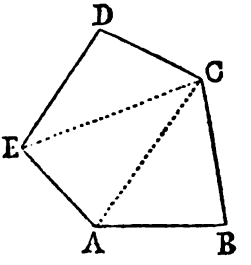
| | | | |
|---|---|---|---|
| Ab | 315 | 350 | bB |
| Ac | 440 | 70 | cC |
| Ad | 585 | 320 | dD |
| Ae | 610 | 50 | eE |
| Af | 990 | 470 | fF |
| AG | 1020 | 0 | |




## PROBLEM VII.

### *To Survey any Field with the Plain Table.*

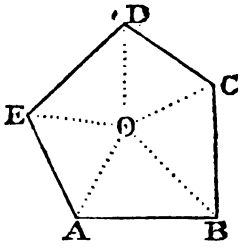
#### 1. *From one Station.*




Plant the table at any angle, as c, from whence all the other angles, or marks ſet up, can be ſeen; and turn the table about till the needle point to the flower-de-luce; and there ſcrew it faſt. Make a point for c on the paper on the table, and lay the edge of the index to c, turning it about c till through the ſights you ſee the mark D; and by the edge of the index draw a dry or obſcure line: then meaſure the diſtance CD, and lay that diſtance down on the line CD. Then turn the index about the point c, till the mark

mark E be ſeen through the ſights, by which draw a line, and meaſure the diſtance to E, laying it on the line from C to E. In like manner determine the poſitions of CA and CB, by turning the ſights ſucceſſively to A and B; and lay the lengths of thoſe lines down. Then connect the points with the boundaries of the field, by drawing the black lines CD, DE, EA, AB, BC.

### 2. *From a Station within the Field.*




When all the other parts cannot be ſeen from one angle, chooſe ſome place O within; or even without, if more convenient, from whence the other parts can be ſeen. Plant the table at O, then fix it with the needle north, and mark the point O on it. Apply the index ſucceſſively to O, turning it round with the ſights to each angle A, B, C, D, E, drawing dry lines to them by the edge of the index, then meaſuring the diſtances OA, OB, &c, and laying them down upon thoſe lines. Laſtly draw the boundaries AB, BC, CD, DE, EA.

### 3. *By going round the Figure.*

When the figure is a wood or water, or from ſome other obſtruction you cannot meaſure lines acroſs it; begin at any point A, and meaſure round it, either within or without the figure, and draw the directions of all the ſides thus: Plant the table at A, turn it with the needle to the north or flower-de-luce, fix it and mark the point A. Apply the index to A, turning it till you can ſee the point E, there draw a line; and then the point B, and there draw a line: then meaſure theſe lines, and lay them down from A to E and B.

Next move the table to B, lay the index along the line AB, and turn the table about till you can ſee the mark A, and ſcrew faſt the table; in which poſition alſo the needle will again point to the flower-de-luce, as it will do indeed at every ſtation when the table is in the right poſition. Here turn the index about B till through the ſights you ſee the mark C; there draw a line, meaſure BC, and lay the diſtance upon that line after you have ſet down the table at C. Turn it then again into its proper poſition, and in like manner find the next line CD. And ſo on quite round by E to A again. Then the proof of the work will be the joining at A: for if the work is all right, the laſt direction EA on the ground, will paſs exactly through the point A on the paper; and the meaſured diſtance will alſo reach exactly to A. If theſe do not coincide, or nearly ſo, ſome error has been committed, and the work muſt be examined over again.

## PROBLEM VIII.

### *To Survey a Field with the Theodolite, &c.*

#### 1. *From one Point or Station.*

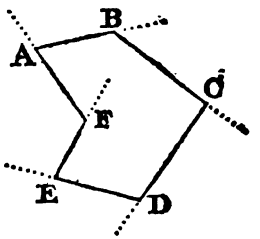
When all the angles can be ſeen from one point, as the angle E (firſt fig. to laſt prob.) place the inſtrument at E, and turn it about till, through the fixed ſights, you ſee the mark B, and there fix it. Then turn the moveable index about till the mark A is ſeen through the ſights, and note the degrees cut on the inſtrument. Next turn the index ſucceſſively to E and D, noting the degrees cut off at each; which gives all the angles BCA, BCE, BCD. Laſtly, meaſure the lines CB, CA, CE, CD; and enter the meaſures in a field-book, or rather againſt the correſponding parts of a rough figure drawn by gueſs to reſemble the field.

2. *From*

### 2. *From a Point within or without.*

Plan the inſtrument at O, (laſt fig.) and turn it about till the fixed ſights point to any object, as A; and there ſcrew it faſt. Then turn the moveable index round till the ſights point ſucceſſively to the other points E, D, C, B, noting the degrees cut off at each of them; which gives all the angles round the point O. Laſtly, meaſure the diſtances OA, OB, OC, OD, OE, noting them down as before, and the work is done.

### 3. *By going round the Field.*




By meaſuring round, either within or without the field, proceed thus. Having ſet up marks at B, C, &c, near the corners as uſual, plant the inſtrument at any point A, and turn it till the fixed index be in the direction AB, and there ſcrew it faſt: then turn the moveable index to the direction AF; and the degrees cut off will be the angle A. Meaſure the line AB, and plant the inſtrument at B, and there in the ſame manner obſerve the angle A. Then meaſure BC, and obſerve the angle C. Then meaſure the diſtance CD, and take the angle D. Then meaſure DE, and take the angle E. Then meaſure EF, and take the angle F. And laſtly meaſure the diſtance FA.

To prove the work; add all the inward angles A, B, C, &c, together, and when the work is right, their ſum will be equal to twice as many right angles as the figure has ſides, wanting 4 right angles. And when there is an angle, as F, that bends inwards, and you meaſure the external angle, which is leſs than two right angles, ſubtract it from 4 right angles, or 360

degrees, to give the internal angle greater than a ſemicircle or 180 degrees.

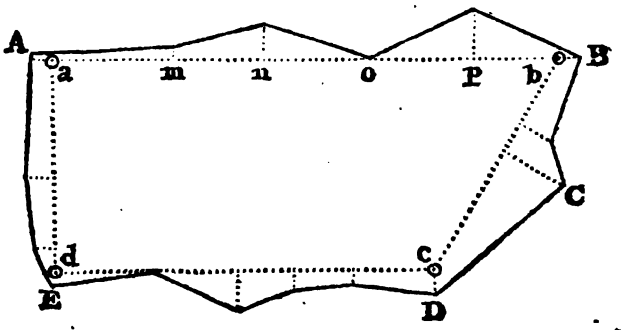
*Otherwiſe.*

Inſtead of obſerving the internal angles, you may take the external angles, formed without the figure by producing the ſides further out. And in this caſe, when the work is right, their ſum altogether will be equal to 360 degrees, But when one of them, as F, runs inwards, ſubtract it from the ſum of the reſt, to leave 360 degrees.

PROBLEM IX.

*To Survey a Field with Crooked Hedges, &c.*

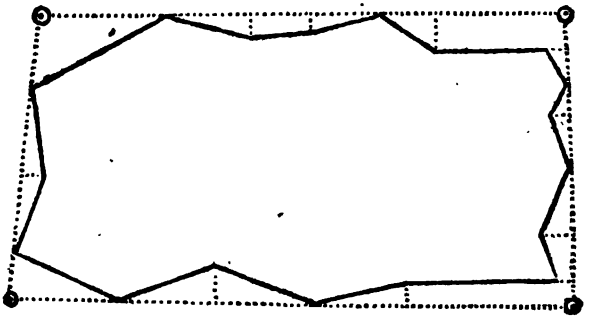
With any of the inſtruments meaſure the lengths and poſitions of imaginary lines running as near the ſides of the field as you can; and in going along them meaſure the offsets in the manner before taught; and you will have the plan on the paper in uſing the plain table, drawing the crooked hedges through the ends of the offsets; but in ſurveying with the theodolite, or other inſtrument, ſet down the meaſures properly in a field-book, or memorandum-book, and plan them after returning from the field, by laying down all the lines and angles.




So

So, in ſurveying the piece ABCDE, ſet up marks a, b, c, d, dividing it into as few ſides as may be. Then begin at any ſtation a, and meaſure the lines ab, bc, cd, da, and take their poſitions, or the angles a, b, c, d; and in going along the lines meaſure all the offsets, as at m, n, o, p, &c, along every ſtation line.

And this is done either within the field, or without, as may be moſt convenient. When there are obſtructions within, as wood, water, hills, &c; then meaſure without, as in the figure here below.



## PROBLEM X.

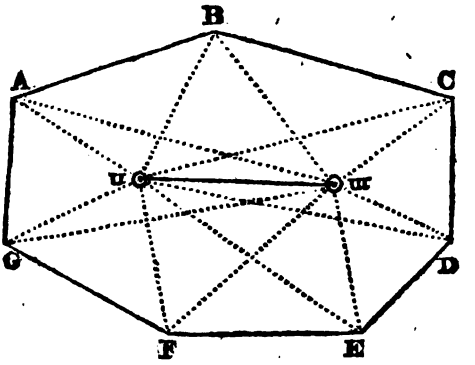
*To Survey a Field or any other Thing, by Two Stations.*

This is performed by chooſing two ſtations, from whence all the marks and objects can be ſeen, then meaſuring the diſtance between the ſtations, and at each ſtation taking the angles formed by every object, from the ſtation line or diſtance.

The two ſtations may be taken either within the bounds, or in one of the ſides, or in the direction of two of the objects, or quite at a diſtance, and without the bounds of the objects, or part to be ſurveyed.

In this manner, not only grounds may be ſurveyed, without even entering them, but a map may be taken of the principal parts of a country, or the chief places

of a town, or any part of a river or coaſt ſurveyed, or any other inacceſſible objects; by taking two ſtations, on two towers, or two hills, or ſuch like.




When the plain table is uſed; plant it at one ſtation m, draw a line m n on it, along which lay the edge of the index, and turn the table about till the ſights point directly to the other ſtation; and there ſcrew it faſt. Then turn the ſights round m ſucceſſively to all the objects ABC, &c, drawing a dry line by the edge of the index at each, as mA, mB, mc, &c. Then meaſure the diſtance to the other ſtation, there plant the table, and lay that diſtance down on the ſtation line from m to n. Next lay the index by the line nm, and turn the table about till the ſights point to the other ſtation m, and there ſcrew it faſt. Then direct the ſights ſucceſſively to all the objects A, B, C, &c, as before, drawing lines each time, as nA, nB, nc, &c; and their interſection with the former lines will give the places of all the objects, or corners, A, B, C, &c.

When the theodolite, or any other inſtrument for taking angles, is uſed; proceed in the ſame way, meaſuring the ſtation diſtance m n, planting the inſtrument firſt at one ſtation, and then at another; then

then placing the fixed ſights in the direction mn, and directing the moveable ſights to every object, noting the degrees cut off at each time. Then, theſe obſervations being planned, the interſections of the lines will give the objects as before.

When all the objects, to be ſurveyed, cannot be ſeen from two ſtations; then three ſtations may be uſed, or four, or as many as is neceſſary; meaſuring always the diſtance from one ſtation to another; placing the inſtrument in the ſame poſition at every ſtation, by means deſcribed before; and from each ſtation obſerving or ſetting every object that can be ſeen from it, by taking its direction or angular poſition, till every object be determined by the interſection of two or more lines of direction, the more the better. And thus may very extenſive ſurveys be taken, as of large commons, rivers, coaſts, countries, hilly grounds, and ſuch like.

## PROBLEM XI.

### *To Survey a Large Eſtate.*

If the eſtate be very large, and contain a great number of fields, it cannot well be done by ſurveying all the fields ſingly, and then putting them together; nor can it be done by taking all the angles and boundaries that incloſe it. For in theſe caſes, any ſmall errors will be ſo multiplied, as to render it very much diſtorted.

1. Walk over the eſtate two or three times, in order to get a perfect idea of it, and till you can carry the map of it tolerably in your head. And to help your memory, draw an eye draught of it on paper, or at leaſt, of the principal parts of it, to guide you.

2. Chooſe two or more eminent places in the eſtate, for your ſtations, from whence you can ſee all the principal parts of it: and let theſe ſtations be as far

diſtant from one another as poſſible; as the fewer ſtations you have to command the whole, the more exact your work will be: and they will be fitter for your purpoſe, if theſe ſtation lines be in or near the boundaries of the ground, and eſpecially if two lines or more proceed from one ſtation.

3. Take what angles, between the ſtations, you think neceſſary, and meaſure the diſtances from ſtation to ſtation, always in a right line: theſe things muſt be done, till you get as many angles and lines as are ſufficient for determining all your points of ſtation. And in meaſuring any of theſe ſtation diſtances, mark accurately where theſe lines meet with any hedges, ditches, roads, lanes, paths, rivulets, &c, and where any remarkable object is placed, by meaſuring its diſtance from the ſtation line, and where a perpendicular from it cuts that line; and always mind, in any of theſe obſervations, that you be in a right line, which you will know by taking backſight and foreſight, along your ſtation line. And thus as you go along any main ſtation line, take offsets to the ends of all hedges, and to any pond, houſe, mill, bridge, &c, omitting nothing that is remarkable. And all theſe things muſt be noted down; for theſe are your data, by which the places of ſuch objects are to be determined upon your plan. And be ſure to ſet marks up at the interſections of all hedges with the ſtation line, that you may know where to meaſure from, when you come to ſurvey theſe particular fields, which muſt immediately be done, as ſoon as you have meaſured that ſtation line, whilſt they are freſh in memory. By this means all your ſtation lines are to be meaſured, and the ſituation of all places adjoining to them determined, which is the firſt grand point to be obtained. It will be proper for you to lay down your work upon paper every night, when you go home, that you may ſee how you go on,

4. As

4. As to the inner parts of the eſtate, they muſt be determined in like manner, by new ſtation lines: for, after the main ſtations are determined, and every thing adjoining to them, then the eſtate muſt be ſubdivided into two or three parts by new ſtation lines; taking inner ſtations at proper places, where you can have the beſt view. Meaſure theſe ſtation lines as you did the firſt, and all their interſections with hedges, and all offsets to ſuch objects as appear. Then you may proceed to ſurvey the adjoining fields, by taking the angles that the ſides make with the ſtation line, at the interſections, and meaſuring the diſtances to each corner, from the interſections. For every ſtation line will be a baſis to all the future operations; the ſituation of all parts being entirely dependant upon them; and therefore they ſhould be taken as long as poſſible; and it is beſt for them to run along ſome of the hedges or boundaries of one or more fields, or to paſs through ſome of their angles. All things being determined for theſe ſtations, you muſt take more inner ſtations, and continue to divide and ſubdivide till at laſt you come to ſingle fields; repeating the ſame work for the inner ſtations, as for the outer ones, till all be done: and cloſe the work as often as you can, and in as few lines as poſſible. And that you may chooſe ſtations the moſt conveniently, ſo as to cauſe the leaſt labour, let the ſtation lines run as far as you can along ſome hedges, and through as many corners of the fields, and other remarkable points, as you can. And take notice how one field lies by another; that you may not miſplace them in the draught.

5. An eſtate may be ſo ſituated, that the whole cannot be ſurveyed together; becauſe one part of the eſtate cannot be ſeen from another. In this caſe, you may divide it into three or four parts, and ſurvey the parts ſeparately, as if they were lands belonging to different perſons; and at laſt join them together.

6. As it is neceſſary to protract or lay down your work as you proceed in it, you muſt have a ſcale of a due length to do it by. To get ſuch a ſcale, you muſt meaſure the whole length of the eſtate in chains; then you muſt conſider how many inches long the map is to be; and from theſe you will know how many chains you muſt have in an inch; then make your ſcale, or chooſe one already made, accordingly.

7. The trees in every hedge row muſt be placed in their proper ſituation, which is ſoon done by the plain table; but may be done by the eye without an inſtrument; and being thus taken by gueſs, in a rough draught, they will be exact enough, being only to look at; except it be ſuch as are at any remarkable places, as at the ends of hedges, at ſtiles, gates, &c, and theſe muſt be meaſured. But all this need not be done till the draught is finiſhed. And obſerve in all the hedges, what ſide the gutter or ditch is on, and to whom the fences belong.

8. When you have long ſtations, you ought to have a good inſtrument to take angles with; and the plain table may very properly be made uſe of, to take the ſeveral ſmall internal parts, and ſuch as cannot be taken from the main ſtations, as it is a very quick and ready inſtrument.

## PROBLEM XII.

### *To Survey a County, or Large Tract of Land.*

1. Chooſe two, three, or four eminent places for ſtations; ſuch as the tops of high hills or mountains, towers, or church ſteeples, which may be ſeen from one another; and from which moſt of the towns, and other places of note, may alſo be ſeen. And let them be as far diſtant from one another as poſſible. Upon theſe places raiſe beacons, or long poles, with flags of different

different colours flying at them; ſo as to be viſible from all the other ſtations.

2. At all the places, which you would ſet down in the map, plant long poles with flags at them of ſeveral colours, to diſtinguiſh the places from one another; fixing them upon the tops of church ſteeples, or the tops of houſes, or in the centres of leſſer towns.

But you need not have theſe marks at many places at once, as ſuppoſe half a ſcore at a time. For when the angles have been taken, at the two ſtations, to all theſe places, the marks may be moved to new ones; and ſo ſucceſſively to all the places you want. Theſe marks then being ſet up at a convenient number of places, and ſuch as may be ſeen from both ſtations; go to one of theſe ſtations, and with an inſtrument to take angles, ſtanding at that ſtation, take all the angles between the other ſtation, and each of theſe marks, obſerving which is blue, which red, &c, and which hand they lie on; and ſet all down with their colours. Then go to the other ſtation, and take all the angles between the firſt ſtation, and each of the former marks, and ſet them down with the others, each againſt his fellow with the ſame colour. You may, if you can, alſo take the angles at ſome third ſtation, which may ſerve to prove the work, if the three lines interſect in that point, where any mark ſtands. The marks muſt ſtand till the obſervations are finiſhed at both ſtations; and then they muſt be taken down, and ſet up at freſh places. And the ſame operations muſt be performed, at both ſtations, for theſe freſh places; and the like for others. Your inſtrument for taking angles muſt be an exceeding good one, made on purpoſe with teleſcopic ſights; and of three, four, or five feet radius. A circumferentor is reckoned a good inſtrument for this purpoſe.

3. And though it is not abſolutely neceſſary to meaſure any diſtance, becauſe any ſtationary line being

laid down from any ſcale, all the other lines will be proportional to it; yet it is better to meaſure ſome of the lines to aſcertain the diſtances of places in miles; and to know how many geometrical miles there are in any length; and from thence to make a ſcale to meaſure any diſtance in miles. In meaſuring any diſtance, it will not be exact enough to go along the high roads; by reaſon of their turnings and windings, and hardly ever lying in a right line between the ſtations, which muſt cauſe infinite reductions, and create endleſs trouble to make it a right line; for which reaſon it can never be exact. But a better way is to meaſure in a right line with a chain, between ſtation and ſtation, over hills and dales or level fields, and all obſtacles. Only in caſe of water, woods, towns, rocks, banks, &c, where onecannot paſs, ſuch parts of the line muſt be meaſured by the methods of inacceſſible diſtances; and beſides, allowing for aſcents and deſcents, when we meet with them. And a good compaſs that ſhews the bearing of the two ſtations, will always direct you to go ſtraight, when you do not ſee the two ſtations; and in your progreſs, if you can go ſtraight, you may take offsets to any remarkable places, likewiſe note the interſection of your ſtationary line with all roads, rivers, &c.

4. And from all your ſtations, and in your whole progreſs, be very particular in obſerving ſea coaſts, river mouths, towns, caſtles, houſes, churches, windmills, watermills, trees, rocks, ſands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, ſluices, floodgates, locks, &c; and in general all things that are remarkable.

5. After you have done with your firſt and main ſtation lines, which command the whole county; you muſt then take inner ſtations, at ſome places already determined; which will divide the whole into ſeveral partitions: and from theſe ſtations you muſt determine

the places of as many of the remaining towns as you can. And if any remain in that part, you must take more stations, at some places already determined; from which you may determine the rest. And thus we must go through all the parts of the county, taking station after station, till we have determined all we want. And in general the station distances must always pass through such remarkable points as have been determined before, by the former stations.

6. Lastly, the position of the station line you measure, or the point of the compass it lies on, must be determined by astronomical observation. Hang up a thread and plummet in the sun, over some part of the station line, and observe when the shadow runs along that line, and at that moment take the sun's altitude; then having his declination, and the latitude, the azimuth will be found by spherical trigonometry. And the azimuth is the angle the station line makes with the meridian; and therefore a meridian may easily be drawn through the map. Or a meridian may be drawn through it by hanging up two threads in a line with the pole star, when he is just north, which may be known from astronomical tables. Or thus; observe the star Alioth, or that in the rump of the great bear, being that next the square; or else Cassiopeia's hip; I say, observe by a line and plummet when either of these stars and the pole star come into a perpendicular; and at that time they are due north. Therefore two perpendicular lines being fixed at that moment, towards these two stars, will give the position of the meridian.

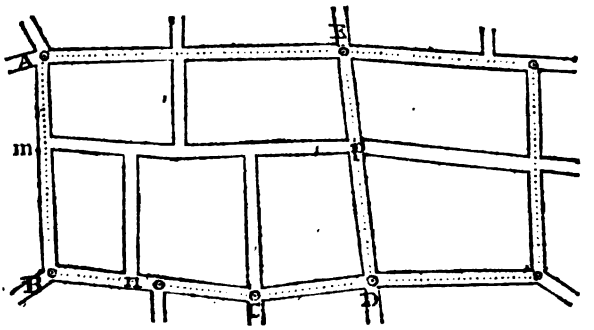
## PROBLEM XIII.

### *To Survey a Town or City.*

This may be done with any of the instruments for taking angles, but best of all with the plain table, where

where every minute part is drawn while in ſight. It is proper alſo to have a chain of 50 feet long, divided into 50 links, and an offset-ſtaff of 10 feet long.

Begin at the meeting of two or more of the principal ſtreets, through which you can have the longeſt proſpects, to get the longeſt ſtation lines. There having fixed the inſtrument, draw lines of direction along thoſe ſtreets, uſing two men as marks, or poles ſet in wooden pedeſtals, or perhaps ſome remarkable places in the houſes at the further ends, as windows, doors, corners, &c. Meaſure theſe lines with the chain, taking offsets with the ſtaff, at all corners of ſtreets, bendings, or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houſes, &c. Then remove the inſtrument to another ſtation along one of theſe lines; and there repeat the ſame proceſs as before. And ſo on till the whole is finiſhed.




Thus, fix the inſtrument at A, and draw lines in the direction of all the ſtreets meeting there; and meaſure A B, noting the ſtreet on the left at m. At the ſecond ſtation B, draw the directions of the ſtreets meeting there; meaſure from B to C, noting the places of the ſtreets at n and o as you paſs by them. At the 3d ſtation C take the direction of all the ſtreets meeting

meeting there, and meaſure CD. At D do the ſame, and meaſure DE, noting the place of the croſs ſtreets at p. And in this manner go through all the principal ſtreets. This done, proceed to the ſmaller and intermediate ſtreets; and laſtly to the lanes, alleys, courts, yards, and every part that it may be thought proper to repreſent.

## CHAPTER III.

### *Of Planning, Caſting-up, and Dividing.*

#### PROBLEM I.

#### *To Lay down the Plan of any Survey.*

IF the ſurvey was taken with a plain table, you have a rough plan of it already on the paper which covered the table. But if the ſurvey was with any other inſtrument, a plan of it is to be drawn from the meaſures that were taken in the ſurvey, and firſt of all a rough plan upon paper.

To do this, you muſt have a ſet of proper inſtruments, for laying down both lines and angles, &c; as ſcales of various ſizes, the more of them, and the more accurate, the better; ſcales of chords, protractors, perpendicular and parallel rulers, &c. Diagonal ſcales are beſt for the lines, becauſe they extend to three figures, or chains and links, which are hundredth parts of chains. But in uſing the diagonal ſcale, a pair of compaſſes muſt be employed to take off the lengths of the principal lines very accurately. But a ſcale with a thin edge divided, is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of thoſe offsets upon the ſtation line; which is done at only one application of the edge of the

the ſcale to that line, and then pricking off all at once the diſtances along it. Angles are to be laid down either with a good ſcale of chords, which is perhaps the moſt accurate way; or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

Very particular directions for laying down all ſorts of figures cannot be neceſſary in this place, to any perſon who has learned practical geometry, and the conſtruction of figures, and the uſe of his inſtruments. It may therefore be ſufficient to obſerve, that all lines and angles muſt be laid down on the plan in the ſame order in which they were meaſured in the field, and in which they are written in the field-book; laying down firſt the angles for the poſition of lines, then the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themſelves, all with dry or obſcure lines; then a black line drawn through the extremities of all the offsets, will be the hedge or bounding line of the field, &c. After the principal bounds and lines are laid down, and made to fit or cloſe properly, proceed next to the ſmaller objects, till you have entered every thing that ought to appear in the plan, as houſes, brooks, trees, hills, gates, ſtiles, roads, lanes, mills, bridges, woodlands, &c, &c.

The north ſide of a map or plan is commonly placed uppermoſt, and a meridian ſomewhere drawn, with the compaſs or flower-de-luce pointing north. Alſo, in a vacant part, a ſcale of equal parts or chains muſt be drawn, and the title of the map in conſpicuous characters, and embelliſhed with a compartment. All hills muſt be ſhadowed, to diſtinguiſh them in the map. Colour the hedges with different colours; repreſent hilly grounds by broken hills and valleys; draw ſingle dotted lines for foot-paths, and double

double ones for horſe or carriage roads. Write the name of each field and remarkable place within it, and, if you chooſe, its content in acres, roods, and perches.

In a very large eſtate, or a county, draw vertical and horizontal lines through the map, denoting the ſpaces between them by letters, placed at the top, and bottom, and ſides, for readily finding any field or other object, mentioned in a table.

In mapping counties, and eſtates that have uneven grounds of hills and valleys, reduce all oblique lines, meaſured up hill and down hill, to horizontal ſtraight lines, if that was not done during the ſurvey, before they were entered in the field-book, by making a proper allowance to ſhorten them. For which purpoſe there is commonly a ſmall table engraven on ſome of the inſtruments for ſurveying.

## PROBLEM. II.

### *To Caſt up the Contents of Fields.*

1. Compute the contents of the figures, whether triangles, or trapeziums, &c, by the proper rules for the ſeveral figures laid down in meaſuring; multiplying the lengths by the breadths, both in links; the product is acres after you have cut off five figures on the right, for decimals; then bring theſe decimals to roods and perches, by multiplying firſt by 4, and then by 40. An example of which is given in the deſcription of the chain, page 508.

2. In ſmall and ſeparate pieces, it is uſual to caſt up their contents from the meaſures of the lines taken in ſurveying them, without making a correct plan of them.

Thus,

Thus, in the triangle in prob. iv, page 522, where we had AP = 794, and AB = 1321
PC = 826

7926
2642
10568

2) 10·91146
5·45573 ac r p
4 Anſ. 32 1 33 nearly

1·82292
40

32·91680

Or the firſt example to prob. v, page 523, thus:

| | |
|---|---|
| AE 214 | 210 ED |
| AF 362 | 306 FB |
| AC 592 | |
| | 516 ſum of perp. |
| | 592 AC |

1032
4644
2580

3·05472 ac r p
4 Anſ. 3 0. 8

·21888
40

8·75520

Or

Or the 2d example to the ſame prob. v, thus:

| | |
|---|---|
| AP 110 | 352 PC |
| AQ 745 | 595 QD |
| AB 1110 | |

PC 352 — AP 110 — 2APC 38720

PC 352 — QD 595 — ſum 947 — PQ 635

4735 — 2841 — 5682 — 2PCDQ 601345

QD 595 — QB 365 — 2975 — 3570 — 1785

217175 = 2QDB

601345 = 2PCDQ

38720 = 2APC

2) 8·57240 = dou. the whole

4·2862

4

1·1448 Anſ. ac r p 4 1 5

40

5·7920

3. In pieces bounded by very crooked and winding hedges, meaſured by offsets, all the parts between the offsets are moſt accurately meaſured ſeparately as ſmall trapezoids. Thus, for the example to prob. III, p. 521, where

| | |
|---|---|
| AC 45 | 62 ch |
| Ad 220 | 84 di |
| Ae 340 | 70 ek |
| Af 510 | 98 fl |
| Ag 634 | 57 gm |
| AB 785 | 91 Bn |

Then

| | | | | | |
|---|---|---|---|---|---|
| AC 45 | ch 62 | di 84 | ek 70 | fl 98 | gm 57 |
| ch 62 | di 84 | ek 70 | fl 98 | gm 57 | BN 91 |
| 90 | 146 | 154 | 168 | 145 | 148 |
| 270 | cd 175 | de 120 | ef 170 | fg 124 | GB 151 |
| 2790 | 730 | 18480 | 11760 | 580 | 148 |
| | 1022 | | 168 | 290 | 740 |
| | 146 | | 28560 | 145 | 148 |
| | 25550 | | | 17980 | 22348 |

2790
25550
18480
28560
17980
22348

2) 1·15708 ac r p
·57854 Content 0 2 12
4

2·31416
40

12·56640

4. Sometimes such pieces as that above, are computed by finding a mean breadth, by dividing the sum of the offsets by the number of them, accounting that for one of them where the boundary meets the station line, as at A; then multiply the length AB by that mean breadth.

Thus:

Thus:

| | | |
|---|---|---|
| 00 | 785 | AB |
| 62 | 66 | mean breadth |
| 84 | 4710 | ac r p |
| 70 | 4710 | Content 0 2 2 by this method, |
| 98 | ·51810 | which is 10 perches too little. |
| 57 | 4 | |
| 91 | 2·07240 | |
| 7) 462 | 40 | And this method is always erroneous, except when the offsets ſtand |
| 66 | 2·89600 | at equal diſtances from one another. |

5. But in larger pieces, and whole eſtates, conſiſting of many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents quite independent of the meaſures of the lines and angles that were taken in ſurveying. For then new lines are drawn in the fields in the plan, ſo as to divide them into trapeziums and triangles, the baſes and perpendiculars of which are meaſured on the plan by means of the ſcale from which it was drawn, and ſo multiplied together for the contents. In this way the work is very expeditiouſly done, and ſufficiently correct; for ſuch dimenſions are taken, as afford the moſt eaſy method of calculation; and, among a number of parts, thus taken and applied to a ſcale, it is likely that ſome of the parts will be taken a ſmall matter too little, and others too great; ſo that they will, upon the whole, in all probability, very nearly balance one another. After all the fields, and particular parts, are thus computed ſeparately, and added all together into one ſum, calculate the whole eſtate independent of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add theſe

alſo together. Then if this ſum be equal to the former, or nearly ſo, the work is right; but if the ſums have any conſiderable difference, it is wrong, and they muſt be examined, and recomputed, till they nearly agree.

A ſpecimen of dividing into one triangle, or one trapezium, which will do for moſt ſingle fields, may be ſeen in the examples to the laſt problem; and a ſpecimen of dividing a large tract into ſeveral ſuch trapeziums and triangles, in prob. VI of chapter II of Surveying, page 524, where a piece is ſo divided, and its dimenſions taken and ſet down; and again at prob. VI of Menſuration of Surfaces, where the contents of the ſame piece are computed.

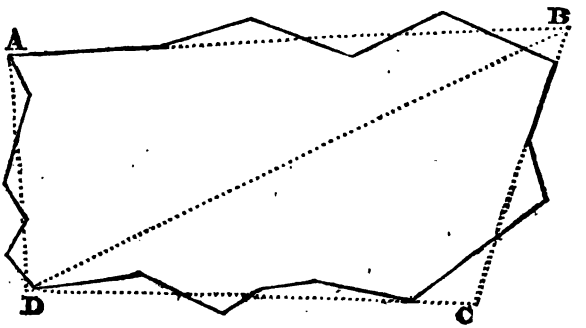
6. But the chief ſecret in caſting up, conſiſts in finding the contents of pieces bounded by curved, or very irregular lines, or in reducing ſuch crooked ſides of fields or boundaries to ſtraight lines, that ſhall incloſe the ſame or equal area with thoſe crooked ſides, and ſo obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now this reducing the crooked ſides to ſtraight ones, is very eaſily and accurately performed thus: Apply the ſtraight edge of a thin, clear piece of lanthorn-horn to the crooked line, which is to be reduced, in ſuch a manner, that the ſmall parts cut off from the crooked figure by it, may be equal to thoſe which are taken in: which equality of the parts included and excluded, you will preſently be able to judge of very nicely by a little practice: then with a pencil draw a line by the ſtraight edge of the horn. Do the ſame by the other ſides of the field or figure. So ſhall you have a ſtraight ſided figure equal to the curved one; the content of which, being computed as before directed, will be the content of the curved figure propoſed.

Or, inſtead of the ſtraight edge of the horn, a horſe-hair

hair may be applied acroſs the crooked ſides in the ſame manner; and the eaſieſt way of uſing the hair, is to ſtring a ſmall ſlender bow with it, either of wire, or cane, or whale-bone, or ſuch like ſlender ſpringy matter; for, the bow keeping it always ſtretched, it can be eaſily and neatly applied with one hand, while the other is at liberty to make two marks by the ſide of it, to draw the ſtraight line by.

EXAMPLE.

Thus, let it be required to find the contents of the ſame figure as in prob. IX of the laſt chapter, page 530, to a ſcale of 4 chains to an inch.




Draw the four dotted ſtraight lines AB, BC, CD, DA, cutting off equal quantities on both ſides of them, which they do as near as the eye can judge: ſo is the crooked figure reduced to an equivalent right-lined one of four ſides ABCD. Then draw the diagonal BD, which by applying a proper ſcale to it, meaſures 1256. Alſo the perpendicular, or neareſt diſtance, from A to this diagonal, meaſures 456; and the diſtance of C from it, is 428.

Then 456
428

884
1256

5024
10048
10048

2) 11·10304
5·55152
4

2·20608
40

8·24320 Content ac ro p
5 2 8

And thus the content of the trapezium, and conſequently of the irregular figure, to which it is equal, is eaſily found to be 5 acres, 2 roods, 8 perches.

## PROBLEM III.

### *To Tranſfer a Plan to another Paper, &c.*

After the rough plan is completed, and a fair one is wanted; this may be done, either on paper or vellum, by any of the following methods.

### FIRST METHOD.

Lay the rough plan upon the clean paper, and keep them always preſſed flat and cloſe together, by weights laid upon them. Then, with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Take them aſunder, and connect the pricked

pricked points on the clean paper, with lines; and it is done. This method is only to be practiſed in plans of ſuch figures as are ſmall and tolerably regular, or bounded by right lines.

### SECOND METHOD.

Rub the back of the rough plan over with black lead powder; and lay the ſaid black part upon the clean paper, upon which the plan is to be copied, and in the proper poſition. Then with the blunt point of ſome hard ſubſtance, as braſs, or ſuch like, trace over the lines of the whole plan; preſſing the tracer ſo much as that the black lead under the lines may be transferred to the clean paper; after which take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink, &c.—Or, inſtead of blacking the rough plan, you may keep conſtantly a blacked paper to lay between the plans.

### THIRD METHOD.

Another way of copying plans, is by means of ſquares. This is performed by dividing both ends and ſides of the plan, which is to be copied, into any convenient number of equal parts, and connecting the correſponding points of diviſion with lines; which will divide the plan into a number of ſmall ſquares. Then divide the paper, upon which the plan is to be copied, into the ſame number of ſquares, each equal to the former when the plan is to be copied of the ſame ſize, but greater or leſs than the others, in the proportion in which the plan is to be increaſed or diminiſhed, when of a different ſize. Laſtly, copy into the clean ſquares, the parts contained in the correſponding ſquares, of the old plan; and you will have the copy either of the ſame ſize, or greater or leſs in any proportion.

### FOURTH METHOD.

A fourth way is by the inſtrument called a pentagraph, which alſo copies the plan in any ſize required.

### FIFTH METHOD.

But the neateſt method of any is this. Procure a copying frame or glaſs, made in this manner; namely, a large ſquare of the beſt window glaſs, ſet in a broad frame of wood, which can be raiſed up to any angle, when the lower ſide of it reſts on a table. Set this frame up to any angle before you, facing a ſtrong light; fix the old plan and clean paper together with ſeveral pins quite around, to keep them together, the clean paper being laid uppermoſt, and upon the face of the plain to be copied. Lay them, with the back of the old plan, upon the glaſs, namely, that part which you intend to begin at to copy firſt; and, by means of the light ſhining through the papers, you will very diſtinctly perceive every line of the plan through the clean paper. In this ſtate then trace all the lines on the paper with a pencil. Having drawn that part which covers the glaſs, ſlide another part over the glaſs, and copy it in the ſame manner. And then another part. And ſo on till the whole be copied.

Then, take them aſunder, and trace all the pencil-lines over with a fine pen and Indian ink, or with common ink.

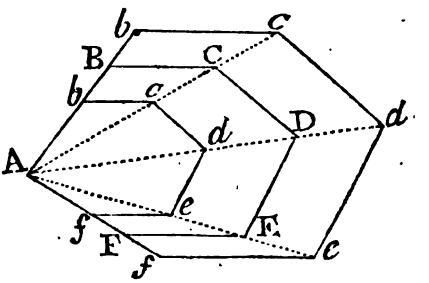
And thus you may copy the fineſt plan, without injuring it in the leaſt.

When the lines, &c, are copied upon the clean paper or vellum, the next buſineſs is to write ſuch names, remarks, or explanations as may be judged neceſſary; laying down the ſcale for taking the lengths of any parts, a flower-de-luce to point out the direction,

tion, and the proper title ornamented with a compartment; and illuſtrating or colouring every part in ſuch manner as ſhall ſeem moſt natural, ſuch as ſhading rivers or brooks with crooked lines, drawing the repreſentations of trees, buſhes, hills, woods, hedges, houſes, gates, roads, &c, in their proper places; running a ſingle dotted line for a foot path, and a double one for a carriage road; and either repreſenting the baſes or the elevations of buildings, &c.

## PROBLEM IV.

### *To change a Figure from One Scale to Another.*




From one angle A draw lines AC, AD, AE, &c, to all the other angles of the figure given; then augment or diminiſh one ſide AB till A*b* be to AB in the given proportion of the ſcales; and by means of a parallel ruler, draw *bc* parallel to BC and meeting AC in *c*, and in the ſame manner *cd* parallel to CD, *de* parallel to DE, *ef* parallel to EF; ſo ſhall A*bcdef*A be the figure required.

## CHAPTER V.

### *Of the Diviſion of Lands.*

In the diviſion of commons, after the whole is ſurveyed and caſt up, and the proper quantities to be allowed for roads, &c, deducted, divide the net quantity remaining among the ſeveral proprietors, by the rule of Fellowſhip, in proportion to the real value

value of their eſtates, and you will thereby obtain their proportional quantities of the land. But as this diviſion ſuppoſes the land, which is to be divided, to be all of an equal goodneſs, you muſt obſerve that if the part in which any one's ſhare is to be marked off, be better or worſe than the general mean quality of the land, then you muſt diminiſh or augment the quantity of his ſhare in the ſame proportion *.

### PROBLEM I.

*It is required to divide any given quantity of ground, or its value, into any given number of parts, and in proportion as any given numbers.*

Divide the given piece, or its value, as in the rule of Fellowſhip, by dividing the whole content or value by the ſum of the numbers expreſſing the proportions of the ſeveral ſhares, and multiplying the quotient ſeverally by the ſaid proportional numbers for the reſpective ſhares required, when the land is all of the ſame quality. But if the ſhares be of different qualities,

* Or, which comes to the ſame thing, divide the ground among the claimants in the direct ratio of the value of their claims, and the inverſe ratio of the quality of the ground allotted to each; that is in proportion to the quotients ariſing from the diviſion of the value of each perſon's eſtate, by the number which expreſſes the quality of the ground in his ſhare.

But theſe regular methods cannot always be put in practice; ſo that, in the diviſion of commons, the uſual way is, to meaſure ſeparately all the land that is of different values, and add into two ſums the contents and the values; then, by the firſt part of the following problem 1, the value of every claimant's ſhare is computed, by dividing the whole value among them in proportion to their eſtates; and, laſtly, by the 2d problem, a quantity is laid out for each perſon, that ſhall be of the value of his ſhare before found.

then divide the numbers expreſſing the proportions or values of the ſhares, by the numbers which expreſs the qualities of the land in each ſhare; and uſe the quotients inſtead of the former proportional numbers.

EXAMPLE I.

If the total value of a common be 2500 pounds, it is required to determine the values of the ſhares of the three claimants A, B, C, whoſe eſtates are of theſe values 10000, and 15000, and 25000 pounds.

The eſtates being in proportion as the numbers 2, 3, 5, whoſe ſum is 10, we ſhall have $2500 \div 10 = 250$; which being ſeverally multiplied by 2, 3, 5, the products 500, 750, 1250, are the values of the ſhares required.

EXAMPLE II.

It is required to divide 300 acres of land among A, B, C, D, E, F, G, and H, whoſe claims upon it are reſpectively in proportion as the numbers 1, 2, 3, 5, 8, 10, 15, 20.

The ſum of theſe proportional numbers is 64, by which dividing 300, the quotient is 4a 2r 30p, which being multiplied by each of the numbers 1, 2, 3, 5, &c, we obtain for the ſeveral ſhares as below:

| | AC. | R. | P. |
|---|---|---|---|
| A = | 4 | 2 | 30 |
| B = | 9 | 1 | 20 |
| C = | 14 | 0 | 10 |
| D = | 23 | 1 | 30 |
| E = | 37 | 2 | 00 |
| F = | 46 | 3 | 20 |
| G = | 70 | 1 | 10 |
| H = | 93 | 3 | 00 |
| Sum = | 300 | 0 | 00 |

EX-

EXAMPLE III.

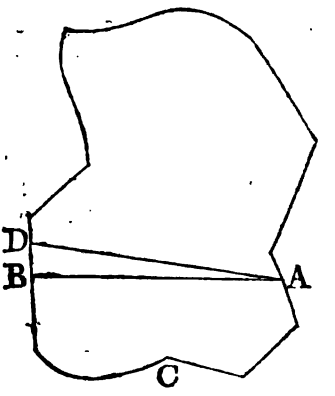
It is required to divide 780 acres among A, B, and C, whoſe eſtates are 1000, 3000, and 4000 pounds a year; the ground in their ſhares being worth 5, 8, and 10 ſhillings the acre reſpectively.

Here their claims are as 1, 3, 4; and the qualities of their land are as 5, 8, 10; therefore their quantities muſt be as $\frac{1}{5}$, $\frac{3}{8}$, $\frac{4}{5}$, or, by reduction, as 8, 15, 16. Now the ſum of theſe numbers is 39; by which dividing the 780 acres, the quotient is 20; which being multiplied ſeverally by the three numbers 8, 15, 16, the three products are 160, 300, 320, for the ſhares of A, B, C, reſpectively.

PROBLEM II.

*To Cut off from a Plan a Given Number of Acres, &c, by a Line drawn from Any Point in the Side of it.*

Let A be the given point in the annexed plan, from which a line is to be drawn cutting off ſuppoſe 5ac. 2r. 14p.




Draw AB cutting off the part ABC as near as can be judged equal to the quantity propoſed; and let the true quantity of ABC, when calculated, be only 4ac. 3r. 20p. which is leſs than 5ac. 2r. 14p. the true quantity, by 0 ac. 2r. 34p. or 71250 ſquare links. Then meaſure AB, which ſuppoſe = 1234 links, and divide 71250, by 617 the half of it, and the quotient 115 links will be the altitude of the triangle to be added, and whoſe baſe is

is AB. Therefore if upon the center B, with the radius 115, an arc be deſcribed; and a line be drawn parallel to AB, touching the arc, and cutting BD in D; and if AD be drawn, it will be the line cutting off the required quantity ADCA.

NOTE. If the firſt piece had been too much, then D muſt have been ſet below B.

In this manner the ſeveral ſhares of commons, to be divided, may be laid down upon the plan, and transferred from thence to the ground itſelf.

Alſo for the greater eaſe and perfection in this buſineſs, the following problems may be added.

### PROBLEM III.

*From an Angle in a Given Triangle, to draw Lines to the Oppoſite Side, dividing the Triangle into any Number of Parts, which ſhall be in any aſſigned Proportion to each other.*

Divide the baſe into the ſame number of parts, and in the ſame proportion, by problem 1; then from the ſeveral points of diviſion draw lines to the propoſed angle, and they will divide the triangle as required *.

### EXAMPLE.

Let the triangle ABC, of 20 acres, be divided into five parts, which ſhall be in proportion to the numbers 1, 2, 3, 5, 9; the lines of diviſion to be drawn from A to CB, whoſe length is 1600 links.

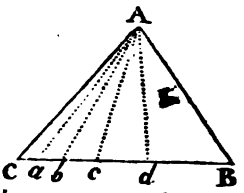
---

* DEMONSTRATION.

For the ſeveral parts are triangles of the ſame altitude, and which therefore are as their baſes, which baſes are taken in the aſſigned proportion.

Here

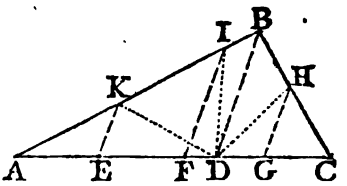
Here 1 + 2 + 3 + 5 + 9 = 20, and 1600 ÷ 20 = 80; which being multiplied by each of the proportional numbers, we have 80, 160, 240, 400, and 720. Therefore I make $ca = 80$, $ab = 160$, $bc = 240$, $cd = 400$, and $dB = 720$; then by drawing the lines A*a*, A*b*, A*c*, A*d*, the triangle is divided as required.




## PROBLEM IV.

*From Any Point in one Side of a Given Triangle, to draw Lines to the other Two Sides, dividing the Triangle into Any Number of Parts which ſhall be in Any Aſſigned Ratio.*

From the given point D, draw DB to the angle oppoſite the ſide AC in which the point is taken; then divide the ſame ſide AC into as many parts AE, EF, FG, GC, and in the ſame proportion with the required parts of the triangle, like as was done in the laſt problem; and from the points of diviſion draw lines EK, FI, GH, parallel to the line BD, and meeting the other ſides of the triangle in K, I, H; laſtly, draw KD, ID, HD, ſo ſhall ADK, KDI, ID, HB, HDC be the parts required*.




The example to this will be done exactly as the laſt.

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* DEMONSTRATION.

The triangles ADK, KDI, IDB, being of the ſame height, are as their baſes AK, KI, IB; which, by means of the parallels EK, FI, DB, are as AE, EF, FD; in like manner, the triangles CDH, HDB are to each other as CG, GD: but the two triangles IDB, BDH, having the ſame baſe BD, are to each other as the diſtances of I and H from BD, or as FD to DG; conſequently the parts DAK, DKI, DIBH, DHC are to each other as AE, EF, FG, GC.

A COL-

# A COLLECTION of QUESTIONS IN SURVEYING.

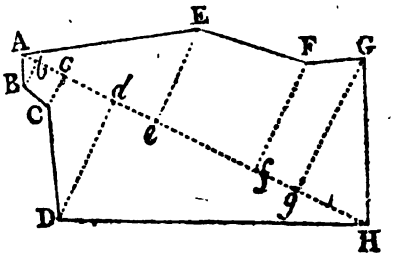
## CHAPTER V.

### QUESTION I.

REQUIRED the area and plot of a figure from the following field-book.

*Note*, That a cipher in the place of a perpendicular, denotes that there the base line touches an angular point. So here the ciphers for the first and last perpendiculars, shew that the base line begins and ends at an angle. Also R denotes right, and L left.

| Field-Book. | |
|---|---|
| Base line | Perpendicular |
| 0 | 0 |
| 19 | 42 R |
| 80 | 62 R |
| 184 | 235 R |
| 307 | 184 L |
| 470 | 220 L |
| 556 | 270 L |
| 697 | 0 |




Anf. 1ac. 3r. 8·908p.

### QUESTION II.

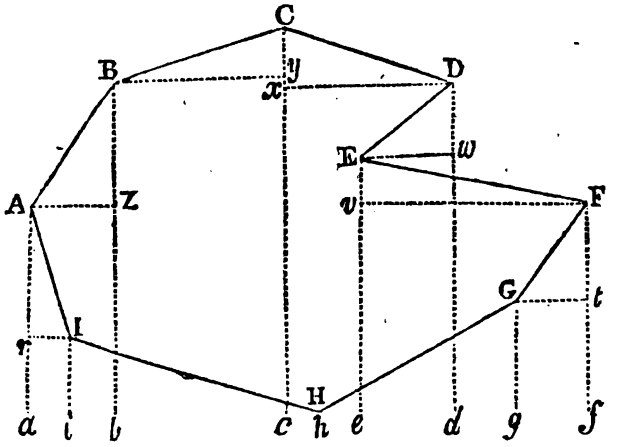
Beginning at the westmost station A, of a large tract of land, and going round towards the north, suppose the lengths of the lines and the angles formed by them and the meridians, to be taken thus: AB = 1550 links, and its

its direction N, $35\frac{1}{2}^{\circ}$ E, that is, $35\frac{1}{2}^{\circ}$ from the north towards the eaſt; then proceeding to the 2d ſtation B, I find the direction of BC to be N E, $72\frac{3}{4}^{\circ}$, and its length = 1870 links. In the ſame manner I find all the other ſides and their directions as expreſſed in order in the 2d and 3d columns of the following table; the 1ſt column containing only the number and mark of each ſtation.

Required the plan and content of this piece.

| Stations | Angles | Sides |
|---|---|---|
| 1. A | NE $35\frac{1}{2}^{\circ}$ | 1550 AB |
| 2. B | NE $72\frac{3}{4}$ | 1870 BC |
| 3. C | SE $70\frac{3}{4}$ | 1870 CD |
| 4. D | SW 53 | 1245 DE |
| 5. E | SE $83\frac{1}{4}$ | 2410 EF |
| 6. F | SW $31\frac{1}{4}$ | 1520 FG |
| 7. G | SW $62\frac{3}{4}$ | 2260 GH |
| 8. H | NW $73\frac{1}{2}$ | 2730 HI |
| 9. I | NW $17\frac{5}{12}$ | 1456 IA |

Anſwer 145ac. 1r. 39·5776p.

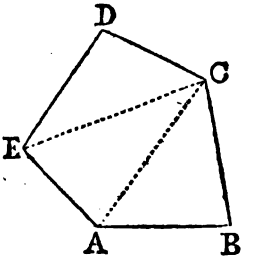



*Note*, A line is wanted to be drawn along the bottom of the figure, through the points *a i b c* H *e d g f*.

QUES-

QUESTION III.

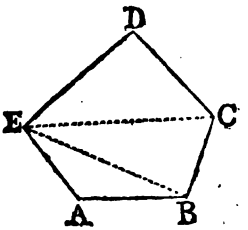
In a pentangular field, beginning with the ſouth ſide, and meaſuring round towards the eaſt firſt, the 1ſt or ſouth ſide was = 2735 links, the 2d = 3115, the 3d = 2370, the 4th = 2925, and the 5th = 2220; alſo the diagonal from the 1ſt angle to the 3d was 3800, and that from the 3d to the 5th was 4010: Required the figure and area.




| | ac | r | p |
|---|---|---|---|
| Anſwer | 117 | 2 | 39·1408 |

QUESTION IV.

In going round a five-ſided field ABCDE, the ſides and angles were thus: The ſide AB = 1940 links, and the angle B 110° 30′; the ſide BC = 1555, and the angle C 117° 45′; the ſide CD = 2125, and the angle D 91° 20′; and the ſide DE = 2741: Required the figure and the content.




| | ac | r | p |
|---|---|---|---|
| Anſwer | 66 | 2 | 24 |

QUESTION V.

In a field I took two ſtations P and Q, at the diſtance of 10 chains from each other:

From P I took the angles

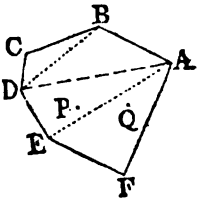
| | | |
|---|---|---|
| QPA | = 21° | 20′ |
| APB | = 49 | 10 |
| BPC | = 57 | 12 |
| CPD | = 29 | 40 |
| DPE | = 64 | 25 |
| EPF | = 79 | 16 |

And from Q the angles

| | | |
|---|---|---|
| PQD | = 10° | 40′ |
| DQC | = 18 | 30 |
| CQB | = 42 | 00 |
| BQA | = 67 | 05 |
| AQF | = 137 | 00 |
| FQE | = 62 | 52 |

Required the figure and area.

| | ac | r | p |
|---|---|---|---|
| Anſwer | 50 | 3 | 39·3136 |




QUESTION VI.

From a ſtation within a field of five ſides I meaſured the diſtances to the ſeveral corners, beginning with that on the weſt, and meaſuring round towards the north, viz. 1ſt diſtance 7345 links, the 2d = 5980, the 3d = 6495, the 4th = 6015, and the 5th = 7050; alſo the angles formed by theſe diſtances in the ſame order were $71\frac{1}{2}$, $55\frac{3}{4}$, $49\frac{1}{4}$, and $81\frac{1}{2}$, degrees. What is the area? Anſ. 979ac. 2r. 35·921p.

QUESTION VII.

From a place O, near the middle of a field ABCDEF, from which I could ſee all the angles, I meaſured the diſtances to the ſeveral corners, and obſerved the quantities of the angles formed at O by thoſe diſtances, as below.

| Distances | Angles |
|---|---|
| OA = 4315 links | AOB = 60° 30′ |
| OB = 2982 | BOC = 47 40 |
| OC = 3561 | COD = 49 50 |
| OD = 5010 | DOE = 57 10 |
| OE = 4618 | EOF = 64 15 |
| OF = 3606 | FOA = 80 35 |

What is the area? Anſ. 412ac. 1r. 17·3224p.

### QUESTION VIII.

Having made choice of a ſtation within a piece of land, I meaſured from thence the ſeveral bearings and diſtances of the corners as below: Required the area.

| | Bearings | Diſtances |
|---|---|---|
| 1ſt | N E | 3540 |
| 2d | N ½ E | 4785 |
| 3d | N w b w | 3915 |
| 4th | s w b s | 4125 |
| 5th | s s E 7° E | 2030 |
| 6th | E b s 3½ E | 2945 |

Anſ. 319ac. 3r. 1·193p.

## SECTION II.

### OF CASK GAUGING.

THE meaning of the word *Gauging* is reſtricted to the meaſuring of caſks, and other things falling under the cognizance of the officers of the

exciſe; and it has received its name from a gauge or rod uſed by the practitioners of the art.

The buſineſs being performed, or the calculations made, commonly by means of the inſtruments, called the gauging or diagonal rod, and the ſliding rule or gauging rule, it will be neceſſary to treat of theſe inſtruments, which I ſhall do as below.

## CHAPTER I.

### *The Deſcription and Uſe of the Sliding Rule.*

This is a ſquare rule, having conſequently four ſides or faces, three of which are furniſhed with ſliding pieces running in grooves. The lines upon them are moſtly logarithmic ones, or diſtances which are proportional to the logarithms of the numbers placed at the ends of them; which kind of lines were placed upon rules, by Mr. *Edmund Gunter*, for expeditiouſly performing arithmetical operations; in which buſineſs he uſed a pair of compaſſes for taking the ſeveral logarithmic diſtances: but inſtead of the compaſſes, ſliding pieces were added, by Mr. *Thomas Everard*, as being more convenient and certain in practice.

Upon the firſt face are three lines, namely, two marked A, B, for multiplying and dividing; and the third, MD, for malt depth, becauſe it ſerves to gauge malt. The middle one B is upon the ſlider, and is a kind of double line, being marked at both the edges of the ſlider, for applying it to both the lines A and MD. Theſe three lines are all of the ſame radius, or diſtance from 1 to 10, each containing twice the length of the radius. A and B are placed and numbered exactly alike, each beginning at 1, which may be either 1, or 10, or 100, &c, or ·1, or ·01, or ·001, &c; but whatever it is, the middle diviſion 10, will be 10 times as much,

much, and the laſt diviſion 100 times as much. But 1 on the line MD is oppoſite 215, or more exactly 2150·4 on the other lines, which number 2150·4 denotes the cubic inches in a malt buſhel, and its diviſions numbered retrograde to thoſe of A and B. Upon theſe two lines are alſo ſeveral other marks and letters: thus, on the line A are MB, for malt buſhel, at the number 2150·4; and A for ale, at 282, the cubic inches in an ale gallon; and upon the line B is W, for wine, at 231, the cubic inches in a wine gallon; alſo *si*, for ſquare inſcribed, at ·707, the ſide of a ſquare inſcribed in a circle whoſe diameter is 1; *se*, for ſquare equal, at ·886, the ſide of a ſquare which is equal to the ſame circle; and C for circumference, at 3·1416, the circumference of the ſame circle.

Upon the ſecond face, or that oppoſite the firſt, are a ſlider and four lines, marked D, C, D, E, at one end, and root, ſquare, root, cube, at the other; the lines C and E containing reſpectively the ſquares and cubes of the oppoſite numbers on the lines D, D; the radius of D being double to that of A, B, C, and triple to that of E: ſo that whatever the firſt 1 on D denotes, the firſt on C is the ſquare of it, and the firſt on E the cube of it,; ſo if D begin with 1, C and E will begin with 1; but if D begin with 10, C will begin with 100, and E with 1000; and ſo on. Upon the line C are marked o*c* at ·0796, for the area of the circle whoſe circumference is 1; and o*d* at ·7854 for the area of the circle whole diameter is 1. Alſo, upon the line D are WG, for wine gauge, at 17·15; and AG for ale gauge, at 18·95; and MR, for malt round, at 52·32; theſe three being the gauge points for round or circular meaſure, and are found by dividing the ſquare roots of 231, 282, and 2150·4 by the ſquare root of ·7854. alſo MS, for malt ſquare, are marked at 46·37, the malt gauge point for ſquare meaſure being the ſquare root of 2150·4.

Upon the third face are three lines, one upon a ſlider marked N; and two on the ſtock, marked ss and SL, for ſegment ſtanding and ſegment lying, which ſerve for ullaging ſtanding and lying caſks.

And upon the fourth, or oppoſite face, are a ſcale of inches, and three other ſcales, marked ſpheroid or 1ſt variety, 2d variety, 3d variety; the ſcale for the 4th, or conic variety, being on the inſide of the ſlider in the third face. The uſe of theſe lines is to find the mean diameters of caſks.

Beſides all thoſe lines, there are two others on the inſides of the two firſt ſliders, being continued from the one ſlider to the other. The one of theſe is a ſcale of inches, from $12\frac{1}{2}$ to 36; and the other is a ſcale of ale gallons between the correſponding numbers ·435 and 3·61; which form a table to ſhew, in ale gallons, the contents of all cylinders whoſe diameters are from $12\frac{1}{2}$ to 36 inches, their common altitude being 1 inch.

As the ſliding rule is for performing, very expeditiouſly, any operations of multiplication, diviſion, and extraction of roots, which may be required by any precept propoſed in words, &c; ſo the manner of making theſe operations will appear in the following problems.

## PROBLEM I.

### *To find the Product of Two Given Numbers, by the Sliding Rule.*

RULE. To either of the given numbers on A ſet 1 on B, then againſt the other number on B is the product on A.

EXAMPLE 1. Required the product of 12 and 25.

By placing 1 on B under 12 on A, above 25 on B ſtands 300 on A; which is the product required.

NOTE. When the 1 on B has been ſet to the one factor

factor on A, if it happen that the other factor on B fall beyond the diviſion, on either A or B, divide it by 10, or 100, &c, till the quotient found on B fall under ſome diviſion on the line A, and multiply this ſaid diviſion by the ſame 10, or 100, &c, for the product required.

EXAMPLE II. So when 250 is to be multiplied by 56: Having ſet 1 on B to 250 on A, although 56 be found on B, it is beyond the end of A; therefore dividing it by 10, I find that oppoſite to the quotient 5·6 on B, is the diviſion 1400 on A; which being multiplied by 10, we obtain 14000 for the product required.

EXAMPLE III. But if 250 were to be multiplied by 1120: Having ſet 1 to 250 as before, 1120 is beyond the end of B, but being divided by 100, oppoſite to the quotient 11·2 on B I find 2800 on A, which being multiplied by 100, we have 280000 for the product required.

## PROBLEM II.

### *To find the Quotient of Two Numbers.*

RULE. Set 1 on B to the diviſor on A, then againſt the dividend on A, is the quotient on B.

EXAMPLE I. To divide 300 by 25. Having ſet 1 on B to 25 on A, oppoſite 300 on A I find 12 on B, the quotient required.

NOTE. When the dividend falls beyond the end of the line A, let it be divided by 10, 100, or ſome other power of 10 till it fall within the line, and uſe the quotient inſtead of it, multiplying the reſult by the ſame power of 10 as before.

EXAMPLE II. So if 14000 muſt be divided by 56. Having ſet 1 to 56, the dividend cannot be found on A till it be divided by 100, the quotient being 140, oppoſite to which I find 2·5 on B, which being

multiplied by 100 we obtain 250 for the quotient required.

### PROBLEM III.

*To work the Rule-of-Three on the Sliding Rule: or having Three Numbers given, to find a Fourth, which ſhall be to the Third as the Second is to the Firſt.*

RULE. Set the firſt term on B, to either the ſecond or third on A; then againſt the remaining term on B, ſtands the fourth term required on A.

EXAMPLE. If 8 yards of cloth coſt 24 ſhillings, what will 96 yards coſt at the ſame rate?

Having ſet 8 on B to 24 on A, oppoſite 96 on B, I find, on A, 288 ſhillings, or 14l 8s, which is the anſwer.

### PROBLEM IV.

*To Extract the Square Root by the Sliding Rule.*

RULE. The firſt 1 on C ſtanding againſt the firſt 1 on D, on the ſtock, oppoſite the given number on C is its root on D.

EXAMPLE. To find the ſide of a ſquare, which ſhall be equal to a triangle, or circle, &c, whoſe area is 225; or, to extract the root of 225.

Here oppoſite 225 on C ſtands 15 on D, which is the anſwer required.

### PROBLEM V.

*To Extract the Cube Root by the Sliding Rule.*

RULE. The line D upon the ſlide being ſet ſtraight with E; find the given number on E, and oppoſite to it will be its cube root on D.

EXAM-

EXAMPLE. To find the ſide of a cube equal to any other ſolid whoſe content is 3375; or to find the cube root of 3375.

Here oppoſite 3375 on E, ſtands 15 on D, which is the anſwer required.

NOTE. It is evident that the ſame lines, as are uſed in theſe two laſt problems, will ſerve to find the ſquare or the cube of any given number, by taking the given number on the contrary lines.

## PROBLEM VI.

### *To find a Mean Proportional between Two Given Numbers.*

RULE. Set one of the given numbers on C to the like or ſame number on D, then againſt the other given number on C, is the number required on D.

EXAMPLE. To find the ſide of a ſquare whoſe area ſhall be equal to that of a parallelogram whoſe length is 9, and its breadth 4 feet; or, to find a mean proportional between 4 and 9.

Having ſet 4 on C to 4 on D, againſt 9 on C ſtands 6 on D, which is the number ſought.

## PROBLEM VII.

### *To find a Number which ſhall be to a Given Number, in a given Duplicate Proportion; or having given Three Numbers, to find a Fourth, which ſhall be to the Third, as the Square of the Second is to the Square of the Firſt.*

RULE. Set the third number on C to the firſt on D, then againſt the ſecond on D, will be found, on C, the fourth required.

EXAM-

EXAMPLE. If the area of a parallelogram, or any other figure, be 120; it is required to find the area of a ſimilar figure, their like dimenſions or ſides being as 2 to 3.

Similar figures being as the ſquares of their like dimenſions, by ſetting 120 on C to 2 on D, againſt 3 on D, ſtands 270 on C, for the number ſought.

### PROBLEM VIII.

*To find a Number which ſhall be to a Given Number, in a given Subduplicate Proportion; or having given Three Numbers, to find a Fourth, which ſhall be to the Third, as the Root of the Second is to the Root of the Firſt.*

RULE. Set the firſt number on C to the third on D, then againſt the ſecond on C, will be found the fourth on D.

EXAMPLE. The ſide of a regular figure is 2, and its area 120; it is required to find the ſide of a ſimilar figure whoſe area is 270.

The roots of the areas of ſimilar figures being as their ſides, we muſt find a number which ſhall be to 2, as the root of 270, is to the root of 120. Therefore, having ſet 120 on C to 2 on D, againſt 270 on C, will be found 3 on D, which is the number ſought.

### PROBLEM IX.

*To find a Number in a given Triplicate Proportion to a Number given; or, having Three Numbers given, to find a Fourth, which ſhall be to the Third, as the Cube of the Second, is to the Cube of the Firſt.*

RULE. Set the firſt number on the ſlide D, to the third

third number on E, then againſt the ſecond on D, is the fourth required on E.

EXAMPLE. If a caſk, whoſe length is 40 inches, contain 100 gallons, what will be the content of a ſimilar caſk whoſe length is 36 inches?

Similar ſolids being as the cubes of their like ſides, the content required muſt be to 100 gallons, as $36^3$ is to $40^3$. Therefore ſetting 40 on D to 100 on E, againſt 36 on D, will be found 72·9 gallons on E, which is the content required.

## PROBLEM X.

*To find a Number in a given Subtriplicate Proportion to a Given Number; or, having Three Numbers given, to find a Fourth, which ſhall be to the Third, as the Cube Root of the Second, is to the Cube Root of the Firſt.*

RULE. Set the third number on D, to the firſt on E, then againſt the ſecond on E, will ſtand the fourth on D.

EXAMPLE. What is the length of a caſk whoſe content is 72·9 gallons, ſuppoſing the length of a ſimilar caſk to be 40 inches, and its content 100 gallons?

Since the dimenſions of ſimilar ſolids are as the cube roots of their contents, we muſt find a number which ſhall be to 40, as the cube root of 72·9 is to the cube root of 100. Therefore, having ſet 40 on D to 100 on E, againſt 72·9 on E, will be found 36 on D, which is the length required.

PRO-

### PROBLEM XI.

*The Length and Breadth of a Parallelogram being given, to find its Area in Malt Buſhels by the Line* MD.

RULE. Set either of the given dimenſions on B, to the other on MD, then againſt 1 on A, is the required area on B.

EXAMPLE. How many malt buſhels can be contained on every inch of the depth of a ciſtern, whoſe length is 180, and breadth 72 inches?

By ſetting 72 on B, to 180 on MD, againſt 1 on A, will appear nearly 6 buſhels on B, which is the quantity required.

### PROBLEM XII.

*To find, by the Line* MD, *the Malt Buſhels which may be contained in a Couch, Floor, or Ciſtern, whoſe length, breadth, and depth are given.*

RULE. Set one of the dimenſions on B to another on MD, then againſt the third on A, will appear the content on B.

EXAMPLE. Required the number of buſhels in the ciſtern whoſe length is 230, breadth 58·2, and depth 5·4 inches.

Having ſet 230, on B to 5·4 on MD, againſt 58·2 on A, I find 33·6 buſhels on B, which is the content nearly.

The uſe of the other parts or marks on the rule will appear in the examples farther on.

## CHAPTER II.

### *Of the Gauging or Diagonal Rod.*

The diagonal rod is a ſquare rule having four ſides

or faces, being generally four feet long, and folding together by means of joints.

It takes its name from its uſe in meaſuring the diagonals of caſks, and computing the contents from the ſaid diagonal only. Where it may be noted, that by the diagonal of a caſk, is meant the line from the bung to the interſection of the head with the ſtave oppoſite to it, and is commonly the longeſt line that can be drawn from the middle of the bung to any part within the caſk.

And, accordingly, upon one face of the rule is a ſcale of inches for taking the meaſure of the diagonal; to which is adapted the areas, in ale gallons, of circles to the correſponding diameters, like the lines on the under ſides of the three ſlides in the ſliding rule.

Upon the oppoſite face are two ſcales of ale and wine gallons, expreſſing the contents of caſks having the correſponding diagonals; and theſe are the lines which chiefly conſtitute the difference between this inſtrument and the ſliding rule; for all the other lines upon it are the ſame with thoſe on that inſtrument, and are to be uſed in the ſame manner.

### EXAMPLE.

Let it be required to find the content of a caſk whoſe diagonal meaſures 34·4 inches, which agrees with the caſk in the following chapter, whoſe head and bung diameters are 32 and 24, and length, 40 inches; for if to the ſquare of 20, half the length, be added the ſquare of 28, half the ſum of the diameters, the ſquare root of the ſum will be 34·4 nearly.

Now to this diagonal 34·4, correſponds, upon the rule, the content 90$\frac{1}{2}$ ale or 111 wine gallons; which differs from all the contents, in the next chapter, obtained by conſidering the caſk as belonging to

to each of the four propoſed varieties; being indeed a kind of medium among them all, and falling in between the ſecond and third variety; and ſo anſwering to the moſt common form of caſks.

## CHAPTER III.

### *Of Caſks conſidered as divided into ſeveral Varieties.*

According to the cuſtom of moſt writers on this ſubject, I ſhall diſtinguiſh caſks into four forms or varieties, viz.

1. The middle fruſtum of a ſpheroid,
2. The middle fruſtum of a parabolic ſpindle,
3. The two equal fruſtums of a paraboloid,
4. The two equal fruſtums of a cone.

I omit here the middle fruſtums of circular, elliptic, and hyperbolic ſpindles, on account of the difficulty of their rules, which renders them unfit for the purpoſe of practical gauging. And indeed ſome of the above four forms are of very little real uſe; for very few, if any, caſks are to be met with which will hold ſo much as the firſt form, or ſo little as the third or fourth; ſo that the ſecond is the moſt generally, if not the only uſeful one, of the four varieties.

*Note*, 282 cubic inches make one ale gallon,
231 — — wine gallon,
2150·42 — — a malt buſhel.

It is alſo to be noted that the dimenſions are ſuppoſed to be inches, in the following rules.

PRO-

## PROBLEM I.

### *To find the Content of a Cask of the First or Spheroidal Variety.*

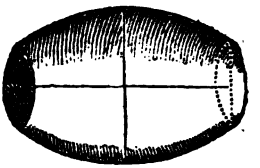
*To the square of the head diameter add double the square of the bung diameter, and multiply the sum by the length of the cask. Then let the product be mult. by ·0009$\frac{1}{4}$, or divided by 1077 for ale gallons, or mult. by ·0011$\frac{1}{3}$, or divided by 882 for wine gallons

### BY THE SLIDING RULE.

Set the length on C to 32·82, for ale, or to 29·7, for wine, on D, and on D find the bung and head diameters, noting the numbers opposite to them on C, then if the latter of these two numbers be added to the double of the former, the sum will be the measure in gallons.

### EXAMPLE.

Required the content of a spheroidal cask, whose bung and head diameters are 32 and 24, and length 40 inches.



*By*

* For, by prob. 12 sect. 5 part 3, the content in inches is $(2B^2 + H^2) \times \frac{1}{3} L n$, which being divided by 282 and 231, becomes $\frac{2B^2 + H^2}{1077{\cdot}157} \times L$ or $(2B^2 + H^2) \times {\cdot}00092837L$ in the one case, and $\frac{2B^2 + H^2}{882{\cdot}355} \times L$ or $(2B^2 + H^2) \times {\cdot}00113333$ in the other; B being the bung and H the head diameter, and L the length of the cask.——And in working by the sliding rule, it need only be remarked that 32·82 and 29·7 are the roots of 1077·157 and 882·355.

*By the Pen.* Here $(2 \times 32^2 + 24^2) \times 40 \times \left\{\begin{matrix}\cdot 0009\frac{1}{4}\\ \cdot 0011\frac{1}{2}\end{matrix}\right\} = \left\{\begin{matrix}97\cdot 44 \text{ ale}\\ 118\cdot 95 \text{ wine}\end{matrix}\right\}$ gallons, the content required.

*By the Sliding Rule.* Having ſet 40 on C to 32·82 on D, againſt 32 and 24 on D, ſtand 38 and 21·3, as near as can be judged, on C; then $2 \times 38 + 21\cdot 3 = 76 + 21\cdot 3 = 97\cdot 3$ ale gallons.

And having ſet 40 on C to 29·7 on D, againſt 32 and 24 on D, ſtand 46·5 and 26·1 on C; then $2 \times 46\cdot 5 + 26\cdot 1 = 93 + 26\cdot 1 = 119\cdot 1$ wine gallons.

## PROBLEM II.

### *To find the Content of a Caſk of the Second or Parabolic Spindle form.*

*To the ſquare of the head diameter, add double that of the bung diameter, and from the ſum take $\frac{2}{5}$ or $\frac{4}{10}$ of the ſquare of the difference of the ſaid diameters; then multiply the remainder by the length, and the product multiplied or divided by the ſame numbers as in the rule to the laſt problem, will give the content.

### BY THE SLIDING RULE.

As in the laſt problem, ſet the length on C to 32·82 or 29·7 on D, and on D find both the bung and head diameters, and alſo their difference, taking out the three numbers oppoſite to them on C; then if to twice the firſt be added the ſecond, and $\frac{4}{10}$ of the

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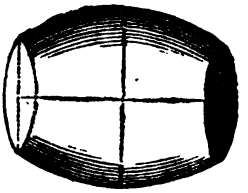
* For, by prob. 18 ſect. 6 part 3, the content in inches is $\frac{8B^2+4BH+3H^2}{15} \times Ln = \frac{Ln}{3} \times \left(\frac{10B^2+4BH+5H^2}{5} - \frac{2B^2+2H^2}{5}\right)$ $= \frac{1}{3}Ln \times (2B^2 + H^2 - \frac{2}{5} \times (B - H)^2)$, and $\frac{1}{3}n$ will give the ſame numbers as in the laſt problem.

third

third be taken from the ſum, the remainder will be the content.

EXAMPLE.

Required the content of a caſk of the ſecond variety, whoſe bung and head diameters are 32 and 24, and length 40 inches.



*By the Pen.* Here $(2 \times 32^2 + 24^2 - \frac{2}{3} \times 8^2) \times 40 \times \left\{\begin{matrix} \cdot 0009\frac{5}{7} \\ \cdot 0011\frac{1}{7} \end{matrix}\right\} = \left\{\begin{matrix} 96{\cdot}49 \text{ ale} \\ 117{\cdot}79 \text{ wine} \end{matrix}\right\}$ gallons, the content required.

*By the Sliding Rule.* Having ſet 40 on C to 32·82 on D, againſt 32, 24, and 8 on D, ſtand 38, 21·3, and 2·4; then $2 \times 38 + 21{\cdot}3 - \frac{2}{3} \times 2{\cdot}4 = 76 + 21{\cdot}3 - 0{\cdot}9 = 96{\cdot}4$ ale gallons.

And having ſet 40 on C to 29·7 on D, againſt 32, 24, and 8 on D, ſtand 46·5, 26·1, and 2·9; then $2 \times 46{\cdot}5 + 26{\cdot}1 - \frac{2}{3} \times 2{\cdot}9 = 93 + 26{\cdot}1 - 1{\cdot}2 = 117{\cdot}9$ wine gallons.

PROBLEM III.

*To find the Content of a Caſk of the Third or Paraboloidal Variety.*

To the ſquare of the bung diameter add the ſquare of the head diameter, and multiply the ſum by the length; then if the product be

mult. by $\left\{\begin{matrix} \cdot 0014 \\ \cdot 0017 \end{matrix}\right\}$ or div. by $\left\{\begin{matrix} 718 \\ 588 \end{matrix}\right\}$ for $\left\{\begin{matrix} \text{ale} \\ \text{wine} \end{matrix}\right\}$,

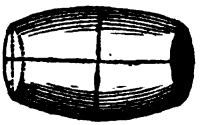
the product or quotient will be the content required.

BY THE SLIDING RULE.

Set the length on c to $\left\{\begin{matrix}26{\cdot}8\\24{\cdot}25\end{matrix}\right\}$ on D $\left\{\begin{matrix}\text{for ale}\\\text{for wine}\end{matrix}\right\}$; then find the bung and head diameters on D, noting the two oppoſite numbers on C, whoſe ſum will be the content required*.

EXAMPLE.

Required the content of a caſk of the third variety, whoſe bung and head diameters are 32 and 24, and length 40 inches.



*By the Pen.* Here $(32^2 + 24^2) \times 40 \times \left\{\begin{matrix}{\cdot}0014\\{\cdot}0017\end{matrix}\right\} = \left\{\begin{matrix}89{\cdot}1 \text{ ale,}\\108{\cdot}8 \text{ wine}\end{matrix}\right\}$ gallons, the content.

*By the Sliding Rule.* Having ſet 40 on C to 26·8 on D, againſt 32 and 24 on D, ſtand 57·3 and 32; whoſe ſum is 89·3 ale gallons.

And having ſet 40 on C to 24·25 on D, againſt 32 and 24 on D, ſtand 69·8 and 39·1, whoſe ſum is 108·9 wine gallons.

PROBLEM IV.

*To find the Content of a Caſk of the Fourth or Conical Variety.*

To three times the ſquare of the ſum of the di-

* For, by prob. 12 ſect. 6 part 3, the content in inches is $\frac{B^2 + H^2}{2} \times L n$; and $\frac{n}{2 \times 282} = \frac{1}{718{\cdot}105} = {\cdot}00139255$, and $\frac{n}{2 \times 231} = \frac{1}{588{\cdot}233} = {\cdot}0017$.

Alſo the numbers 26·8 and 24·25 are the roots of the numbers 718·105 and 588·233.

ameters, add the ſquare of the difference of the diameters; then if the ſum be

mult. by $\left\{\begin{matrix}\cdot00023\frac{1}{3}\\ \cdot00028\frac{1}{3}\end{matrix}\right\}$ or div. by $\left\{\begin{matrix}4308\\ 3529\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{ale}\\ \text{wine}\end{matrix}\right.$

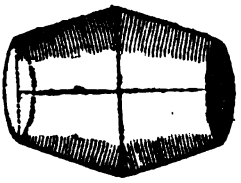
the product or quotient will be the content required.

### BY THE SLIDING RULE.

Set the length on C to $\left\{\begin{matrix}65\cdot64\\ 59\cdot41\end{matrix}\right\}$ for $\left\{\begin{matrix}\text{ale}\\ \text{wine}\end{matrix}\right\}$ on D, and on D find the ſum and the difference of the diameters, noting their oppoſite numbers on C; then if the ſecond be added to three times the firſt, the ſum will be the content*.

### EXAMPLE.

Required the content of a caſk of the fourth variety, whoſe bung and head diameters are 32 and 24, and length 40 inches.



*By the Pen.* Here $(3 \times 56^2 + 8^2) \times 40 \times \left\{\begin{matrix}\cdot00023\frac{1}{3}\\ \cdot00028\frac{1}{3}\end{matrix}\right\} = \left\{\begin{matrix}87\cdot9342 \text{ ale}\\ 107\cdot348 \text{ wine}\end{matrix}\right\}$ gallons, the content.

*By the Sliding Rule.* Having ſet 40 on C to 65·64 on D, againſt 56 and 8 on D, ſtand 29·1 and 0·6; then $3 \times 29\cdot1 + 0\cdot6 = 87\cdot3 + 0\cdot6 = 87\cdot9$ ale gallons.

* For, by prob. 8 ſect. 1 part 3, the content in inches is $\frac{B^2 + BH + H^2}{3} \times Ln = \frac{Ln}{12} \times [3 \times (B + H)^2 + (B - H)^2]$; where $\frac{\pi}{12 \times 282} = \frac{1}{4308\cdot628} = \cdot00023209$, and $\frac{\pi}{12 \times 231} = \frac{1}{3529\cdot42} = \cdot00028333$. Alſo 65·64 and 59·41 are the roots of the numbers 4308·628 and 3529·42.

And, having ſet 40 on C to 59·41 on D, againſt 56 and 8 on D, ſtand 35·6 and 0·7; then $3 \times 35{\cdot}6 + 0{\cdot}7 = 107{\cdot}5$ wine gallons.

## CHAPTER IV.

### *Of Gauging Caſks by their Mean Diameters.*

#### PROBLEM I.

*To find the Mean Diameter of a Caſk of Any of the Four Varieties, having Given the Bung and Head Diameters.*

*Divide the head diameter by the bung diameter, and find the quotient in the firſt column of the following table, marked Qu. Then if the bung diameter be multiplied by the number upon the ſame line with it, and in the column anſwering to the proper variety, the product will be the true mean diameter, or the diameter of a cylinder of the ſame content with the caſk propoſed, cutting off four figures for decimals.

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* *The Inveſtigation of the numbers of this table.*

By the 3d part it appears that if the bung diameter be repreſented by 1, and the head diameter by $b$, then the content for the 1ſt, 2d, 3d, and 4th varieties will be reſpectively $ln$ drawn into

$\frac{2+bb}{3}$, $\frac{8+4b+3bb}{15}$, $\frac{1+bb}{2}$, $\frac{1+b+bb}{3}$;

but the content muſt alſo be equal to $ln$ drawn into the ſquare of the mean diameter; and conſequently the ſaid mean diameter, or multiplier in the table, will be reſpectively equal to

$\sqrt{\frac{2+bb}{3}}$, $\sqrt{\frac{8+4b+3bb}{15}}$, $\sqrt{\frac{1+bb}{2}}$, $\sqrt{\frac{1+b+bb}{3}}$.

Then by writing, in theſe forms, the ſeveral values of $b$, viz. ·50, ·51, ·52, &c, there will reſult the correſponding numbers of the table.

Qu.

| Qu | 1 Var | 2 Var | 3 Var | 4 Var | Qu | 1 Var | 2 Var | 3 Var | 4 Var |
|---|---|---|---|---|---|---|---|---|---|
| 50 | 8660 | 8465 | 7905 | 7637 | 76 | 9270 | 9227 | 8881 | 8827 |
| 51 | 8680 | 8493 | 7937 | 7681 | 77 | 9296 | 9258 | 8924 | 8874 |
| 52 | 8700 | 8520 | 7970 | 7725 | 78 | 9324 | 9290 | 8967 | 8922 |
| 53 | 8720 | 8548 | 8002 | 7769 | 79 | 9352 | 9320 | 9011 | 8970 |
| 54 | 8740 | 8576 | 8036 | 7813 | 80 | 9380 | 9352 | 9055 | 9018 |
| 55 | 8760 | 8605 | 8070 | 7858 | 81 | 9409 | 9383 | 9100 | 9066 |
| 56 | 8781 | 8633 | 8104 | 7902 | 82 | 9438 | 9415 | 9144 | 9114 |
| 57 | 8802 | 8662 | 8140 | 7947 | 83 | 9467 | 9446 | 9189 | 9163 |
| 58 | 8824 | 8690 | 8174 | 7992 | 84 | 9496 | 9478 | 9234 | 9211 |
| 59 | 8846 | 8720 | 8210 | 8037 | 85 | 9526 | 9510 | 9280 | 9260 |
| 60 | 8869 | 8748 | 8246 | 8082 | 86 | 9556 | 9542 | 9326 | 9308 |
| 61 | 8892 | 8777 | 8282 | 8128 | 87 | 9586 | 9574 | 9372 | 9357 |
| 62 | 8915 | 8806 | 8320 | 8173 | 88 | 9616 | 9606 | 9419 | 9406 |
| 63 | 8938 | 8835 | 8357 | 8220 | 89 | 9647 | 9638 | 9466 | 9455 |
| 64 | 8962 | 8865 | 8395 | 8265 | 90 | 9678 | 9671 | 9513 | 9504 |
| 65 | 8986 | 8894 | 8433 | 8311 | 91 | 9710 | 9703 | 9560 | 9553 |
| 66 | 9010 | 8924 | 8472 | 8357 | 92 | 9740 | 9736 | 9608 | 9602 |
| 67 | 9034 | 8954 | 8511 | 8404 | 93 | 9772 | 9768 | 9656 | 9652 |
| 68 | 9060 | 8983 | 8551 | 8450 | 94 | 9804 | 9801 | 9704 | 9701 |
| 69 | 9084 | 9013 | 8590 | 8497 | 95 | 9836 | 9834 | 9753 | 9751 |
| 70 | 9110 | 9044 | 8631 | 8544 | 96 | 9868 | 9867 | 9802 | 9800 |
| 71 | 9136 | 9074 | 8672 | 8590 | 97 | 9901 | 9900 | 9851 | 9850 |
| 72 | 9162 | 9104 | 8713 | 8637 | 98 | 9933 | 9933 | 9900 | 9900 |
| 73 | 9188 | 9135 | 8754 | 8685 | 99 | 9966 | 9966 | 9950 | 9950 |
| 74 | 9215 | 9166 | 8796 | 8732 | 100 | 10000 | 10000 | 10000 | 10000 |
| 75 | 9242 | 9196 | 8838 | 8780 | | | | | |

EXAMPLE.

Supposing the diameters to be 32 and 24, it is required to find the mean diameter for each variety.

Dividing 24 by 32 we obtain ·75, which being found in the column of quotients, opposite thereto stand the numbers

| | | | |
|---|---|---|---|
| ·9242 | which being each | 29·5744 | for the corres- |
| ·9196 | multiplied by 32, | 29·4272 | ponding mean |
| ·8838 | produce respec- | 28·2816 | diameters re- |
| ·8780, | tively | 28·0960 | quired. |

BY THE SLIDING RULE.

Find the difference between the bung and head diameters upon the fourth face of the rule, or inſide of the third ſlider, and oppoſite thereto is, for each variety, a number to be added to the head diameter, for the mean diameter required.

So in the above example, againſt 8, the difference of the diameters, are found the numbers

| | | | |
|---|---|---|---|
| 5·60 | which be- | 29·60 | for the reſpective mean |
| 5·10 | ing added | 29·10 | diameters: all of which |
| 4·56 | to 24, there | 28·56 | are too great except the |
| 4 12 | reſult | 28·12 | 2d, which is too little. |

So that this method does not give the true mean diameter.

PROBLEM II.

*To find the Content of a Caſk by the Mean Diameter on the Sliding Rule.*

Set the length on C to the *gauge point, 18·95 for ale, or 17·15 for wine, on D; then againſt the mean diameter on D, is the content on C.

EXAMPLE.

If the bung diameter be 32, the head 24; and the length 40 inches.

Having found the mean diameters as in the laſt problem, and ſet 40 on C to 18·95 or 17·15 on D, againſt

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* The above gauge points are found thus, viz. $\sqrt{\frac{282}{\cdot 785398}}$ = 18·95, and $\sqrt{\frac{231}{\cdot 785398}} = 17\cdot 15$.

against $\left\{\begin{matrix}29\cdot57\\29\cdot43\\28\cdot28\\28\cdot10\end{matrix}\right\}$ on D is $\left\{\begin{matrix}97\cdot4\\96\cdot5\\89\cdot1\\88\cdot0\end{matrix}\right\}$ or $\left\{\begin{matrix}119\cdot5\\118\cdot0\\108\cdot8\\107\cdot3\end{matrix}\right\}$ on C, as near as can be judged; which agree nearly with the contents determined in the preceding chapter.

## CHAPTER V.

### *Of the Gauging of all Caſks in general by means of Four Dimenſions, viz. the Length, the Bung and Head Diameters, and the Diameter taken in the Middle between the Bung and Head.*

#### GENERAL PROBLEM.

### *To find the Content of Any Caſk in Ale or Wine Gallons, by Four Dimenſions.*

*Add into one ſum, the ſquare of the bung diameter, the ſquare of the head diameter, and the ſquare of

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* This rule is taken from prob. 3 ſect. 1 part 4; the number $\cdot0005\frac{2}{3}$ being $= \frac{\cdot785398}{6\times231}$ very nearly, and $\frac{\cdot785398}{6\times282} = \cdot0004641844 = \cdot0004\frac{7}{15}$ nearly $= \cdot000\frac{51}{77}$ more nearly.

And with regard to the operation by the ſliding rule, it may be obſerved that 42 is $= \sqrt{\frac{6\times231}{\cdot785398}}$, and $46\cdot4 = \sqrt{\frac{6\times282}{\cdot785398}}$.

The above rule, by the ſaid prob. 3 ſect. 1 part 4, was proved to be accurately true, not only for all the four different varieties of caſks, but alſo for all caſks and ſolids generated from any conic ſection whatever; and although the caſk be not preciſely in the form of any ſuch curve, the rule will give the content very near the truth; ſo that whatever be the form of the caſk, we may in all caſes be pretty ſure of the content to within $\frac{1}{10}$ of a gallon, or perhaps leſs, ſuppoſing the dimenſions to be truly taken. So that more perfect than this, both with reſpect to truth and expedition, nothing can be expected, or indeed wiſhed for, in gauging

of double the middle diameter, and multiply that ſum by the length of the caſk; then the product

mult. by $\left\{\begin{matrix}\cdot0004\frac{17}{11}\\ \cdot0005\frac{2}{3}\end{matrix}\right\}$ $\left\{\begin{matrix}\text{for ale}\\ \text{for wine}\end{matrix}\right\}$ will give the content.

BY THE SLIDING RULE.

Set the length on C to $\left\{\begin{matrix}46\cdot4 & \text{for ale}\\ 42\cdot0 & \text{for wine}\end{matrix}\right\}$ on D; and find both the bung, head and middle diameters on

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which makes me hope that one day this method will come into general uſe with the practitioners in the exciſe; and till then, I am fully perſuaded that much of their practice muſt be mere gueſs work. The pretended difficulty of taking the middle diameter may perhaps deter ſome from uſing this method; but there cannot be any real difficulty in taking this diameter, except when a wooden hoop may happen to be at the part where the diameter ought to be taken; but that will very rarely happen.

To find the fourth dimenſion or diameter in the middle between the bung and head; upon one ſide of a ſquare rule let be drawn a ſcale of quarter inches, numbered both ways from nothing; and let the middle or 0 diviſion be applied to the bung or middle of the caſk, as in the margin, parallel or nearly parallel to the axe, and in the direction of the ſtaff; then whatever number of inches are in the whole length of the caſk, it is evident that, from the nature of the ſcale, the ſame diviſion, or number of quarter inches, will be oppoſite to the part where the middle diameter muſt be taken, which here ſuppoſe to be 10; and at 10, on each part of the rule, meaſure the diſtance *rs*; then if the ſum of *rs* and *rs* be taken from the bung diameter, there will remain the required middle diameter, excepting the allowance for the thickneſs of the ſtaff, which muſt be ſubtracted.

The

on D, noting the three numbers againſt them on C; then the ſum of the firſt and ſecond with 4 times the third will be the content required.

EXAMPLE

Let the length of a caſk be 40 inches, the bung diameter 32, the head 24, and middle diameter 30·2 inches

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The following collection of caſks happened in real practice; and their dimenſions were carefully taken; but their contents were computed by a ſliding rule, and ſo may not all be preciſely true.

From hence it appears that a ſpheroidal caſk is a mere imaginary thing, the contents of real caſks being leſs than is aſſigned to them by that form; as indeed they ought, from the nature of the curves; for a ſpheroidal caſk would be leaſt curved in the middle, and the moſt at the ends; whereas a real caſk is the leaſt curved at the ends, if it be any thing curved there at all; and indeed there is reaſon to think it is not, as will appear in chap. 7.

*Caſks gauged by four dimenſions.*

| Numb | Length | Head diam. | Bung diam. | Mid. diam. | Ale gallons | Diff. leſs than ſphr. | Numb | Length | Head diam. | Bung diam. | Mid. diam. | Ale gallons | Diff. leſs than ſphr. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 28·3 | 23·2 | 27·7 | 26·3 | 53·6 | 1·0 | 19 | 48·8 | 24·2 | 32·1 | 29·4 | 114·8 | 5·5 |
| 2 | 29·8 | 22·2 | 26·0 | 24·8 | 50·2 | 1·1 | 20 | 51·2 | 23·3 | 31·0 | 28·2 | 111·3 | 5·8 |
| 3 | 30·8 | 23·2 | 27·5 | 26·1 | 57·7 | 1·1 | 21 | 49·3 | 23·8 | 32·6 | 29·5 | 117·0 | 6·1 |
| 4 | 32·2 | 24·5 | 30·1 | 28·4 | 70·6 | 1·3 | 22 | 48·0 | 28·2 | 33·8 | 31·4 | 137·3 | 6·1 |
| 5 | 30·0 | 24·7 | 29·2 | 27·6 | 62·6 | 1·8 | 23 | 45·2 | 26·6 | 33·2 | 30·4 | 115·6 | 6·5 |
| 6 | 32·5 | 23·8 | 28·2 | 26·8 | 63·6 | 1·9 | 24 | 51·6 | 36·6 | 41·6 | 39·6 | 223·3 | 6·7 |
| 7 | 34·3 | 26·3 | 33·5 | 31·1 | 90·4 | 2·9 | 25 | 44·2 | 28·1 | 36·4 | 33·3 | 134·6 | 6·8 |
| 8 | 34·5 | 26·4 | 33·0 | 30·7 | 89·0 | 3·0 | 26 | 57·0 | 32·7 | 42·0 | 39·1 | 236·6 | 6·8 |
| 9 | 41·0 | 26·3 | 32·2 | 30·2 | 102·2 | 3·0 | 27 | 51·0 | 33·1 | 38·1 | 35·7 | 181·0 | 8·0 |
| 10 | 37·0 | 26·1 | 31·8 | 29·9 | 90·3 | 3·1 | 28 | 51·5 | 33·3 | 40·0 | 37·2 | 197·0 | 8·7 |
| 11 | 44·5 | 34·4 | 40·8 | 38·8 | 183·8 | 3·2 | 29 | 54·0 | 34·8 | 44·8 | 41·5 | 253·5 | 8·8 |
| 12 | 47·0 | 26·3 | 33·8 | 31·4 | 126·3 | 3·5 | 30 | 50·0 | 34·3 | 40·5 | 37·7 | 197·6 | 9·4 |
| 13 | 34·2 | 27·2 | 33·8 | 31·4 | 92·3 | 3·8 | 31 | 49·0 | 29·5 | 36·0 | 33·0 | 148·1 | 9·4 |
| 14 | 47·0 | 25·3 | 32·0 | 29·7 | 113·1 | 4·3 | 32 | 51·0 | 33·5 | 39·2 | 36·4 | 189·6 | 9·7 |
| 15 | 45·5 | 30·7 | 38·0 | 35·5 | 157·0 | 4·7 | 33 | 51·0 | 33·4 | 39·8 | 37·0 | 194·0 | 9·8 |
| 16 | 44·6 | 24·7 | 32·2 | 29·6 | 106·6 | 4·7 | 34 | 55·5 | 30·6 | 40·6 | 37·0 | 207·2 | 10·2 |
| 17 | 48·6 | 24·2 | 32·1 | 29·4 | 114·4 | 4·9 | 35 | 45·6 | 28·0 | 34·6 | 32·4 | 134·8 | 12·0 |
| 18 | 46·0 | 25·7 | 34·7 | 31·7 | 125·3 | 5·5 | 36 | 55·0 | 35·8 | 48·0 | 43·2 | 282·2 | 17·8 |

inches nearly $= \sqrt{912}$, which is taken upon the ſuppoſition that the caſk is ſpheroidal.

Then $(32^2 + 24^2 + 4 \times 912) \times 40 \times \cdot 0004\frac{7}{11} = 97 \cdot 44$ ale gallons; or multiplied by $\cdot 0005\frac{2}{3}$, gives 118·95 wine gallons, for the content, the ſame as at prob. 1 chap 3.

*By the Sliding Rule.* Having ſet 40 on C to 46·4 on D, againſt 32, 24, and 30·2 on D, ſtand 19, 10·5, and 17 on C; then $19 + 10 \cdot 5 + 4 \times 17 = 97 \cdot 50$, ale gallons.

And, having ſet 40 on C to 42 on D, againſt 32, 24, and 30·2 on D, ſtand 23·2, 13, and 20·7 on C; then $23 \cdot 2 + 13 + 4 \times 20 \cdot 7 = 119$ wine gallons, for the content as before, nearly.

## CHAPTER VI.

### *A New, Eaſy, and Expeditious Method of computing the Content of a Caſk from Three Dimenſions only, of Whatever Variety it may be.*

*In the column of diameters, or middle column, in

* The method of gauging here propoſed, is no other than the rule for one form of ſolids, corrected ſo as to make it agree, in content, with the moſt common or general form of caſks: and, for the more expedition, the ſquares of all diameters, within certain limits, are divided by the proper conſtant diviſor, and the quotients arranged in the table oppoſite to their correſponding diameters; that in practice nothing more may be required than to multiply theſe quotients by any aſſigned length of a caſk, for the content in gallons.

Now it hath been found that wine hogſheads generally contain about a gallon and a quarter leſs than the content aſſigned to them by the rule for ſpheroidal caſks; to correct the rule for that

in the following table, find the bung and head diameters, taking out the number on the left of the head diameter, and the number on the right of the bung diameter; then the ſum of thoſe two numbers multiplied by the length of the caſk will give the content

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that form of caſks, then, we muſt increaſe the diviſor 882·355 in the proportion of 61¾ to 63; by which proportion we obtain 900, very nearly, to be uſed inſtead of 882·355, and then the rule will become $\frac{H^2 + 2B^2}{900} \times L$ for the content in gallons.

Then as the leaſt diameter of a wine hogſhead is about 19 inches, and the greateſt bung of a pipe about 35, to all diameters, in inches and tenths, within thoſe limits, I have computed the values of the quantities $\frac{H^2}{900}$ and $\frac{2B^2}{900}$, and diſpoſed them in the columns titled head and bung areas; which it is evident need only be taken out of the table, for any example of bung and head diameters, and multiply their ſum by the length for the content. So that we have here a general and eaſy rule by which any perſon, in half a minute, may compute the content of any caſk whoſe bung and head diameters, and length are given, and that generally nearer to the truth than by any rule now commonly uſed.

I ſhall here inſert the method by which I made this table; for although I have as above explained the principle upon which the table is founded, viz. that the numbers in it conſiſt of every diameter ſquared, and then divided by 900, I did not find the numbers in that manner, but by the following method in perhaps a tenth part of the time required by the former.

Since the 1ſt, 2d, 3d, &c, numbers are

$$\frac{19{\cdot}1^2}{900} = \frac{19^2}{900} + \frac{3{\cdot}81 \times 1}{900} \text{ or } + \frac{3{\cdot}81}{900} = A,$$

$$\frac{19{\cdot}2^2}{900} = \frac{19^2}{900} + \frac{3{\cdot}82 \times 2}{900} = A + \frac{3{\cdot}83}{900} = A + {\cdot}00425\dot{} = B,$$

$$\frac{19{\cdot}3^2}{900} = \frac{19^2}{900} + \frac{3{\cdot}83 \times 3}{900} = B + \frac{3{\cdot}85}{900} = B + {\cdot}00427\dot{} = C,$$

&c,

it is evident that the differences of the terms form an arithmetical ſeries, whoſe common difference is $\frac{{\cdot}02}{900}$ or ·00002̇, and therefore by writing

tent in wine gallons. And the wine gallons being multiplied by 77, and the product divided by 94, the quotient will be the ale gallons; or multiply by 9 and divide by 11 for ale gallons, because $\frac{2}{3}\frac{1}{8}\frac{1}{2} = \frac{77}{94} = \frac{9}{11}$ very nearly.

---

writing down a column of differences, and to the first term adding the first difference, to obtain the second term: then to the second term adding the second difference, for the third term; and so on as appears in the margin; the terms will evidently be found in a very expeditious manner: also by doubling these head areas, we obtain the corresponding bung areas.——In the above computations the last figures, having a point above them, are infinite repetends.

| Diff. | Areas. |
|---|---|
| | 40111 |
| 425 | 40536 |
| 427 | 40964 |
| 430 | 41394 |
| 432 | 41826 |
| 434 | 42261 |
| 436 | 42697 |
| 438 | 43136 |

Farther, it is evident that in this method, the gauge point will be 30 or $\sqrt{900}$, which is to be used, in the rule for spheroidal casks, instead of 29·7, and the correct content will be obtained.

| Head area | Diameter | Bung area | Head area | Diameter | Bung area | Head area | Diameter | Bung area |
|---|---|---|---|---|---|---|---|---|
| ·4054 | 19·1 | ·8107 | ·6615 | 24·4 | 1·3230 | ·9801 | 29·7 | 1·9602 |
| ·4096 | 19·2 | ·8192 | ·6669 | 24·5 | 1·3339 | ·9867 | 29·8 | 1·9734 |
| ·4139 | 19·3 | ·8278 | ·6724 | 24·6 | 1·3448 | ·9933 | 29·9 | 1·9867 |
| ·4183 | 19·4 | ·8364 | ·6779 | 24·7 | 1·3558 | 1·0000 | 30·0 | 2·0000 |
| ·4226 | 19·5 | ·8450 | ·6834 | 24·8 | 1·3668 | 1·0067 | 30·1 | 2·0134 |
| ·4270 | 19·6 | ·8537 | ·6889 | 24·9 | 1·3778 | 1·0134 | 30·2 | 2·0268 |
| ·4314 | 19·7 | ·8624 | ·6944 | 25·0 | 1·3888 | 1·0201 | 30·3 | 2·0402 |
| ·4356 | 19·8 | ·8712 | ·7000 | 25·1 | 1·4000 | 1·0268 | 30·4 | 2·0537 |
| ·4400 | 19·9 | ·8800 | ·7056 | 25·2 | 1·4112 | 1·0336 | 30·5 | 2·0772 |
| ·4444 | 20·0 | ·8889 | ·7112 | 25·3 | 1·4224 | 1·0404 | 30·6 | 2·0808 |
| ·4489 | 20·1 | ·8978 | ·7168 | 25·4 | 1·4337 | 1·0472 | 30·7 | 2·0944 |
| ·4534 | 20·2 | ·9068 | ·7225 | 25·5 | 1·4450 | 1·0540 | 30·8 | 2·1081 |
| ·4579 | 20·3 | ·9158 | ·7282 | 25·6 | 1·4564 | 1·0609 | 30·9 | 2·1218 |
| ·4624 | 20·4 | ·9248 | ·7339 | 25·7 | 1·4678 | 1·0676 | 31·0 | 2·1356 |
| ·4669 | 20·5 | ·9339 | ·7396 | 25·8 | 1·4792 | 1·0747 | 31·1 | 2·1494 |
| ·4715 | 20·6 | ·9430 | ·7453 | 25·9 | 1·4907 | 1·0816 | 31·2 | 2·1632 |
| ·4761 | 20·7 | ·9522 | ·7511 | 26·0 | 1·5022 | 1·0885 | 31·3 | 2·1771 |
| ·4807 | 20·8 | ·9614 | ·7569 | 26·1 | 1·5138 | 1·0955 | 31·4 | 2·1919 |
| ·4853 | 20·9 | ·9707 | ·7627 | 26·2 | 1·5254 | 1·1025 | 31·5 | 2·2050 |
| ·4900 | 21·0 | ·9800 | ·7685 | 26·3 | 1·5371 | 1·1095 | 31·6 | 2·2190 |
| ·4947 | 21·1 | ·9894 | ·7744 | 26·4 | 1·5488 | 1·1165 | 31·7 | 2·2331 |
| ·4994 | 21·2 | ·9988 | ·7803 | 26·5 | 1·5606 | 1·1236 | 31·8 | 2·2472 |
| ·5041 | 21·3 | 1·0082 | ·7862 | 26·6 | 1·5724 | 1·1307 | 31·9 | 2·2614 |
| ·5088 | 21·4 | 1·0177 | ·7921 | 26·7 | 1·5842 | 1·1378 | 32·0 | 2·2756 |
| ·5136 | 21·5 | 1·0272 | ·7980 | 26·8 | 1·5961 | 1·1449 | 32·1 | 2·2898 |
| ·5184 | 21·6 | 1·0368 | ·8040 | 26·9 | 1·6080 | 1·1520 | 32·2 | 2·3041 |
| ·5232 | 21·7 | 1·0464 | ·8100 | 27·0 | 1·6200 | 1·1592 | 32·3 | 2·3184 |
| ·5280 | 21·8 | 1·0561 | ·8160 | 27·1 | 1·6320 | 1·1664 | 32·4 | 2·3328 |
| ·5329 | 21·9 | 1·0658 | ·8220 | 27·2 | 1·6441 | 1·1736 | 32·5 | 2·3472 |
| ·5378 | 22·0 | 1·0756 | ·8281 | 27·3 | 1·6562 | 1·1808 | 32·6 | 2·3617 |
| ·5427 | 22·1 | 1·0854 | ·8342 | 27·4 | 1·6684 | 1·1881 | 32·7 | 2·3762 |
| ·5476 | 22·2 | 1·0952 | ·8403 | 27·5 | 1·6806 | 1·1954 | 32·8 | 2·3908 |
| ·5525 | 22·3 | 1·1051 | ·8464 | 27·6 | 1·6928 | 1·2027 | 32·9 | 2·4054 |
| ·5575 | 22·4 | 1·1150 | ·8525 | 27·7 | 1·7051 | 1·2100 | 33·0 | 2·4200 |
| ·5625 | 22·5 | 1·1250 | ·8587 | 27·8 | 1·7174 | 1·2173 | 33·1 | 2·4347 |
| ·5675 | 22·6 | 1·1350 | ·8649 | 27·9 | 1·7298 | 1·2247 | 33·2 | 2·4494 |
| ·5725 | 22·7 | 1·1451 | ·8711 | 28·0 | 1·7422 | 1·2321 | 33·3 | 2·4642 |
| ·5776 | 22·8 | 1·1552 | ·8773 | 28·1 | 1·7547 | 1·2395 | 33·4 | 2·4790 |
| ·5827 | 22·9 | 1·1654 | ·8836 | 28·2 | 1·7672 | 1·2469 | 33·5 | 2·4939 |
| ·5878 | 23·0 | 1·1756 | ·8899 | 28·3 | 1·7798 | 1·2544 | 33·6 | 2·5088 |
| ·5929 | 23·1 | 1·1858 | ·8962 | 28·4 | 1·7924 | 1·2619 | 33·7 | 2·5238 |
| ·5980 | 23·2 | 1·1961 | ·9025 | 28·5 | 1·8050 | 1·2694 | 33·8 | 2·5388 |
| ·6032 | 23·3 | 1·2064 | ·9088 | 28·6 | 1·8177 | 1·2769 | 33·9 | 2·5538 |
| ·6084 | 23·4 | 1·2168 | ·9152 | 28·7 | 1·8304 | 1·2844 | 34·0 | 2·5689 |
| ·6136 | 23·5 | 1·2272 | ·9216 | 28·8 | 1·8432 | 1·2920 | 34·1 | 2·5840 |
| ·6188 | 23·6 | 1·2377 | ·9280 | 28·9 | 1·8560 | 1·2996 | 34·2 | 2·5992 |
| ·6241 | 23·7 | 1·2482 | ·9344 | 29·0 | 1·8689 | 1·3072 | 34·3 | 2·6144 |
| ·6294 | 23·8 | 1·2588 | ·9409 | 29·1 | 1·8818 | 1·3148 | 34·4 | 2·6297 |
| ·6347 | 23·9 | 1·2694 | ·9474 | 29·2 | 1·8948 | 1·3225 | 34·5 | 2·6450 |
| ·6400 | 24·0 | 1·2800 | ·9539 | 29·3 | 1·9078 | 1·3302 | 34·6 | 2·6604 |
| ·6453 | 24·1 | 1·2907 | ·9604 | 29·4 | 1·9208 | 1·3379 | 34·7 | 2·6758 |
| ·6507 | 24·2 | 1·3014 | ·9669 | 29·5 | 1·9339 | 1·3456 | 34·8 | 2·6912 |
| ·6561 | 24·3 | 1·3122 | ·9735 | 29·6 | 1·9470 | 1·3533 | 34·9 | 2·7067 |

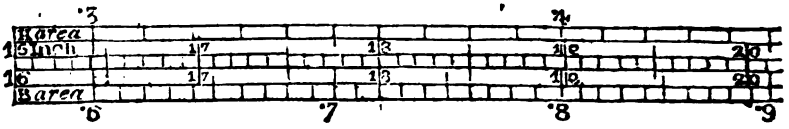
EXAMPLE.

Required the content of a caſk whoſe length is 40 inches, and bung and head diameters 32 and 24 inches.

| | | |
|---|---|---|
| To the right of 32, the bung diameter, is | 2·2756 | the bung area; |
| and to the left of 24, the head diameter, is | ·6400 | the head area, |
| the ſum | 2·9156 | |
| multiplied by | 40 | the length |
| produces | 116·624 | wine gallons for the content required. |

To work by the ſliding rule, we may proceed as in prob. 1 chap. 3, uſing 30 as the gauge point inſtead of 29·7; thus, having ſet 40 on C to 30 on D, againſt

A ſtill farther improvement is to be made of this method, by transferring the table to ſcales upon a rod, or the inſide of a Branan's rule, inſtead of the two uſeleſs diagonal ſcales, a ſcale of inches and tenths correſponding to the ſcale of areas, A part of ſuch a ſcale is repreſented below.




And for the making of ſuch a ſcale I ſhall here ſubjoin a table of diameters computed from aſſumed equi-different areas, which is much fitter for this purpoſe than the foregoing table of areas, which was computed from aſſumed equi-different diameters. This table of diameters was computed by *Mr.* AB. CROCKER, then of *Ilminſter*, but now of *Frome*, *Somerſetſhire*, and who alſo communicated the ſubſtance of this chapter, together with the collection of caſks at the end of the laſt.

*A Table*

againſt 32 and 24 on D, ſtand 45·6 and 25·5 on C; then $2 \times 45{\cdot}6 + 25{\cdot}5 = 91{\cdot}2 + 25{\cdot}5 = 116{\cdot}7$ wine gallons the content nearly as before.

And for the content in ale gallons, we have $116{\cdot}6 \times \frac{9}{11} = 95{\cdot}4$

CHAP-

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## *A Table of Diameters for Making Scales.*

| Head area | Diameter | Bung area | Head area | Diameter | Bung area | Head area | Diameter | Bung area | Head area | Diameter | Bung area |
|---|---|---|---|---|---|---|---|---|---|---|---|
| ·30 | 16·43 | ·60 | ·58 | 22·85 | 1·16 | ·86 | 27·82 | 1·72 | 1·14 | 32·03 | 2·28 |
| ·31 | 16·70 | ·62 | ·59 | 23·04 | 1·18 | ·87 | 27·98 | 1·74 | 1·15 | 32·17 | 2·30 |
| ·32 | 16·97 | ·64 | ·60 | 23·24 | 1·20 | ·88 | 28·14 | 1·76 | 1·16 | 32·31 | 2·32 |
| ·33 | 17·24 | ·66 | ·61 | 23·43 | 1·22 | ·89 | 28·30 | 1·78 | 1·17 | 32·45 | 2·34 |
| ·34 | 17·50 | ·68 | ·62 | 23·62 | 1·24 | ·90 | 28·45 | 1·80 | 1·18 | 32·59 | 2·36 |
| ·35 | 17·76 | ·70 | ·63 | 23·81 | 1·26 | ·91 | 28·61 | 1·82 | 1·19 | 32·73 | 2·38 |
| ·36 | 18·00 | ·72 | ·64 | 24·00 | 1·28 | ·92 | 28·78 | 1·84 | 1·20 | 32·86 | 2·40 |
| ·37 | 18·26 | ·74 | ·65 | 24·18 | 1·30 | ·93 | 28·93 | 1·86 | 1·21 | 33·00 | 2·42 |
| ·38 | 18·50 | ·76 | ·66 | 24·37 | 1·32 | ·94 | 29·09 | 1·88 | 1·22 | 33·14 | 2·44 |
| ·39 | 18·73 | ·78 | ·67 | 24·56 | 1·34 | ·95 | 29·24 | 1·90 | 1·23 | 33·27 | 2·46 |
| ·40 | 18·97 | ·80 | ·68 | 24·74 | 1·36 | ·96 | 29·40 | 1·92 | 1·24 | 33·41 | 2·48 |
| ·41 | 19·21 | ·82 | ·69 | 24·92 | 1·38 | ·97 | 29·55 | 1·94 | 1·25 | 33·54 | 2·50 |
| ·42 | 19·44 | ·84 | ·70 | 25·10 | 1·40 | ·98 | 29·70 | 1·96 | 1·26 | 33·67 | 2·52 |
| ·43 | 19·67 | ·86 | ·71 | 25·28 | 1·42 | ·99 | 29·85 | 1·98 | 1·27 | 33·81 | 2·54 |
| ·44 | 19·90 | ·88 | ·72 | 25·45 | 1·44 | 1·00 | 30·00 | 2·00 | 1·28 | 33·94 | 2·56 |
| ·45 | 20·12 | ·90 | ·73 | 25·63 | 1·46 | 1·01 | 30·15 | 2·02 | 1·29 | 34·07 | 2·58 |
| ·46 | 20·35 | ·92 | ·74 | 25·81 | 1·48 | 1·02 | 30·30 | 2·04 | 1·30 | 34·20 | 2·60 |
| ·47 | 20·57 | ·94 | ·75 | 25·98 | 1·50 | 1·03 | 30·45 | 2·06 | 1·31 | 34·33 | 2·62 |
| ·48 | 20·79 | ·96 | ·76 | 26·15 | 1·52 | 1·04 | 30·60 | 2·08 | 1·32 | 34·47 | 2·64 |
| ·49 | 21·00 | ·98 | ·77 | 26·33 | 1·54 | 1·05 | 30·74 | 2·10 | 1·33 | 34·60 | 2·66 |
| ·50 | 21·21 | 1·00 | ·78 | 26·50 | 1·56 | 1·06 | 30·88 | 2·12 | 1·34 | 34·73 | 2·68 |
| ·51 | 21·42 | 1·02 | ·79 | 26·67 | 1·58 | 1·07 | 31·03 | 2·14 | 1·35 | 34·86 | 2·70 |
| ·52 | 21·63 | 1·04 | ·80 | 26·84 | 1·60 | 1·08 | 31·18 | 2·16 | 1·36 | 34·98 | 2·72 |
| ·53 | 21·84 | 1·06 | ·81 | 27·00 | 1·62 | 1·09 | 31·32 | 2·18 | 1·37 | 35·11 | 2·74 |
| ·54 | 22·05 | 1·08 | ·82 | 27·17 | 1·64 | 1·10 | 31·47 | 2·20 | 1·38 | 35·25 | 2·76 |
| ·55 | 22·25 | 1·10 | ·83 | 27·33 | 1·66 | 1·11 | 31·61 | 2·22 | 1·39 | 35·37 | 2·78 |
| ·56 | 22·45 | 1·12 | ·84 | 27·50 | 1·68 | 1·12 | 31·75 | 2·24 | 1·40 | 35·50 | 2·80 |
| ·57 | 22·65 | 1·14 | ·85 | 27·66 | 1·70 | 1·13 | 31·90 | 2·26 | 1·41 | 35·62 | 2·82 |

Now as to the method of forming this table; ſince any area is equal to the ſquare of its diameter divided by 900, the diameter muſt be equal to the root of the product of 900 drawn into the

## CHAPTER VII.

### *A new and very exact Method of Computing the Content of a Caſk of any Form from Three Dimenſions only.*

*Add into one ſum 39 times the ſquare of the bung diameter, 25 times the ſquare of the head diameter,

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the area; ſo that there will belong
to the firſt area ·30, the diam. $\sqrt{\cdot 30 \times 900}$ or $\sqrt{30 \times 9} = \sqrt{A}$;
to the 2d area ·31, the diam. $\sqrt{31 \times 9} = \sqrt{A + 9} = \sqrt{B}$;
to the 3d area ·32, the diam. $\sqrt{32 \times 9} = \sqrt{B + 9} = \sqrt{C}$, &c.

Hence it is evident that the ſucceſſive products, out of which the ſquare root is to be extracted, will be obtained by the continual addition of 9; and then extracting the roots, by means of the table of logarithms, will give the ſeveral diameters required.

But if you have a table of ſquare roots by you, ſuch as that at the end of my *Mathematical Miſcellany*, ſince the root of 9 is 3, the moſt expeditious method of making the table, is to take the ſquare roots of 30, 31, 32, &c, out of the table of roots, and then multiply them by 3 for the ſeveral correſponding diameters.

* This rule I have made out from a method of inveſtigating the contents of caſks, which was hinted by *Mr.* JAMES DAVIDSON, *Teacher of the Mathematics at Dundee, in North Britain*; the invention of which take as below.

It appearing reaſonable that, in determining a proper rule for computing the contents of caſks, we ought to have regard to the general methods which are uſed by coopers in conſtructing them; application was therefore made to ſeveral ingenious workmen in that way, from whoſe account it appeared that, ordinarily, the ſides or edges of each ſtave of a caſk, for about one-third of its length next each end, are made tapering in a ſtraight line, and that for the middle third part they were curved, or made convex, to form the bulge or middle of the caſk.

From this plain account then, we are naturally led to conſider one-third of a caſk, at each end, as the fruſtum of a cone; and the middle part is moſt properly conſidered as in the form of a parabolic curve, for theſe reaſons; viz. 1ſt, that, on account of the ſhortneſs of this curved part, it may be conſidered but as a ſmall part near the vertex of the curve; 2d, that all curves near the vertex differ inſenſibly from the parabola; and 3d, that its rules

are

meter, and 26 times the product of the diameters; multiply the sum by the length, and the product by ·00034; then the last product divided by 9 will give the wine gallons, and divided by 11 will give the ale gallons.

EX-

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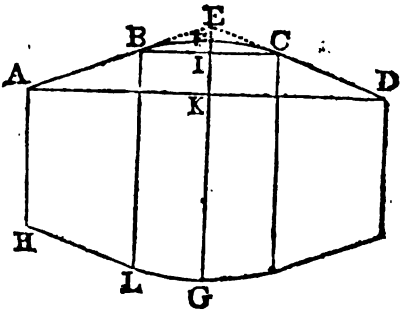
are most easily adapted to practical uses. We shall therefore suppose the middle third part of a cask to be the middle frustum of a parabolic spindle, and the ends as two frustums of a cone, and investigate its contents upon those principles in the following manner.




Let AB and CD be the two right-lined parts, and BC the parabolic part; then produce AB and DC to meet in E, and draw the lines as are evident by the figure. Put the length AD of the cask = L, the bung diameter EG = B, and the head diameter AH = H. Then since AB has the same direction as EB at B, ABE will be a tangent to the parabola BF, and therefore FI = $\frac{1}{2}$EI; but BI = $\frac{1}{3}$AK, and hence by similar triangles EI = $\frac{1}{3}$EK; consequently FI = $\frac{1}{2}$EI = $\frac{1}{3}$EK = $\frac{1}{3}$FK = $\frac{B-H}{10}$; so that the common diameter BL = FG − 2FI = $B - \frac{B-H}{5} = \frac{4B+H}{5}$; which call c.

Now, by the common rules for parabolic spindles and conic frustums, we obtain $\frac{8B^2+4BC+3C^2}{15} \times \frac{L\pi}{3} = \frac{328B^2+44BH+3H^2}{25\times 45} \times L\pi$ for the parabolic, or middle part; and $\frac{C^2+CH+H^2}{3} \times \frac{2L\pi}{3} = \frac{160B^2+280BH+310H^2}{25\times 45} \times L\pi$ for the two ends; and the sum of these two give $(488B^2+324BH+313H^2) \times \frac{L\pi}{1125} = (39B^2+26BH+25H^2) \times \frac{L\pi}{90}$ nearly, for the content in inches; $\pi$ being = ·785398. And the quantity

EXAMPLE

Required the content of a caſk whoſe length is 40, bung diameter 32, and head diameter 24 inches.

Here $(39 \times 32^2 + 26 \times 32 \times 24 + 25 \times 24^2) \times 40 \times .00034 = 1010.5$; which being divided by 9 and by 11, we obtain 112·3 wine gallons, or 91·9 ale gallons, for the content required. Agreeing nearly with the content found by the Diagonal Rod in page 573.

## CHAPTER VIII.

### *Of the Ullage of Caſks.*

The ullage of a caſk is now generally underſtood to be either the empty part, or the part filled, of a caſk which is not quite full.

In

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$\frac{.785398}{90}$ being divided by 231 gives $\frac{.00034}{9}$ the multiplier for wine gallons; and ſince 231 is to 282 as 9 to 11 very nearly, $\frac{.00034}{11}$ will be the multiplier for ale gallons, as in the rule.

And thus we have an eaſy and expeditious rule, which has not only the property of agreeing with the content, as computed from the method by four dimenſions, when thoſe dimenſions are actually taken; but alſo agrees well with the real contents of caſks; as hath been proved by ſeveral caſks which have actually been filled with a true gallon meaſure, after their contents have been computed by this method.

To ſhew its agreement with the four-dimenſion method, take any one of the examples out of the collection at the end of chapter 5, as for inſtance No. 14, whoſe content by that method is 113·1 gallons; and by this method it will be found to come out nearly 112. Again, if we take any other, as ſuppoſe the laſt, whoſe content was 282·2, it will come out by this method 283·2. So that in theſe two inſtances, taken at random, although the contents be very large, they agree to within a gallon; the one method exceeding in the former caſe, and the other in the latter.

In this buſineſs, caſks are conſidered as in two poſitions, viz. with their axes either parallel or perpendicular to the horizon, that is, either lying along, or ſtanding upright upon one end. The ullage of a caſk, either ſtanding or lying, may be determined for any particular variety, by the preceding parts of this work; but the rules thence obtained are too complex for ordinary practice, eſpecially for a lying caſk; I ſhall not, therefore, introduce thoſe rules into this place, but ſhall content myſelf with ſetting down ſuch approximations as have been found to agree beſt with expedition and accuracy; together with the univerſal method by three diameters, which is not only ſufficiently eaſy, but alſo the moſt accurate for this purpoſe.

### PROBLEM I.

*To find the Ullage of a Standing and Lying Caſk by the Lines* SS *and* SL *on the Sliding Rule.*

By ſome of the preceding chapters, find the whole content of the caſk.

Then ſet the length on N to 100 on SS for a ſeg. ſtanding; or ſet the bung diameter on N to 100 on SL for a ſeg. lying; then againſt the wet or dry inches on N, is a number to be reſerved. Then ſet 100 on B to the reſerved number on A; and againſt the whole content on B, will be found the ullage on A. And this ullage will be either the empty part, or the part filled, according as the dry or wet inches were uſed, in finding the reſerved number.

### EXAMPLE I.

Required the ullage of a ſtanding caſk, whoſe length is 40, bung diameter 32, head diameter 24, wet inches 10, and conſequently the dry inches 30.

By the laſt chapter, the content was nearly 92 ale gallons: Hence, having ſet 40 on N to 100 on SS, and againſt 10 on N found 23 on SS; then ſet 100 on B to 23 on A, and againſt 92 on B is found 21·2 ale gallons on A, for the quantity remaining in the caſk.

If the dry inches 30 be uſed, the reſerved ſegment will be found to be 76·7; and then the correſponding ullage is 70·2, for the gallons drawn off.

And the ſum of theſe two parts is 91·4 gallons, which is about half a gallon leſs than the whole content; which error would be inconſiderable if it were to be divided between the two parts; but inſtead of that, commonly the one part is too little, and the other too great, by which it happens that the error in each part moſtly exceeds that of their ſum.

EXAMPLE. II.

Let the dimenſions and content be the ſame in the caſe of the lying caſk, alſo the wet inches 8, and conſequently the dry inches 24.

Having ſet 32 on N to 100 on SL, againſt 8 and 24 on N, are the reſerved ſegments 17·8 and 82·5 on SL. Then having ſet 100 on B to 92 on A, againſt 17·8 and 82·5 on B, are 16·4 and 76 on A; which are the parts filled and empty reſpectively, and whoſe ſum is 92·4, near half a gallon too much.

PRO-

## PROBLEM II.

### *To find the Ullage of a Standing Caſk by the Pen.*

#### RULE I.

*Add all together the ſquare of the diameter at the ſurface of the liquor, the ſquare of the diameter of the neareſt end, and the ſquare of double the diameter taken in the middle between the other two; then multiply the ſum by the length between the ſurface and neareſt end, and the product again by $\cdot 0004\frac{2}{7}$ for ale gallons, or by $\cdot 0005\frac{2}{7}$ for wine gallons, in the leſs part of the caſk, whether empty or filled.

#### EXAMPLE.

Taking the above diameters 24, 27, and 29 inches, and the wet inches 10; we ſhall have $(24^2 + 54^2 + 29^2) \times 10 \times \cdot 0004\frac{2}{7} = 4333 \times \cdot 004\frac{2}{7} = 20 \cdot 2$ gallons for the ullage required.

#### RULE II.

As the ſquare of the length of the caſk, is to the ſquare of the difference between the ſaid length and wet or dry inches, viz. the leſs of them; ſo is the difference between the bung and head diameters, to a number, which being taken from the bung diameter, will leave the diameter of a cylinder of the ſame



* The diameters at the ſurface of the liquor, and in the middle between it and the neareſt head, will eaſily be obtained, by ſuſpending a plummet by a rule laid over the middle of the head, ſo as juſt to touch the bulge of the caſk, and then meaſuring the diſtance of the ſtring from the ſide of the caſk at the ſurface of the liquor and in the middle between it and the neareſt head; for then the doubles of theſe meaſures taken from the bung diameter, will leave the ſaid diameters required.

length with, and nearly equal to the part filled or empty, viz. the leſs of them*.

EXAMPLE.

Taking the ſame example as in problem 1, we have as $40^2 : (40 - 10)^2 :: 32 - 24 : 4\frac{1}{2}$; and $32 - 4\frac{1}{2} = 27\frac{1}{2}$ the mean diameter.

Then $27{\cdot}5^2 \times 10 \times {\cdot}002\frac{7}{9} = 21$ ale gallons is the ullage required.

*Note.* The number ·0027851 or $002\frac{7}{9}$ nearly, is the conſtant multiplier $\frac{{\cdot}7853\,98\ \&c}{282}$.

PROBLEM III.

*To find the Ullage of a Lying Caſk by the Pen.*

Divide the wet inches by the bung diameter; find the quotient in the firſt column of the table of circular ſegments, at the end of the book, and take out the ſegment oppoſite to it; then multiply this ſegment by the whole content of the caſk, and the product again by $1\frac{1}{4}$ for the ullage nearly†.

EX-

* This rule is founded on the reaſonable ſuppoſition, that for a ſmall part of caſk, the diameter in the middle may be conſidered as a mean diameter; which is ſtrictly true when the caſk is paraboloidal, and very near the truth when parabolic-ſpindular; but the ſaid middle diameter, as given in the rule, is computed from the latter form, becauſe the rule is thereby rendered not only eaſier; but generally more exact.

Thus, by the property of the parabola, $L^2 : (L - 1)^2 :: B - H : \frac{B - H}{L^2} \times (L - 1)^2$, and $B - \frac{B - H}{L^2} \times (L - 1)^2 = M$; where B is the bung, H the head, and M the mean diameter; alſo L is the length of the caſk, and 1 the wet or dry inches.

† This method is evidently nothing elſe than taking the whole content, in ſuch proportion to the ullage, as the whole bung circle bears to the ſegment of it cut off by the ſurface of the liquor; and is nearer to the truth than any other practical rule that I can find.

EXAMPLE.

Taking the ſame caſk as before, whoſe length is 40, bung diameter 32, head diameter 24; and ſuppoſing the wet inches to be 8.

By chapter 7, the whole content is nearly 92 ale gallons. Then $\frac{8}{32} = \frac{1}{4} = .25$; oppoſite to which, in the table of areas, is the ſegment .15354621; hence $92 \times .15354621 \times 1\frac{1}{4} = 18$ ale gallons, the ullage required.

SCHOLIUM.

Having delivered the neceſſary rules for meaſuring caſks, &c, I do not ſuppoſe that any thing more of the ſubject of gauging is neceſſary to be given in this book. For as to ciſterns, couches, &c, tuns, coolers, &c, coppers, ſtills, &c, which are firſt ſuppoſed to be in the form of ſome of the ſolids in the former parts of this work, and then meaſured accordingly, no perſon can be at a loſs concerning them, who knows any thing of ſuch ſolids in general; and to treat of them here, would induce me to a long and tedious repetition, only for the ſake of pointing out the proper multipliers or diviſors; which is, I think, a reaſon very inadequate to ſo cumberſome an increaſe of my book.

I ſhall only juſt obſerve, that when tuns, &c, of oval baſes are to be gauged; as thoſe baſes really meaſure to more than true ellipſes of the ſame length and breadth, they ought to be meaſured by the equi-diſtant ordinate method, delivered in ſection 2 of part 4.

And that when caſks are met with, which have different head diameters, they may be deemed incomplete caſks, and their contents conſidered and meaſured as the ullage of a caſk.

## SECTION III.

### OF THE WORKS OF ARTIFICERS.

THE artificers whoſe works are here to be treated of, are 1 Bricklayers, 2 Carpenters and Joiners, 3 Glaziers, 4 Maſons, 5 Painters, 6 Pavers, 7 Plaſterers, 8 Plumbers, and 9 Slaters and Tilers.

Artificers compute the contents of their works by ſeveral different meaſures.

As glazing and maſonry by the foot:

Painting, plaſtering, paving, &c, by the yard of 9 ſquare feet:

Flooring, partitioning, roofing, tiling, &c, by the ſquare, of 100 ſquare feet:

And brickwork either by the yard of 9 ſquare feet, or by the perch, or ſquare rod or pole, containing $272\frac{1}{4}$ ſquare feet, or $30\frac{1}{4}$ ſquare yards, being the ſquare of the rod or pole of $16\frac{1}{2}$ feet or $5\frac{1}{2}$ yards long.

As this number $272\frac{1}{4}$ is a troubleſome number to divide by, the $\frac{1}{4}$ is often omitted in practice, and the content in feet divided only by the 272. But as this is not exact, it will be both better and eaſier to multiply the feet by 4, and then divide ſucceſſively by 9, 11, and 11. Alſo to divide ſquare yards by $30\frac{1}{4}$, firſt multiply them by 4, and then divide twice by 11.

All works, whether ſuperficial or ſolid, are computed by the rules proper to the figure of them, whether it be a triangle, or rectangle, a parallelopiped, or any other figure.

For taking meaſures, the moſt common inſtrument is what is called the carpenters rule, of which it will be neceſſary here to give a deſcription.

CHAP-

## CHAPTER I.

### *Of the Common or Carpenters Rule.*

This inſtrument is otherwiſe called the ſliding rule; and it is much uſed in timber meaſuring and artificers works, both for taking the dimenſions, and caſting up the contents.

The inſtrument conſiſts of two equal pieces, each a foot in length, which are connected together by a folding joint.

One ſide or face of the rule, is divided into inches, and half-quarters, or eighths. On the ſame face alſo are ſeveral plane ſcales, divided into twelfth parts by diagonal lines; which are uſed in planning dimenſions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely each foot into 10 equal parts, and each of theſe into 10 parts again: ſo that by means of this laſt ſcale, dimenſions are taken in feet and tenths and hundredths, and multiplied as common decimal numbers, which is the beſt way.

On the one part of the other face are four lines, marked A, B, C, D, the two middle ones B and C being on a ſlider, which runs in a groove made in the ſtock. The ſame numbers ſerve for both theſe two middle lines, the one being above the numbers, and the other below.

Theſe four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from 1 to 10. The other or loweſt line D, is a ſingle one, proceeding from 4 to 40 It is alſo called the girt line, from its uſe in caſting up the contents of trees and timber; and upon it are marked WG at 17·15, and AG at 18·95, the wine and ale gauge points, to make this inſtrument ſerve the purpoſe of a gauging rule.

Upon the other part of this face there is a table of the

the value of a load, or 50 cubit feet, of timber, at all prices, from 6 pence to 2 ſhillings a foot.

When 1 at the beginning of any line is accounted 1, then the 1 in the middle will be 10, and the 10 at the end 100; and when the 1 at the beginning is accounted 10, then 1 in the middle is 100, and the 10 at the end 1000; and ſo on. And all the ſmaller diviſions are altered proportionally.

### PROBLEM I.

### *To Multiply Numbers together.*

Suppoſe the two numbers 24 and 13.—Set 1 on B to 13 on A; then againſt 24 on B ſtands 312 on A, which is the required product of the two given numbers 24 and 13.

*Note.* In any operations, when a number runs beyond the end of the line, ſeek it on the other radius, or other part of the line, that is, take the 10th part of it, or the 100th part of it, &c, and increaſe the reſult proportionally 10 fold, or 100 fold, &c.

In like manner the product of 35 and 19 is 665,
and the product of 270 and 54 is 14580.

### PROBLEM II.

### *To Divide by the Sliding Rule.*

As ſuppoſe to divide 312 by 24.—Set the diviſor 24 on B to the dividend 312 on A; then againſt 1 on B ſtands 13 the quotient on A.

Alſo 396 divided by 27 gives 14·6,
and 741 divided by 42 gives 17·6.

PRO-

PROBLEM III.

*To Square any Number.*

Suppoſe to ſquare 23.—Set 1 on B to 23 on A; then againſt 23 on B ſtands 529 on A, which is the ſquare of 23.

Or, by the other two lines, ſet 1 or 100 on C to the 10 on D, then againſt every number on D ſtands its ſquare in the line C. So againſt 23 ſtands 529,
againſt 20 ſtands 400,
againſt 30 ſtands 900,
and ſo on.

If the given number be hundreds, &c, reckon the 1 on D for 100, or 1000, &c; then the correſponding 1 on C is 10000, or 1000000, &c. So the ſquare of 230 is found to be 52900.

PROBLEM IV.

*To Extract the Square Root.*

Set 1 or 100, &c, on, C to 1 or 10, &c, on D; then againſt every number found on C, ſtands its ſquare root on D.

So, againſt 529 ſtands its root 23,
againſt 400 ſtands its root 20,
againſt 900 ſtands its root 30,
againſt 300 ſtands its root 17·3,
and ſo on.

PROBLEM V.

*To find a Mean Proportional between two Numbers.*

As ſuppoſe between 29 and 430.—Set the one number 29 on C to the ſame on D; then againſt the other

other number 430 on C, ſtands their mean proportional 111 on D.

Alſo, the mean between 29 and 320 is 96·3,
and the mean between 71 and 274 is 139.

### PROBLEM VI.

*To find a Third Proportional to two Numbers.*

Suppoſe to 21 and 32.—Set the firſt 21 on B to the ſecond 32 on A; then againſt the ſecond 32 on B, ſtand 48·8 on A, which is the third proportional ſought.

Alſo the 3d proportional to 17 and 29 is 49·4,
and the 3d proportional to 73 and 14 is 2·5.

### PROBLEM VII.

*To find a Fourth Proportional to three Numbers.*

*Or, to perform the Rule-of-Three.*

Suppoſe to find a fourth proportional to 12, 28, and 114.—Set the firſt term 12 on B to the 2d term 28 on A; then againſt the third term 114 on B, ſtands 266 on A, which is the 4th proportional ſought.

Alſo the 4th proportional to 6, 14, 29, is 67·6,
and the 4th proportional to 27, 20, 73, is 54·0.

## CHAPTER II.

*Of the different Meaſures uſed by different Artificers.*

144 ſquare inches = a ſquare foot,
9 ſquare feet = a ſquare yard,
63 ſquare feet = 7 ſquare yards = a rood,
100 ſquare feet = a ſquare,
272¼ ſquare feet = 30¼ ſquare yards = a rod, perch, or ſquare pole.

The

The above denominations are thoſe by which moſt kinds of work are valued; but ſome particular articles are valued by the foot running, or lineal meaſure.

*A Table for changing hundredth Parts of Feet into Inches and Parts, and the contrary; by Means of which, Dimenſions taken in one of theſe, may be readily changed to the other.*

| Centeſms, or hundredth parts of a foot | Inches | 12th pts. | Inches | 8th pts. or half quarters | Centeſms |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 0 | 1 | 1 |
| 2 | 0 | 3 | 0 | 2 | 2 |
| 3 | 0 | 4 | 0 | 3 | 3 |
| 4 | 0 | 6 | 0 | 4 | 4 |
| 5 | 0 | 7 | 0 | 5 | 5 |
| 6 | 0 | 9 | 0 | 6 | 6 |
| 7 | 0 | 10 | 0 | 7 | 7 |
| 8 | 1 | 0 | 1 | 0 | 8 |
| 9 | 1 | 1 | 2 | 0 | 17 |
| 10 | 1 | 2 | 3 | 0 | 25 |
| 20 | 2 | 5 | 4 | 0 | 33 |
| 30 | 3 | 7 | 5 | 0 | 42 |
| 40 | 4 | 10 | 6 | 0 | 50 |
| 50 | 6 | 0 | 7 | 0 | 58 |
| 60 | 7 | 2 | 8 | 0 | 67 |
| 70 | 8 | 5 | 9 | 0 | 75 |
| 80 | 9 | 7 | 10 | 0 | 83 |
| 90 | 10 | 10 | 11 | 0 | 92 |
| 100 | 12 | 0 | 12 | 0 | 100 |

EXAMPLE I.

If it be required to change 44 centeſms into inches and 12ths.——Againſt 4 and 40 in the firſt column, ſtand

ſtand reſpectively 6 parts, and 4 in. 10 pts, in the ſecond; then the ſum of theſe two is 5 inches 4 parts, the number required.

EXAMPLE II.

If it be required to change 5 inches and $\frac{3}{4}$ or 6 eighths to centeſms.——Againſt 6 eighths and 5 inches, in the third column, ſtand reſpectively 6 and 42 in the laſt column; the ſum of which is 48 centeſms.

The reaſon why I have ſet 8ths in the third column, and not 12ths, is that the meaſurers who calculate by 12ths take the dimenſions in 8ths, and change them into 12ths; becauſe the inches upon the rule are divided into 8ths, and not 12ths.

*Note.* You may convert centeſms into inches and parts, and the contrary, without the table, by the following rule, viz.

Multiply centeſms by 12 for inches, cutting off two figures; and multiply theſe two figures by 12 again for parts, cutting off two figures here alſo. And divide 8ths or 12ths, with two cyphers annexed, by 8 or by 12 for inches; and divide inches by 12 for centeſms.

EXAMPLE I.

Thus, taking the firſt example above.

44 centeſms
12

Inches 5·28 That is 44 centeſms =
12 5 inches 3 parts.

Parts 3·36

EX-

EXAMPLE II.

And in the ſecond example above,

| | |
|---|---|
| 8 | 6·00 |
| 12 | 5·75 |
| | 48 centeſms. |

I ſhall now proceed to the different kinds of works, in treating of which ſeparately, I ſhall note the methods of meaſuring the ſeveral articles, with the allowances, deductions, &c; and then, at the end of the whole, illuſtrate them all in the ſeveral parts of a real building, whoſe plan and elevation ſhall be laid down, with its ſcale annexed, for comparing the dimenſions, &c.

## CHAPTER III.

### *Of Bricklayers Work.*

Brickwork is eſtimated at the rate of a brick and a half thick; and if a wall be more or leſs than this ſtandard thickneſs, it muſt be reduced to it, as follows:

Multiply the ſuperficial content of the wall by the number of half bricks in the thickneſs, and divide the product by 3.

The dimenſions of a building are uſually taken by meaſuring half round on the outſide, and half round it on the inſide; the ſum of theſe two gives the compaſs of the wall, to be multiplied by the height for the content of the materials.

Chimneys are by ſome meaſured as if they were ſolid, deducting only the vacuity from the hearth to the mantle, on account of the trouble of them.

And by others they are girt or meaſured round for their

their breadth, and the height of the ſtory is their height, taking the depth of the jambs for their thickneſs. And in this caſe no reduction is made for the vacuity from the floor to the mantle-tree, becauſe of the gathering of the breaſt and wings, to make room for the hearth in the next ſtory.

To meaſure the chimney ſhafts, which appear above the building; girt them about with a line for the breadth, to multiply by their height. And account their thickneſs half a brick more than it really is, in conſideration of the plaſtering and ſcaffolding.

All windows, doors, &c, are to be deducted out of the contents of the walls in which they are placed. But this deduction is made only with regard to materials; for the whole meaſure is taken for workmanſhip, and that all outſide meaſure too, namely, meaſuring quite round the outſide of the building, being in conſideration of the trouble of the returns or angles. There are alſo ſome other allowances, ſuch as double meaſure for feathered gable ends, &c.

EXAMPLES.

1. How many yards and rods of ſtandard brickwork are in a wall, whoſe length or compaſs is 57 feet 3 inches, and height 24 feet 6 inches; the walls being $2\frac{1}{2}$ bricks, or 5 half bricks, thick?

| *Decimals.* | | *Duodecimals.* |
|---|---|---|
| 57·25 | | 57 3 |
| 24·5 | | 24 6 |
| 28625 | | 234 0 |
| 22900 | | 114 |
| 11450 | | 28 7 6 |
| 1402·625 | | 1402 7 6 |
| 5 | half bricks thick | 5 |
| 3 \| 7013·125 | 3 | 7013 1 6 |
| 9 \| 2337·708$\frac{1}{2}$ | sq. feet 9 | 2337, 8 6 |
| 259·745$\frac{10}{27}$ | yds | 259 6 8 6 |
| 4 | | 4 |
| 11 \| 1038·981$\frac{12}{27}$ | 11 | 1036 |
| 11 \| 94·4528 | 11 | 94 2 |
| rods 8·5866 | Anf. | 8 r 17yds 6f 8′ 6″ |

*By the Sliding Rule.*

B A B A
As 1 : 24$\frac{1}{2}$ : : 57$\frac{1}{4}$ : 1403.

Ex. 2. Required the content of a wall 62 feet 6 inches long, and 14 feet 8 inches high, and 2$\frac{1}{2}$ bricks thick. Anſ. 169·753 yards.

Ex. 3. A triangular gable is raiſed 17$\frac{1}{2}$ feet high, on an end wall whoſe length is 24 feet 9 inches, the thickneſs being 2 bricks: required the reduced content.
Anſ. 32·08$\frac{1}{4}$ yds.

Ex. 4. The end wall of a houſe is 28 feet 10 inches long, and 55 feet 8 inches high to the eaves; 20 feet high is 2$\frac{1}{2}$ bricks thick, other 20 feet high is 2 bricks thick, and the remaining 15 feet 8 inches is 1$\frac{1}{2}$ brick thick;

thick; above which is a triangular gable, which riſes 42 courſes of bricks, of which every 4 courſes make a foot. What is the whole content in ſtandard meaſure?

Anſ. 253·626 yards.

## CHAPTER IV.

### *Of Maſons Work.*

To maſonry belongs all ſort of ſtone work; and the meaſure made uſe of is a foot, either ſuperficial or ſolid.

Walls, columns, blocks of ſtone or marble, &c, are meaſured by the cubic foot; and pavements, ſlabs, chimney pieces, &c, by the ſuperficial or ſquare foot.

Cubic or ſolid meaſure is uſed for the materials, and ſquare meaſure for the workmanſhip.

In the ſolid meaſure, the true length, breadth, and thickneſs, are taken, and multiplied continually together. In the ſuperficial, there muſt be taken the length and breadth of every part of the projection, which is ſeen without the general upright face of the building.

EXAMPLES.

1. Required the ſolid content of a wall, 53 feet 6 inches long, 12 feet 3 inches high, and 2 feet thick.

| *Decimals.* | | *Duodecimals.* |
|---|---|---|
| 53·5 | | 53 6 |
| 12¼ | | 12 3 |
| 642·0 | | 642 0 |
| 13·375 | | 13 4 6 |
| 655·375 | | 655 4 6 |
| 2 | | 2 |
| 1310·750 | anſ. | 1310 9 0 |

By

*By the Sliding Rule.*

| B | A | B | A |
|---|---|---|---|
| 1 : | $53\frac{1}{2}$ :: | $12\frac{1}{4}$ : | 655 |
| 1 : | 655 :: | 2 : | 1310 |

Ex. 2. What is the ſolid content of a wall, the length being 12 feet 3 inches, height 10 feet 9 inches, and 2 feet thick? Anſ. 521·375 feet.

Ex. 3. Required the value of a marble ſlab, at 8s per foot; the length being 5 feet 7 inches, and breadth 1 foot 10 inches. Anſ. £4 1 $10\frac{1}{2}$.

Ex. 4. In a chimney-piece, ſuppoſe the
length of the mantle and ſlab, each 4f 6in
breadth of both together - - 3 2
length of each jamb - - 4 4
breadth of both together - 1 9
Required the ſuperficial content. Anſ. 21f 10in.

## CHAPTER V.

### *Of Carpenters and Joiners Work.*

To this branch belongs all the wood-work of a houſe, ſuch as flooring, partitioning, roofing, &c.

*Note.* Large and plain articles are uſually meaſured by the ſquare foot or yard, &c; but inriched mouldings, and ſome other articles, are often eſtimated by running or lineal meaſure, and ſome things are rated by the piece.

In meaſuring of joiſts, it is to be obſerved, that only one of their dimenſions is the ſame with that of the floor; and the other will exceed the length of the room by the thickneſs of the wall, and $\frac{1}{3}$ of the ſame, becauſe each end is let into the wall about $\frac{1}{3}$ of its thickneſs.

No deductions are made for hearths, on account of the additional trouble and waſte of materials.

*Partitions* are meaſured from wall to wall for one dimenſion, and from floor to floor, as far as they extend, for the other.

No deduction is made for door-ways, on account of the trouble of framing them.

In meaſuring of joiners work, the ſtring is made to ply cloſe to every part of the work over which it paſſes.

*The meaſure of centering for cellars* is found by making a ſtring paſs over the ſurface of the arch for the breadth, and taking the length of the cellar for the length; but in groin-centering, it is uſual to allow double meaſure, on account of their extraordinary trouble.

*In roofing*, the length of the houſe in the inſide, together with $\frac{2}{7}$ of the thickneſs of one gable, is to be conſidered as the length; and the breadth is equal to double the length of a ſtring which is ſtretched from the ridge down the rafter, and along the eaves-board, till it meets with the top of the wall.

*For ſtair-caſes*, take the breadth of all the ſteps, by making a line ply cloſe over them, from the top to the bottom, and multiply the length of this line by the length of a ſtep for the whole area.—By the length of a ſtep is meant the length of the front and the returns at the two ends; and by the breadth is to be underſtood the girt of its two upper ſurfaces, or the tread and riſer.

*For the baluſtrade*, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel poſt, for the length; and twice the length of the baluſter upon the landing, with the girt of the hand-rail, for the breadth.

*For wainſcotting*, take the compaſs of the room for the length; and the height from the floor to the ceiling, making the ſtring ply cloſe into all the mouldings, for the breadth.—Out of this muſt be made deductions for windows, doors and chimneys, &c: but workmanſhip

is

is counted for the whole, on account of the extraordinary trouble.

*For doors*, it is uſually to allow for their thickneſs, by adding it into both the dimenſions of length and breadth, and then multiply them together for the area. —If the door be pannelled on both ſides, take double its meaſure for the workmanſhip; but if one ſide only be pannelled, take the area and its half for the workmanſhip.—*For the ſurrounding architrave*, girt it about the outermoſt part for its length; and meaſure over it, as far as it can be ſeen when the door is open, for the breadth.

*Window-ſhutters, baſes*, &c, are meaſured in the ſame manner.

In the meaſuring of roofing for workmanſhip alone, all holes for chimney ſhafts and ſky-lights are generally deducted.

But in meaſuring for work and materials, they commonly meaſure in all ſky-lights, luthern-lights, and holes for the chimney ſhafts, on account of their trouble and waſte of materials.

EXAMPLES.

1. Required the content of a floor 48 feet 6 inches long, and 24 feet 3 inches broad.

| *Decimals.* | | | *Duodecimals.* | | |
|---|---|---|---|---|---|
| 48·5 | | | 48 | 6 | |
| 24¼ | | | 24 | 3 | |
| 1940 | | | 204 | 0 | |
| 970 | | | 96 | | |
| 12·125 | | | 12 | 1 | 6 |
| 1176·125 | feet | | 1176 | 1 | 6 |
| 11·76125 | ſquares | Anſ. | 11 | 76 | 1 6 |

Ex. 2. A floor being 36 feet 3 inches long, and 16 feet 6 inches broad, how many ſquares are in it?

Anſ. 5ſq. 98⅛feet.

Ex. 3. How many ſquares are there in 173 feet 10 inches in length, and 10 feet 7 inches height, of partitioning? Anſ. 18·3972 ſquares.

Ex. 4. What coſt the roofing of a houſe at 10s 6d a ſquare; the length, within the walls, being 52 feet 8 inches, and the breadth 30 feet 6 inches; reckoning the roof $\frac{1}{2}$ of the flat? Anſ. £12 12 11$\frac{3}{4}$.

Ex. 5. To how much, at 6s per ſquare yard, amounts the wainſcotting of a room; the height, taking in the cornice and mouldings, being 12 feet 6 inches, and the whole compaſs 83 feet 8 inches; alſo the three window ſhutters are each 7 feet 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the door and ſhutters, being worked on both ſides, are reckoned work and half work? Anſ. £36 12 2$\frac{1}{2}$

## CHAPTER VI.

### *Of Slaters and Tilers Work.*

In theſe articles, the content of a roof is found by multiplying the length of the ridge by the girt over from eaves to eaves; making allowance in this girt for the double row of ſlates at the bottom, or for how much one row of ſlates or tiles is laid over another.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip. And in tiling and ſlating, it is common to add the length of the valley or hip to the content in feet.

Deductions are ſeldom made for chimney ſhafts or ſmall window holes.

### EXAMPLES.

1. Required the content of a ſlated roof, the length being 45 feet 9 inches, and whole girt 34 feet 3 inches.

*Decimals.*

| *Decimals.* | | *Duodecimals.* |
|---|---|---|
| 45·75 | | 45 9 |
| 34¼ | | 34 3 |
| 18300 | | 205 6 |
| 13725 | | 135 |
| 114375 | | 11′ 5 3 |
| 9) 1566·9375 | feet | 9) 1566 11 3 |
| yds 174·104 | Anſwer | 174ʸ 11ⁱ 3″ |

Ex. 2. To how much amounts the tiling of a houſe, at 25s 6d per ſquare; the length being 43 feet 10 inches, and the breadth on the flat 27 feet 5 inches, alſo the eaves projecting 16 inches on each ſide?

Anſ. £24 9 5½.

## CHAPTER VII.

### *Of Plaſterers Work.*

Plaſterers work is of two kinds, namely, ceiling, which is plaſtering upon laths; and rendering, which is plaſtering upon walls: which are meaſured ſeparately.

The contents are eſtimated either by the foot, or yard, or ſquare of 100 feet. Inriched mouldings, &c, are rated by running or lineal meaſure.

Deductions are to be made for chimneys, doors, windows, &c. But the windows are ſeldom deducted, as the plaſtered returns at the top and ſides are allowed to compenſate for the window opening.

### EXAMPLES.

1. How many yards contains the ceiling, which is 43 feet 3 inches long, and 25 feet 6 inches broad?

| *Decimals.* | *Duodecimals.* |
|---|---|
| 43·25 | 43 3 |
| 25½ | 25 6 |
| 21625 | 221 3 |
| 8650 | 86 |
| 21625 | 21 7 6 |
| 9) 1102·875 | 9) 1102 10 6 |
| yds 122·541 | Anſwer 122ʸ 4ᶠ 10ⁱ 6″ |

Ex. 2. To how much amounts the ceiling of a room, at 10d per yard; the length being 21 feet 8 inches, and the breadth 14 feet 10 inches? Anſ. £1 9 8¾

Ex. 3. The length of a room is 18 feet 6 inches, the breadth 12 feet 3 inches, and height 10 feet 6 inches; to how much amounts the ceiling and rendering, the former at 8d and the latter at 3d per yard; allowing for the door of 7 feet by 3 feet 8, and a fire place of 5 feet ſquare? Anſ. £8 5 6½.

Ex. 4. Required the quantity of plaſtering in a room, the length being 14 feet 5 inches, breadth 13 feet 2 inches, and height 9 feet 3 inches to the under ſide of the cornice, which girts 8½ inches, and projects 5 inches from the wall on the upper part next the ceiling: deducting only for a door 7 feet by 4.

| | | | |
|---|---|---|---|
| Anſ. 53ʸᵈ | 5ᶠ | 3ⁱ | of rendering |
| 18 | 5 | 6 | of ceiling |
| | 39 | 0 1 1/2 | of cornice. |

## CHAPTER VIII.

### *Of Painters Work.*

Painters work is computed in ſquare yards. Every part is meaſured where the colour lies; and the mea-

meaſuring line is forced into all the mouldings and corners.

Windows are done at ſo much a piece. And it is uſual to allow double meaſure for carved mouldings, &c.

EXAMPLES.

1. How many yards of painting contains the room which is 65 feet 6 inches in compaſs, and 12 feet 4 inches high?

| *Decimals.* | | *Duodecimals.* | |
|---|---|---|---|
| 65·5 | | 65 | 6 |
| $12\frac{1}{2}$ | | 12 | 4 |
| 786·0 | | 786 | 0 |
| 21·83 | | 21 | 10 |
| 9) 807·83 | | 9) 807 | 10 |
| 89·788 | Anſwer | 89 | 6 10 |

Ex. 2. The length of room being 20 feet, its breadth 14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting are in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two windows each 6 feet by 3 feet 2 inches? Anſ. $73\frac{2}{17}$ yards.

Ex. 3. What coſt the painting of a room, at 6d per yard; its length being 24 feet 6 inches, its breadth 16 feet 3 inches, and height 12 feet 9 inches; alſo the door is 7 feet by 3 feet 6, and the window ſhutters to two windows each 7 feet 9 by 3 feet 6, but the breaks of the windows themſelves are 8 feet 6 inches high, and 1 foot 3 inches deep: deducting the fire-place of 5 feet by 5 feet 6? Anſ. £3 3 $10\frac{1}{2}$.

CHAP.

## CHAPTER IX.

### *Of Glaziers Work.*

Glaziers take their dimenſions either in feet, inches, and parts, or feet, tenths, and hundredths. And compute their work in ſquare feet.

In taking the length and breadth of a window, the croſs bars between the ſquares are included. Alſo windows of round or oval forms are meaſured as ſquare, meaſuring them to their greateſt length and breadth, on account of the waſte in cutting the glaſs.

EXAMPLES.

1. How many ſquare feet contains the window which is 4·25 feet long, and 2·75 feet broad?

| *Decimals.* | | *Duodecimals.* |
|---|---|---|
| 2·75 | | 2 9 |
| 4¼ | | 4 3 |
| 11·00 | | 11 0 |
| 6875 | | 8 3 |
| 11·6875 | Anſwer | 11 8 3 |

2. What will the glazing a triangular ſky-light come to at 10d per foot; the baſe being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches?

Anſ. £4 7 2¼.

3. There is a houſe with three tier of windows, three windows in each tier, their common breadth 3 feet 11 inches;

now the height of the firſt tier is 7ᶠ 10ⁱⁿ
of the ſecond 6 8
of the third 5· 4

Required the expence of glazing at 14d per foot.

Anſ. £13 11 10½.

Ex. 4.

Ex. 4. Required the expence of glazing the windows of a houſe at 13d a foot; there being three ſtories, and three windows in each ſtory;

the height of the lower tier is $7^{f}\ 9^{in}$
of the middle - - 6 6
of the upper - - 5 $3\frac{1}{4}$
and of an oval window over the door 1 $10\frac{1}{2}$

The common breadth of all the windows being 3 feet 9 inches. Anſ. £ 12 5 6.

## CHAPTER X.

### *Of Pavers Work.*

Pavers work is done by the ſquare yard. And the content is found by multiplying the length by the breadth.

EXAMPLES.

1. What coſt the paving a foot-path at 3s 4d a yard; the length being 35 feet 4 inches, and breadth 8 feet 3 inches?

*Decimals.*

35·3
$8\frac{1}{4}$
282·66
8·83
9)291·5
32·38
2s is $\frac{1}{10}$ 3·2388
1 is $\frac{1}{2}$ 1·6194
4d is $\frac{1}{3}$ 5398
l 5·3981
20
s 7·9620
12
d 11·5440

*Duodecimals.*

35 4
8 3
282 8
8 10
9)291 6
Content 32 3 6
£ 0 3 4
4
0 13 4
8
5 6 8
$3^{rd}$ is $\frac{1}{3}$ 1 $1\frac{1}{4}$
$6^{f}$ is $\frac{1}{6}$ $2\frac{1}{4}$
Anſwer £ 5 7 $11\frac{1}{2}$

Ex. 2. What coſt the paving a court at 3s 2d per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches? Anſ. £7 4 $4\frac{1}{7}$.

Ex. 3. What will be the expence of paving a rectangular court yard, whoſe length is 63 feet, and breadth 45 feet; in which there is laid a foot-path of 5 feet 3 inches broad, running the whole length, with broad ſtones at 3s a yard; the reſt being paved with pebbles, at 2s 6d a yard? Anſ. £40 5 $10\frac{1}{2}$.

## CHAPTER XL.

### *Of Plumbers Work.*

Plumbers work is commonly done by the pound or hundred weight; but I ſhall below inſert a method of diſcovering the weight by the meaſure of it.

Sheet-lead uſed in roofing, guttering, &c, is commonly between 7 and 12 pound weight to the ſquare foot; but the following table ſhews by inſpection the particular weight of a ſquare foot for each of ſeveral thickneſſes.

| Thickneſs. | lb. to a ſq. foot | Thickneſs. | lb. to a ſq. foot | Thickneſs. | lb. to a ſq. foot. | Thickneſs. | lb. to a ſq. foot. |
|---|---|---|---|---|---|---|---|
| ·10 | 5·899 | $\frac{1}{8}$ | 7·373 | ·15 | 8·848 | ·18 | 10·618 |
| ·11 | 6·489 | ·13 | 7·668 | ·16 | 9·438 | ·19 | 11·207 |
| $\frac{1}{9}$ | 6·554 | ·14 | 8·258 | $\frac{1}{6}$ | 9·831 | ·2 = $\frac{1}{5}$ | 11·797 |
| ·12 | 7·078 | $\frac{1}{7}$ | 8·427 | ·17 | 10·028 | ·21 | 12·387 |

In this table the thickneſs is ſet down in tenths and hundredths, &c, of an inch; and the annexed correſponding numbers are the weights in avoirdupois pounds, and thouſandth parts of a pound. So the weight of a ſquare foot of $\frac{1}{10}$ or $\frac{100}{1000}$ of an inch

thick, is 5 pounds and 899 thouſandth parts of a pound; and the weight of a ſquare foot to $\frac{1}{9}$ of an inch thickneſs, is 6 pounds and $\frac{554}{1000}$ of a pound.

Leaden pipe of an inch bore, is commonly 13 or 14 lb to the yard in length.

### EXAMPLES.

1. How much weighs the lead which is 39 feet 6 inches long, and 3 feet 3 inches broad, at $8\frac{1}{2}$lb to the ſquare foot?

| *Decimals.* | | *Duodecimals.* |
|---|---|---|
| 39·5 | | 39 6 |
| $3\frac{1}{4}$ | | 3 3 |
| 118·5 | | 118 6 |
| 9·875 | | 9 10 6 |
| 128·375 | | 128 4 6 |
| $8\frac{1}{2}$ | | $8\frac{1}{2}$ |
| 1027·000 | | 1024 |
| 64·1875 | | 64 |
| | | $2\frac{5}{6}$ |
| | | $0\frac{17}{48}$ |
| 1091·1875 | Anſwer | $1091\frac{9}{48}$ lb. |

Ex. 2. What coſt the covering and guttering a roof with lead at 18s the cwt; the length of the roof being 43 feet, and breadth or girt over it 32 feet; the guttering 57 feet long, and 2 feet wide; the former 9·831lb, and the latter 7·373lb to the ſquare foot? Anſ. £115 9 $1\frac{1}{2}$.

CHAP-

## CHAPTER XII.

## OF VAULTED AND ARCHED ROOFS.

ARCHED roofs are either vaults, domes, ſaloons, or groins.

Vaulted roofs are formed by arches ſpringing from the oppoſite walls, and meeting in a line at the top.

Domes are made by arches ſpringing from a circular or polygonal baſe, and meeting in a point at the top.

Saloons are formed by arches connecting the ſide walls to a flat roof or ceiling in the middle.

Groins are formed by the interſection of vaults with each other.

Vaulted roofs are commonly of the three following ſorts:

1. Circular roofs, or thoſe whoſe arch is ſome part of the circumference of a circle.

2. Elliptical or oval roofs, or thoſe whoſe arch is an oval, or ſome part of the circumference of an ellipſis.

3. Gothic roofs, or thoſe which are formed by two circular arcs, ſtruck from different centers, and meeting in a point directly over the middle of the breadth, or ſpan of the arch.

### PROBLEM I.

### *To find the Surface of a Vaulted Roof.*

Multiply the length of the arch by the length of the vault, and the product will be the ſuperficies.

*Note.* To find the length of the arch, make a line or ſtring ply cloſe to it, quite acroſs from ſide to ſide.

EXAM-

EXAMPLES.

1. Required the ſurface of a vaulted roof, the length of the arch being 31·2 feet, and the length of the vault 120 feet.

31·2
120

Anſ. 3744·0 ſquare feet.

Ex. 2. How many ſquare yards are in the vaulted roof, whoſe arch is 42·4 feet, and the length of the vault 106 feet? Anſ. 499·37 yds.

PROBLEM II.

*To find the Content of the Concavity of a Vaulted Roof.*

Multiply the length of the vault by the area of one end, that is, by the area of a vertical tranſverſe ſection, for the content.

NOTE. When the arch is an oval, multiply the ſpan by the height, and the product by ·7854 for the area.

EXAMPLES.

1. Required the content of the concavity of a ſemicircular vaulted roof, the ſpan or diameter being 30 feet, and the length of the vault 150 feet.

·7854
900 the ſquare of 30
2)706·86
353·43 area of the end
150 the length
1767150
35343
5301·450 the anſwer.

Ex. 2.

Ex. 2. What is the content of the vacuity of an oval vault, whoſe ſpan is 30 feet, and height 12 feet; the length of the vault being 60 feet? Anſ. 1694·64.

Ex. 3. Required the content of the vacuity of a gothic vault, whoſe ſpan is 50 feet, the chord of each arc 50 feet, and the diſtance of each arc from the middle of theſe chords 15 feet; alſo the length of the vault 20. Anſ. 42988·8.

## PROBLEM III.

### *To find the Superficies of a Dome.*

Find the area of the baſe, and double it; then ſay, as the radius of the baſe, is to the height of the dome, ſo is the double area of the baſe, to the ſuperficies.

NOTE. For the ſuperficies of a hemiſpherical dome, take the double area of the baſe only.

### EXAMPLES.

1. To how much comes the painting of an octagonal ſpherical dome, at 8d per yard; each ſide of the baſe being 20 feet?

4·828427 tabular area
400 ſquare of 20

1931·3708 area of the baſe
2

9) 3862·7416 ſuperficies in feet
429·1934 yards
8

13 | 3433·5472
20 | 28·6 1½d
£14 6 1½ anſwer

Ex. 2.

Ex. 2. Required the ſuperficies of a hexagonal ſpherical dome, each ſide of the baſe being 10 feet.
Anſ. 519·6152.

Ex. 3. What is the ſuperficies of a dome with a circular baſe, whoſe circumference is 100 feet, and height 20 feet? Anſ. 2000 feet.

PROBLEM IV.

*To find the Solid Content of a Dome.*

Multiply the area of the baſe by $\frac{2}{3}$ of the height.

EXAMPLES.

1. Required the ſolid content of an octagonal dome, each ſide of the baſe being 20 feet, and the height 21 feet.

4·828427
400
1931·3708 area of the baſe
14 $\frac{2}{3}$ of the height
77254832
19313708
27039·1912 anſwer

Ex. 2. What is the ſolid content of a ſpherical dome, the diameter of whoſe circular baſe is 30 feet?
Anſ. 7068·6 feet.

PROBLEM V.

*To find the Superficies of a Saloon.*

Find its breadth by applying a ſtring cloſe to it

acroſs the ſurface. Find alſo its length by meaſuring along the middle of it, quite round the room.

Then multiply theſe two together for the ſurface.

EXAMPLE.

The girt acroſs the face of a ſaloon being 5 feet, and its mean compaſs about 100 feet, required the area or ſuperficies.

100
5
500 anſwer.

PROBLEM VI.

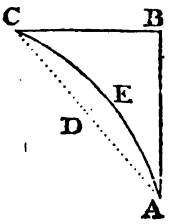
*To find the Solid Content of a Saloon.*

Multiply the area of a tranſverſe ſection by the compaſs taken round the middle part. Subtract this product from the whole vacuity of the room, ſuppoſing the walls to go upright all the height to the flat ceiling. And the difference will be the anſwer.

EXAMPLE.

If the height AB of the ſaloon be 3·2 feet, the chord ADC of its front 4·5, and the diſtance DE of its middle part from the arch be 9 inches; required the ſolidity, ſuppoſing the mean compaſs to be 50 feet.

2)4·5




2 ) 4·5

2·25 AD
2·25

1125
450
450

5·0625 AD²
·5625 DE²

5·6250 (2·37 AE
4 4

43 | 162 3) 9·48
3 | 129 3·16 = $\frac{4}{3}$ AE
46 | 33 4·50 AC

7·66
·3 = $\frac{4}{10}$ DE
2·298 area seg. ADCEA.

0·75 DE
0·75

375
525

·5625

Again 3·2 4·5
3·2 4·5
64 225
96 180
10·24 20·25 AC²
10·24 AB²

10·01 ( 3·16 = BC
9 1·6 = $\frac{1}{2}$ AB
61 | 1·01 1896
1 | 61 316
40 5·056 area of triangle ABC
2·298 area seg.
2·758 area of section AECBA
50 compass
137·900 content of the solid part

Then this taken from the whole upright ſpace, will leave the content of the vacuity contained within the room.

### PROBLEM VII.

*To find the Concave Superficies of a Groin.*

To the area of the baſe, add $\frac{1}{7}$ part of itſelf, for the ſuperficial content.

### EXAMPLES.

1. What is the ſuperficial content of the groin arch, raiſed on a ſquare baſe of 15 feet on each ſide?

15
15
---
75
15
---
7 ) 225 area of the baſe
$32\frac{1}{7}$ its 7th part
---
$257\frac{1}{7}$ anſwer

Ex. 2. Required the ſuperficies of a groin arch, raiſed on a rectangular baſe, whoſe dimenſions are 20 feet by 16. Anſ. $365\frac{5}{7}$.

### PROBLEM VIII.

*To find the Solid Content of a Groin Arch.*

Multiply the area of the baſe by the height; from the product ſubtract $\frac{1}{10}$ of itſelf, and the remainder will be the content of the vacuity.

EXAM-

EXAMPLES.

1. Required the content of the vacuity within a groin arch, ſpringing from the ſides of a ſquare baſe, each ſide of which is 16 feet.

| | |
|---|---|
| 16 | |
| 16 | |
| 96 | |
| 16 | |
| 256 | area of baſe |
| 8 | height or radius |
| 2048 | |
| $204\frac{4}{5}$ | $\frac{1}{10}$ ſubtract |
| $1843\frac{1}{5}$ | anſwer |

2. What is the content of the vacuity below an oval groin, the ſide of its ſquare baſe being 24 feet, and its height 8 feet? Anſ. $4147\frac{1}{5}$.

NOTES.

1. To find the ſolid content of the brick or ſtone work, which forms any arch or vault: Multiply the area of the baſe by the height, including the work over the top of the arch; and from the product ſubtract the content of the vacuity found by the foregoing problems; and the remainder will be the content of the ſolid materials.

2. In groin arches however, it is uſual to take the whole as ſolid, without deducting the vacuity, on account of the trouble and waſte of materials, attending the cutting and fitting them to the arch.

## CHAPTER XIII.

### GENERAL ILLUSTRATION.

HAVING gone through the meaſurable parts of building, and noted the methods of meaſuring them, and of computing the contents; I proceed now to the general illuſtration of the whole, by aſſigning the dimenſions of a houſe, and from thence computing the contents of the ſeveral different kinds of works ſuppoſed to be uſed in it; in the performing of which are ſhewn the methods of ruling the book, and entering the dimenſions, with the contents; then the method of abſtracting the contents, and laſtly of forming the bills of expences of the work and materials.

At the end of this chapter is given a plate, containing the elevation and plans of the ſeveral ſtories of the houſe, accurately delineated from a particular ſcale, inſerted in the plate; by which the reader may, with a pair of compaſſes, meaſure and compare the dimenſions of moſt of the articles, as he proceeds through them. In this example is uſed a houſe of only two ſtories, and two rooms on each floor, as thoſe are very ſufficient for explaining the ſeveral methods and works here treated of.

The columns of numbers in the following forms, are ſufficiently explained by the titles at the tops of them; excepting the figures 2, 3, &c, in the firſt column, and prefixed to the dimenſions; the meaning of which figures is, that the contents ariſing from the dimenſions to which they are prefixed, are to be multiplied by 2, 3, &c, and then ſet down in the column of contents.

Of

## *Of the* DIGGING *of the* CELLARS.

| *Feet* | *In.* | |
|---|---|---|
| 54 | 0 | length |
| 28 | 0 | breadth |
| 8 | 0 | depth |

Which three dimenſions, being multiplied continually together, produce 12096 cubic feet; which being divided by 27, the cubic feet in a yard, there reſult 448 cubic yards of digging.

Thus:

```
  54
  28
 ----
 432
 108
 ----
1512
   8
 ----
27 ) 12096 ( 448
     108
    ----
     129
     108
    ----
      216
      216
     ----
```

## *The* BRICK WORK.

| | Dimensions. | | Half bricks thick | Contents | | Titles |
|---|---|---|---|---|---|---|
| | *Feet* | *In.* | | *Feet* | *In.* | |
| | 155 | 0 | 6 | 1550 | 0 | The cellar walls |
| | 10 | 0 | | | | |
| | 23 | 6 | 8 | 470 | 0 | Middle walls of ditto |
| | 10 | 0 | | | | |
| 2 | 6 | 8 | 8 | 48 | 10 | Deduct doors in ditto |
| | 3 | 8 | | | | |
| 2 | 23 | 6 | 1 | 799 | 0 | Paving the cellars, brick on edge |
| | 17 | 0 | | | | |
| | 23 | 6 | 1 | 211 | 6 | Ditto between the cellars |
| | 9 | 0 | | | | |
| 2 | 24 | 0 | 8 | 840 | 0 | Groin arches over the two cellars |
| | 17 | 6 | | | | |
| | 24 | 0 | 8 | 228 | 0 | Common arch under the paſſage |
| | 9 | 6 | | | | |
| | 23 | 6 | 4 | 211 | 6 | Deduct vacuity of ditto |
| | 9 | 0 | | | | |
| | 155 | 0 | 5 | 3797 | 6 | Out walls of the principal and attic ſtory |
| | 24 | 6 | | | | |
| 2 | 24 | 0 | 7 | 1176 | 0 | Middle walls of ditto |
| | 24 | 6 | | | | |
| 2 | 7 | 0 | 6 | 112 | 0 | Chimney ſhafts within the roof |
| | 8 | 0 | | | | |

Di-

| | Dimen-sions. | | Half bricks thick | Products | | Titles |
|---|---|---|---|---|---|---|
| | *Feet* | *In.* | | *Feet* | *In.* | *Deduct as follows, viz.* |
| 2 | 8 | 8 | 5 | 80 | 10 | Windows in front |
| | 4 | 8 | | | | |
| 3 | 4 | 8 | 5 | 65 | 4 | Ditto |
| | 4 | 8 | | | | |
| | 11 | 0 | 5 | 66 | 0 | Front door |
| | 6 | 0 | | | | |
| 4 | 8 | 0 | 5 | 128 | 0 | End windows |
| | 4 | 0 | | | | |
| 2 | 4 | 0 | 5 | 32 | 0 | Ditto |
| | 4 | 0 | | | | |
| 2 | 4 | 0 | 1 | 32 | 0 | Ditto blind or falſe |
| | 4 | 0 | | | | |
| | 4 | 0 | 5 | 32 | 0 | Stair-caſe window |
| | 8 | 0 | | | | |
| 6 | 6 | 8 | 7 | 146 | 8 | Doors in middle walls |
| | 3 | 8 | | | | |
| 2 | 6 | 8 | 1 | 48 | 10 | Ditto falſe |
| | 3 | 8 | | | | |
| 2 | 4 | 6 | 5 | 40 | 6 | Receſs of chimneys |
| | 4 | 6 | | | | |
| 2 | 3 | 10 | 5 | 29 | 4 | Ditto |
| | 3 | 10 | | | | |
| 2 | 3 | 6 | 5 | 24 | 6 | Ditto |
| | 3 | 6 | | | | |

To

To abſtract the foregoing particulars, that is, to collect together the contents of the ſame kind, next to reduce their ſums to the ſtandard thickneſs of a brick and a half, and then, having made the deductions, to collect the reduced ſums, to obtain the whole in one ſum, I proceed thus: I make only two columns for the whole contents, and two for the deductions of the ſame thickneſs, viz. one column for the one-brick contents, and the other for the brick-and-half contents; and diſpoſe the ſuperior denominations in one or both of theſe columns, by placing them down more than once; thus, the contents of 2 bricks thick, I ſet down twice in the one-brick column; thoſe of $2\frac{1}{2}$ bricks, I ſet down once in the one-brick column, and once in the brick-and-half column; thoſe of 3 bricks, twice in the brick-and-half column; thoſe of $3\frac{1}{2}$ bricks, once in the brick-and-half, and twice in the one-brick column; and thoſe of four bricks, twice in the brick-and-half, and once in the one-brick column; and ſo on, if there were higher denominations; and, laſtly, for the contents of half a brick thick, I take the halves of them, and ſet in the one-brick column. Then, having added up every column, I reduce the ſums of the one-brick thick, to the brick-and-half thickneſs; and add this reduced content to the brick-and-half ſums, both in the ſolids and deductions; then, laſtly, taking the one ſum from the other, for the whole content of the brick work, as follows.

*Abſtract*

*Abstract of the Brick Work.*

| 1½ br. thick | | 1 brick thick | | 1½ deduct | | 1 br. deduct | |
|---|---|---|---|---|---|---|---|
| *Feet* | *In.* | *Feet* | *In.* | *Feet* | *In.* | *Feet* | *In.* |
| 1550 | 0 | 470 | 0 | 48 | 10 | 48 | 10 |
| 1550 | 0 | 399 | 6 | 48 | 10 | 211 | 6 |
| 470 | 0 | 105 | 9 | 80 | 10 | 211 | 6 |
| 470 | 0 | 840 | 0 | 65 | 4 | 80 | 10 |
| 840 | 0 | 228 | 0 | 66 | 0 | 65 | 4 |
| 840 | 0 | 3797 | 6 | 128 | 0 | 66 | 0 |
| 228 | 0 | 1176 | 0 | 32 | 0 | 128 | 0 |
| 228 | 0 | 1176 | 0 | 32 | 0 | 32 | 0 |
| 3797 | 6 | 8192 | 9 | 146 | 8 | 16 | 0 |
| 1176 | 0 | | 2 | 40 | 6 | 32 | 0 |
| 112 | 0 | 3)16385 | 6 | 29 | 4 | 146 | 8 |
| 112 | 0 | 5461 | 10 | 24 | 6 | 146 | 8 |
| 11373 | 6 | | | 742 | 10 | 24 | 5 |
| 5461 | 10 | | | 869 | 5 | 40 | 6 |
| 16835 | 4 | | | 1612 | 3 | 29 | 4 |
| 1612 | 3 | | | | | 24 | 6 |
| 9)15223 | 1 | | | | | 1304 | 1 |
| 1691⅘ yds | | | | | | | 2 |
| | | | | | | 3)2608 | 2 |
| | | | | | | 869 | 5 |

*Feet*

| Feet | In. | |
|---|---|---|
| 2 | 0 | thick. of walls |
| | 4 | multiply |
| 8 | 0 | |
| 24 | 6 | height of 2 ſtories |
| 196 | 0 | |
| | 5 | half bricks thick |
| 3)980 | 0 | |

| Feet | In. | |
|---|---|---|
| 211 | 6 | vacu. arch. |
| | 4 | half br. thick |
| 3)846 | 0 | |
| 282 | 0 | reduced content of the vacuity of the com. arch. |

| | | |
|---|---|---|
| 326 | 8 | reduced cont. of the 4 quoins of the houſe |
| 1612 | 3 | the deductions |
| 1933 | 11 | ſum |
| 282 | 0 | vacuity of arch under the paſſage, deduct |
| 9)1656 | 11 | |
| 184 | | yards to be added for workmanſhip only. |

*So that in the Bill we ſhall have*

| | £ | s | d |
|---|---|---|---|
| 1691 yards 4 feet of brick-and-half wall, work and materials at — per yard | — | — | — |
| 184 yards for workmanſhip, in doors and windows, &c, at — per yard | — | — | — |
| £ | — | — | — |

The

## *The* MASON'S WORK.

| Dimenſions | | Contents | | Titles. |
|---|---|---|---|---|
| *Feet* | *In.* | *Feet* | *In.* | |
| 157 | 0 | 340 | 2 | Stone baſe |
| 2 | 2 | | | |
| 157 | 0 | 104 | 8 | Facia |
| 0 | 8 | | | |
| 6 4 | 8 | 21 | 0 | Window ſoles |
| 0 | 9 | | | |
| 27 | 0 | 63 | 0 | Jambs and head of front door, plain. |
| 2 | 4 | | | |
| 2 9 | 0 | 54 | 0 | Columns to front door |
| 3 | 0 | | | |
| 4 3 | 8 | 22 | 0 | Baſe and capitals to ditto (commonly taken double meaſure) |
| 1 | 6 | | | |
| 11 | 6 | 10 | 6 | Architrave |
| 0 | 11 | | | |
| 9 | 6 | 7 | 11 | Soffit of ditto |
| 0 | 10 | | | |
| 11 | 6 | 7 | 8 | Frize |
| 0 | 8 | | | |
| 13 | 10 | 20 | 9 | Level cornice |
| 1 | 6 | | | |
| 12 | 0 | 30 | 0 | Cornice of pediment |
| 2 | 6 | | | |
| 2 1 | 6 | 1 | 0 | Returns of cima-recta |
| 0 | 4 | | | |

Di-

| | Dimenſions | | Contents | | Titles |
|---|---|---|---|---|---|
| | *Feet* | *In.* | *Feet* | *In.* | |
| 2 | 22 | 8 | 98 | 2 | Architraves of low windows |
| | 2 | 2 | | | |
| 2 | 5 | 8 | 6 | 1 | Frize of ditto |
| | 0 | $6\frac{1}{2}$ | | | |
| 2 | 7 | 9 | 21 | 11 | Cornice of ditto |
| | 1 | 5 | | | |
| 3 | 21 | 4 | 138 | 8 | Architrave of attic windows |
| | 2 | 2 | | | |
| | 120 | 2 | 250 | 4 | Cornice to front and ends of the houſe |
| | 2 | 1 | | | |
| | 109 | 6 | 200 | 9 | Blocking courſe to front and ends |
| | 1 | 10 | | | |
| 4 | 5 | 6 | 34 | 10 | Steps to front |
| | 1 | 7 | | | |
| 2 | 8 | 6 | 34 | 0 | Ends of ditto |
| | 2 | 0 | | | |
| 2 | 3 | 0 | 8 | 0 | Ditto |
| | 1 | 4 | | | |
| | 9 | 0 | 27 | 0 | Top of the landing |
| | 3 | 0 | | | |
| 2 | 14 | 0 | 98 | 0 | Chimney tops of ſtone |
| | 3 | 6 | | | |
| 2 | 15 | 4 | 51 | 1 | Baſe and facias of ditto |
| | 1 | 8 | | | |

| | Dimenſions | | Contents | | Titles |
|---|---|---|---|---|---|
| | *Feet* | *In.* | *Feet* | *In.* | |
| 2 | 5 | 0 | 21 | 8 | Poliſhed ſlabs to the chimneys in the low rooms |
| | 2 | 2 | | | |
| 2 | 3 | 9 | 13 | 9 | Back ſlabs to ditto |
| | 1 | 10 | | | |
| 2 | 12 | 1 | 22 | 1 | Jambs and mantles to ditto |
| | 0 | 11 | | | |
| 4 | 4 | 4 | 26 | 0 | Coves to ditto |
| | 1 | 6 | | | |
| 2 | 4 | 8 | 18 | 8 | Slabs to chimneys in the attic ſtory |
| | 2 | 0 | | | |
| 2 | 3 | 6 | 12 | 10 | Back ſlabs |
| | 1 | 10 | | | |
| 2 | 11 | 2 | 20 | 5 | Jambs and mantles |
| | 0 | 11 | | | |
| 4 | 4 | 0 | 24 | 0 | Coves |
| | 1 | 6 | | | |
| 2 | 4 | 2 | 16 | 8 | Slabs |
| | 2 | 0 | | | |
| 2 | 3 | 0 | 11 | 0 | Back ſlabs |
| | 1 | 10 | | | |
| 2 | 10 | 4 | 17 | 2 | Jambs and mantles |
| | 0 | 10 | | | |
| 4 | 3 | 6 | 21 | 0 | Coves |
| | 1 | 6 | | | |
| 12 | 3 | 6 | 56 | 0 | Steps down to the cellar |
| | 1 | 4 | | | |

*Ab-*

*Abſtract of the Maſonry.*

In making this abſtract, the articles of the ſame price muſt be collected together. So in the firſt column, are collected together the baſe, facias, window ſoles, jambs and head of door, blocking courſe and chimney tops, as being of the ſame value; in the 2d are collected architraves, frizes, cornices, mouldings about the door, columns, baſes, and capitals; and ſo for other things of another value.

| Baſe, facias, &c | | Architraves, &c | | Chimney ſlabs, &c | | Front ſteps | | Chimney jambs, &c | | Cornice of the houſe | | Cellar ſteps | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| *Feet* | *I* | *Feet* | *I* | *Feet* | *I* | *Feet* | *I* | *Feet* | *I* | *Feet* | *I* | *Feet* | *I* |
| 340 | 2 | 54 | 0 | 21 | 8 | 34 | 10 | 22 | 1 | 250 | 4 | 56 | 0 |
| 104 | 8 | 22 | 0 | 13 | 9 | 34 | 0 | 20 | 5 | | | | |
| 21 | 0 | 10 | 6 | 26 | 0 | 8 | 0 | 17 | 2 | | | | |
| 63 | 0 | 7 | 11 | 18 | 8 | 27 | 0 | 59 | 8 | | | | |
| 200 | 9 | 7 | 8 | 12 | 10 | 103 | 10 | | | | | | |
| 98 | 0 | 20 | 9 | 24 | 0 | | | | | | | | |
| 51 | 1 | 30 | 0 | 16 | 8 | | | | | | | | |
| 878 | 8 | 1 | 0 | 11 | 0 | | | | | | | | |
| | | 98 | 2 | 21 | 0 | | | | | | | | |
| | | 6 | 1 | 165 | 7 | | | | | | | | |
| | | 21 | 11 | | | | | | | | | | |
| | | 138 | 8 | | | | | | | | | | |
| | | 418 | 8 | | | | | | | | | | |

*The Maſon's Bill.*

| *Feet* | *In.* | | |
|---|---|---|---|
| 878 | 8 | Baſe, facias, &c. at — a foot | - - - |
| 418 | 8 | Columns, mouldings, &c. at — a foot | - - - |
| 165 | 7 | Chimney ſlabs and coves, at — a foot | - - - |
| 103 | 10 | Steps to front door, at — a foot | - - - |
| 59 | 8 | Chim. jambs and mantles, at — a foot | - - - |
| 250 | 4 | Cornice of the houſe, at — a foot | - - - |
| 56 | 0 | Cellar ſteps, at — a foot | - - - |
| | | | £ - - - |

The

## *The* CARPENTERS *and* JOINERS WORK.

| | Dimen-sions | | Contents | | Titles |
|---|---|---|---|---|---|
| | *Feet* | *In.* | *Feet* | *In.* | |
| 2 | 24 | 0 | 840 | 0 | Common joiſt. in the low rooms |
| | 17 | 6 | | | |
| 2 | 24 | 6 | 857 | 6 | Framing the upper floors, with girders, binding & bridging joiſt |
| | 17 | 6 | | | |
| 2 | 11 | 9 | 282 | 0 | Common joiſting of the ſtair-caſe, allowing the joiſt one foot hold of the wall at each end |
| | 12 | 0 | | | |
| | | | 180 | 0 | Plates over doors and windows, running meaſure |
| | 59 | 3 | 2133 | 0 | Roofing |
| | 36 | 0 | | | |
| | 109 | 0 | 218 | 0 | Gutter boarding at front and ends |
| | 2 | 0 | | | |
| 4 | 24 | 0 | 1680 | 0 | Ceiling joiſts to both ſtories |
| | 17 | 6 | | | |
| 2 | 17 | 6 | 385 | 0 | Stoothed partitions |
| | 11 | 0 | | | |
| 2 | 24 | 0 | 840 | 0 | Deal flooring |
| | 17 | 6 | | | |
| 2 | 23 | 6 | 822 | 6 | Ditto upper ſtory |
| | 17 | 6 | | | |
| 2 | 12 | 0 | 234 | 0 | Ditto in ſtair-caſe |
| | 9 | 9 | | | |

| | Dimenſions. | | Contents | | Titles |
|---|---|---|---|---|---|
| | *Feet* | *In.* | *Feet* | *In.* | |
| 11 | 4 | 8 | 51 | 4 | Ditto in window ways |
| | 1 | 0 | | | |
| 7 | 3 | 8 | 64 | 2 | Ditto in door ways |
| | 2 | 6 | | | |
| 11 | 4 | 0 | 66 | 0 | Deal ſteps to ſtair-caſe, 1ſt and 2d flights |
| | 1 | 6 | | | |
| 11 | 1 | 0 | 6 | 5 | Ends of ditto |
| | 0 | 7 | | | |
| 2 | 4 | 0 | 36 | 0 | Half paces to ditto |
| | 4 | 6 | | | |
| 9 | 4 | 0 | 108 | 0 | Steps of the 3d flight |
| | 3 | 0 | | | |
| 9 | 1 | 0 | 5 | 3 | Ends of ditto |
| | 0 | 7 | | | |
| | 9 | 9 | 5 | 8 | Face of the landing |
| | 0 | 7 | | | |
| 7 | 8 | 4 | 272 | 2 | Saſhes |
| | 4 | 8 | | | |
| 6 | 4 | 8 | 130 | 8 | Ditto |
| | 4 | 8 | | | |
| 6 | 16 | 8 | 250 | 0 | Inſide door-caſings, pannelled. |
| | 2 | 6 | | | |
| 3 | 16 | 8 | 33 | 4 | Ditto, plain |
| | 0 | 8 | | | |

Di-

| | Dimenſions | | Contents | | Titles |
|---|---|---|---|---|---|
| | *Feet* | *In.* | *Feet* | *In.* | |
| | 25 | 0 | 18 | 9 | Frame to front door |
| | 0 | 9 | | | |
| | 25 | 0 | 29 | 2 | Inſide caſing to ditto, pannelled |
| | 1 | 2 | | | |
| 9 | 6 | 8 | 200 | 0 | Doors, 6 pannelled |
| | 3 | 4 | | | |
| | 10 | 0 | 50 | 0 | Front door |
| | 5 | 0 | | | |
| 7 | 8 | 0 | 224 | 0 | Window ſhutters on the firſt floor and ſtair-caſe window |
| | 4 | 0 | | | |
| 5 | 4 | 0 | 80 | 0 | Ditto to attic windows |
| | 4 | 0 | | | |
| 12 | 4 | 6 | 63 | 0 | Soffits to windows, pannelled |
| | 1 | 2 | | | |
| 7 | 22 | 8 | 145 | 5 | Architraves to low windows and ſtair-caſe window |
| | 0 | 11 | | | |
| | 28 | 4 | 28 | 4 | Ditto to front door |
| | 1 | 0 | | | |
| 5 | 20 | 8 | 94 | 8 | Ditto to upper windows |
| | 0 | 11 | | | |
| 15 | 18 | 9 | 210 | 11 | Ditto to all the doors |
| | 0 | 9 | | | |
| 2 | 72 | 0 | 288 | 0 | Plain dados to low rooms |
| | 2 | 0 | | | |

| | Dimenſions | | Contents | | Titles |
|---|---|---|---|---|---|
| | *Feet* | *In* | *Feet* | *In.* | |
| 2 | 72 | 0 | 108 | 0 | Baſe and ſur-baſe to ditto |
| | 0 | 9 | | | |
| 2 | 72 | 0 | 72 | 0 | Plain plinth to ditto |
| | 0 | 6 | | | |
| | 22 | 6 | 183 | 9 | Baluſtrade in the ſtair-caſe |
| | 8 | 2 | | | |
| | 68 | 0 | 17 | 0 | Baſe moulding to ſtair-caſe |
| | 0 | 3 | | | |
| | 68 | 0 | 34 | 0 | Plain plinth to ditto |
| | 0 | 6 | | | |
| 5 | 6 | 4 | 95 | 0 | Plain backs and elbows to attic windows |
| | 3 | 0 | | | |

*Abſtract of the Carpenters and Joiners Work.*

| Joiſting | Ceiling joiſts, &c. | Flooring | Stairs | Saſhes | Window ſhutters, &c. | Dados, &c. | 6 pannelled doors | Architraves, &c. |
|---|---|---|---|---|---|---|---|---|
| *Feet I* | *Feet I* | *Feet I* | *Feet I* | *Feet I* | *Feet I* | *Feet I* | *Feet I* | *Feet I* |
| 840 0 | 1680 0 | 840 0 | 66 0 | 272 2 | 250 0 | 33 4 | 200 0 | 145 5 |
| 282 0 | 385 0 | 822 6 | 6 5 | 130 8 | 29 2 | 18 9 | 50 0 | 28 4 |
| | | 234 0 | 36 0 | | 224 0 | 288 0 | | 94 8 |
| 1122 0 | 2065 0 | 51 4 | 108 0 | 402 10 | 80 0 | 72 0 | 250 0 | 210 11 |
| | | 64 2 | 5 3 | | 63 0 | 34 0 | | 108 0 |
| | | | 5 8 | | | 95 0 | | 17 0 |
| 224 0 | | 2012 0 | | | 646 2 | | | |
| 8 0 | | | 227 4 | | | 541 1 | | 604 4 |
| 2)304 0 | | | | | | | | |
| 152 0 | Half of the wind. ſhut, being work & half | | | | | | | |
| 250 0 | Doors, being double work | | | | | | | |
| 402 0 | Sum, to be charged for workmanſhip, beſides being charged once for work and materials. | | | | | | | |

The

*The Carpenters and Joiners Bill.*

| *Sq.* | *Feet* | *I* | | £ | *s* | *d* |
|---|---|---|---|---|---|---|
| 11 | 22 | 0 | Common joiſting, at — a ſquare | - | - | - |
| 8 | 57 | 6 | Framing the upper floors, at — a ſq. | - | - | - |
| | 180 | 0 | Plates over doors and windows, running meaſure, at — a foot | - | - | - |
| 21 | 33 | 0 | Roofing, at — a ſquare | - | - | - |
| | 218 | 0 | Gutter boarding, at — a foot | - | - | - |
| 20 | 65 | 0 | Ceiling joiſts and ſtoothed partitions, at — a ſquare | - | - | - |
| 20 | 12 | 0 | Flooring, at — a ſquare | - | - | - |
| | 227 | 4 | Steps in the ſtair-caſe, at — a foot | - | - | - |
| | 402 | 10 | Saſhes, at — a foot | - | - | - |
| | 646 | 2 | Window-ſhutters, ſoffits, & pannelled door-caſes, at — a foot | - | - | - |
| | 541 | 1 | Plain door-caſes, frame to front door, plain plinths, and plain backs and elbows, at — a foot | - | - | - |
| | 250 | 0 | Doors, 6 pannelled, at — a foot | - | - | - |
| | 604 | 4 | Architraves, &c at — a foot | - | - | - |
| | 402 | 0 | Workmanſhip only in doors and window-ſhutters | - | - | - |
| | | | | £ | | |

*Note.* All the articles except the laſt are for both work and materials.

*The* SLATERS WORK.

The Slater's work will be the ſame with the article of roofing in the carpenter's work, which is 2133 ſquare feet, or

| *Sqrs.* | *Feet* | | |
|---|---|---|---|
| 21 | 33 | at — a ſquare | £ - - - |

## *The* PLASTERER'S WORK.

| Dimenſions | | | Contents | | Titles |
|---|---|---|---|---|---|
| | *Feet* | *In.* | *Feet* | *In.* | |
| 2 | 23 | 0 | 782 | 0 | Ceil. to the low rooms, 3 coats |
| | 17 | 0 | | | |
| 2 | 83 | 0 | 166 | 0 | Plain cornice to ditto |
| | 1 | 0 | | | |
| 8 | 83 | 0 | 664 | 0 | Enriched mouldings in ditto, running meaſure |
| | 9 | 0 | 108 | 0 | Ceiling in the lower part of the ſtair-caſe, 3 coats |
| | 12 | 0 | | | |
| | 43 | 6 | 32 | 7 | Plain cornice in ditto |
| | 0 | 9 | | | |
| | 83 | 0 | 747 | 0 | Walls in one of the low rooms, hard-finiſhing |
| | 9 | 0 | | | |
| 2 | 4 | 4 | 37 | 6 | Deduct doors from ditto |
| | 4 | 4 | | | |
| 3 | 8 | 8 | 138 | 8 | Deduct windows |
| | 5 | 4 | | | |
| | 5 | 0 | 10 | 0 | Deduct part of chimney-piece |
| | 2 | 0 | | | |
| | | | 747 | 0 | The walls plaſtering of the other low rooms, 2 coats, the dimenſions and deductions the ſame as in the former |
| | | | 186 | 2 | The deductions together |
| | 22 | 2 | 254 | 11 | Walls of the ſtair-caſe, hard-finiſhing. |
| | 11 | 6 | | | |

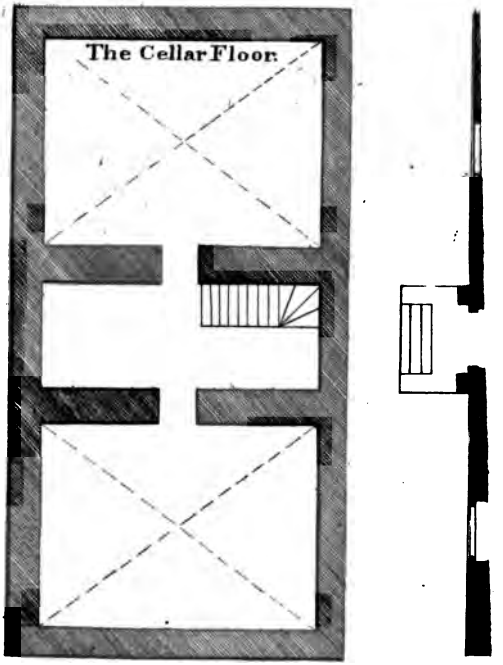
Di-

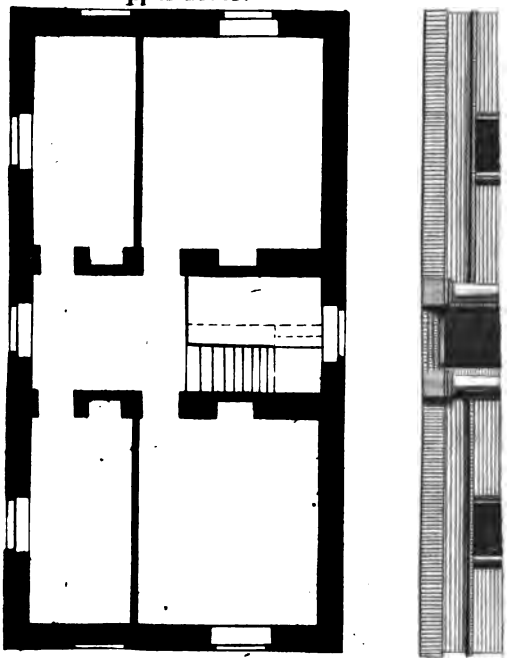
| Dimenſions | | Contents | | Titles |
|---|---|---|---|---|
| *Feet* | *In.* | *Feet* | *In.* | |
| 7 | 6 | 22 | 6 | Walls of the ſtair-caſe, hard-finiſhing |
| 3 | 0 | | | |
| 7 | 6 | 71 | 3 | Ditto |
| 9 | 6 | | | |
| 6 | 0 | 34 | 6 | Ditto |
| 5 | 9 | | | |
| 17 | 9 | 124 | 3 | Ditto |
| 7 | 0 | | | |
| 67 | 0 | 569 | 6 | Walls of upper part, ditto |
| 8 | 6 | | | |
| 10 | 0 | 66 | 8 | Deduct front door from ditto |
| 6 | 8 | | | |
| 6 | 0 | 168 | 0 | Ditto other doors |
| 4 | 0 | | | |
| 8 | 8 | 46 | 2 | Ditto window |
| 5 | 4 | | | |
| 7 | 4 | 39 | 1 | Ditto attic window |
| 5 | 4 | | | |
| 23 | 0 | 201 | 3 | Stair-caſe ceiling, 3 coats |
| 8 | 9 | | | |
| 67 | 8 | 67 | 8 | Plain cornice to ditto |
| 1 | 0 | | | |

| | Dimen-sions | | Contents | | Titles |
|---|---|---|---|---|---|
| | *Feet* | *In.* | *Feet* | *In.* | |
| 2 | 17 | 0 | 493 | 0 | Ceilings of upper rooms, 3 coats |
| | 14 | 6 | | | |
| 2 | 17 | 0 | 246 | 6 | Ditto |
| | 7 | 3 | | | |
| 2 | 65 | 0 | 108 | 4 | Plain cornices |
| | 0 | 10 | | | |
| 2 | 50 | 6 | 84 | 2 | Ditto |
| | 0 | 10 | | | |
| 4 | 17 | 6 | 595 | 0 | Partitions, 2 coats on laths |
| | 8 | 6 | | | |
| 2 | 47 | 6 | 807 | 6 | Ditto walls, 2 coats |
| | 8 | 6 | | | |
| 2 | 33 | 6 | 569 | 6 | Ditto |
| | 8 | 6 | | | |
| 4 | 7 | 4 | 156 | 5 | Deduct windows from ditto |
| | 5 | 4 | | | |
| 4 | 6 | 0 | 104 | 8 | Deduct doors |
| | 4 | 4 | | | |
| 2 | 4 | 6 | 36 | 0 | Deduct chimney pieces |
| | 4 | 0 | | | |
| | 3 | 4 | 13 | 4 | Ditto |

The

To Face Page 648



*The Abſtract of the Plaſtering.*

| Ceiling | | Rendering | | Deductions | | Plain | Enrich'd |
|---|---|---|---|---|---|---|---|
| 3 Coats | 2 Coats | Hard-finiſhing | 2 Coats | Hard-finiſh. | 2 Coats Walls | Cornices | Mouldings |
| *Feet I* | *Feet* | *Feet I* | *Feet I* | *Feet I* | *Feet I* | *Feet I* | *Feet I* |
| 782 0 | 9) 595 1 | 747 0 | 747 0 | 37 6 | 186 2 | 166 0 | 664 0 |
| 108 0 | 0 | 254 11 | 807 6 | 138 8 | 156 5 | 32 7 | |
| 201 3 | *Y F* | 22 6 | 569 0 | 10 0 | 104 0 | 67 8 | |
| 493 0 | 66 1 | 71 3 | | 66 8 | 36 0 | 108 4 | |
| 246 6 | | 34 6 | 2123 6 | 168 0 | 13 4 | 84 2 | |
| | | 124 3 | 495 11 | 46 2 | | | |
| 9) 1830 9 | | 569 6 | | 39 1 | 495 11 | 458 9 | |
| | | | 9) 1627 7 | | | | |
| *Y F I* | | 1823 11 | | 506 1 | | | |
| 203 3 9 | | 506 1 | *Y F I* | | | | |
| | | | 180 7 7 | | | | |
| | | 9) 1317 10 | | | | | |
| | | *Y F I* | | | | | |
| | | 146 3 10 | | | | | |

*The Plaſterer's Bill.*

| *Yds* | *F* | *I* | | *£* | *s* | *d* |
|---|---|---|---|---|---|---|
| 203 | 3 | 9 | Ceiling, 3 coats, at — a yard | - | - | - |
| 66 | 1 | 0 | Ditto, 2 coats, at — a yard | - | - | - |
| 146 | 3 | 10 | Plaſtering, hard-finiſhing, at - a yd. | - | - | - |
| 180 | 7 | 7 | Ditto, 2 coats, at — a yard | - | - | - |
| | 458 | 9 | Plain cornice, at — a foot | - | - | - |
| | 664 | 0 | Enriched mouldings, at — a foot | - | - | - |
| | | | | *£* - | - | - |

The annexed plate contains the elevation and plans of the houſe to which all the foregoing examples belong.

I ſhall now deliver a ſhort but complete tract on Timber Meaſuring, with which I ſhall conclude theſe applications, as propoſed.

SEC-

## SECTION IV.

## TIMBER MEASURING.

### PROBLEM I.

*To find the Area or Superficial Feet in a Board or Plank.*

MULTIPLY the length by the mean breadth.

*Note.* When the board is tapering, add the breadths at the two ends together, and take half the ſum for the mean breadth.

*By the Sliding Rule.*

Set 12 on B to the breadth in inches on A; then againſt the length in feet on B, is the content on A, in feet and fractional parts.

EXAMPLES.

1. What is the value of a plank, whoſe length is 12 feet 6 inches, and mean breadth 11 inches?

*By Decimals.*

12·5
11
137·5
12 | 11·46
1½d is ⅛ | 1s 5d anſ.

*By Duodecimals.*

12 6
11
11 5 6
1½d is ⅛ | 1s 4½d
5′ is | 0 ½
1s 5d anſ.

BY

BY THE SLIDING RULE.

As 12 B : 11 A :: $12\frac{1}{2}$ B : $11\frac{1}{2}$ A.

That is, as 12 on B is to 11 on A, ſo is $12\frac{1}{2}$ on B to $11\frac{1}{2}$ on A.

Ex. 2. Required the content of a board, whoſe length is 11 feet 2 inches, and breadth 1 foot 10 inches. Anſ. $20^f\ 5^i\ 8''$.

Ex. 3. What is the value of a plank, which is 12 feet 9 inches long, and 1 foot 3 inches broad, at $2\frac{1}{2}$d a foot? Anſ. 3s $3\frac{3}{4}$d.

Ex. 4. Required the value of 5 oaken planks, at 3d per foot, each of them being $17\frac{1}{2}$ feet long; and their ſeveral breadths are as follows, namely, two of $13\frac{1}{2}$ inches in the middle, one of $14\frac{1}{2}$ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and $11\frac{1}{4}$ at the narrower.
Anſ. £1 5 $8\frac{1}{4}$.

PROBLEM II.

*To find the Solid Content of Squared or Four-ſided Timber.*

Multiply the mean breadth by the mean thickneſs, and the product again by the length, and the laſt product will give the content.

BY THE SLIDING RULE.

C D D C

As length : 12 or 10 : : quarter girt : ſolidity.

That is, as the length in feet on C, is to 12 on D when the quarter girt is in inches, or to 10 on D when it is in tenths of feet; ſo is the quarter girt on D, to the content on C.

*Note* 1. If the tree taper regularly from the one end to the other, either take the mean breadth and thickneſs in the middle, or take the dimenſions at the two ends, and half their ſum will be the mean dimenſions.

2. If the piece do not taper regularly, but is unequally thick in ſome parts and ſmall in others, take ſeveral different dimenſions, add them all together, and divide their ſum by the number of them, for the mean dimenſions.

3. The quarter girt is a geometrical mean proportional between the mean breadth and thickneſs, that is the ſquare root of their product. Sometimes unſkilful meaſurers uſe the arithmetical mean inſtead of it, that is, half their ſum; but this is always attended with error, and the more ſo as the breadth and depth differ the more from each other.

EXAMPLES.

1. The length of a piece of timber is 18 feet 6 inches, the breadths at the greater and leſs end 1 foot 6 inches and 1 foot 3 inches, and the thickneſs at the greater and leſs end 1 foot 3 inches and 1 foot: required the ſolid content.

*Decimals.*

| *Decimals.* | | *Duodecimals.* |
|---|---|---|
| 1·5 | | 1 6 |
| 1·25 | | 1 3 |
| 2 ) 2·75 | | 2 ) 2 9 |
| 1·375 | mean breadth | 1 4 6 |
| 1·25 | | 1 3 |
| 1·0 | | 1 0 |
| 2 ) 2·25 | | 2 ) 2 3 |
| 1·125 | mean depth | 1 1 6 |
| 1·375 | mean breadth | 1 4 6 |
| 5625 | | 1 1 6 |
| 7875 | | 4 6 |
| 3375 | | 6 6 |
| 1125 | | |
| 1·546875 | | 1 6 6 6 |
| 18·5 | length | 18 6 |
| 7734375 | | 27 9 9 0 |
| 12375000 | | 9 3 3 |
| 1546875 | | |
| 28·6171875 | content | 28 7 0 3 |

BY THE SLIDING RULE.

B A B A
As 1 : $13\frac{6}{7}$ :: $16\frac{4}{7}$ : 223, the mean ſquare.

C D C D
As 1 : 1 :: 223 : 14·9, quarter girt.

C D D C
As $18\frac{1}{2}$ : 12 :: 14·9 : 28·6, the content.

Ex. 2. What is the content of the piece of timber, whose length is $24\frac{1}{2}$ feet, and the mean breadth and thickness each 1·04 feet? Ans. $26\frac{1}{2}$ feet.

Ex. 3. Required the content of a piece of timber, whose length is 20·38 feet, and its ends unequal squares, the side of the greater being $19\frac{1}{2}$, and the side of the less $9\frac{7}{8}$. Ans. 29·838 feet.

Ex. 4. Required the content of the piece of timber, whose length is 27·36 feet; at the greater end the breadth is 1·78, and thickness 1·23; and at the less end the breadth is 1·04, and thickness 0·91. Ans. 41·278 feet.

## PROBLEM III.

### *To find the Solidity of Round or Unsquared Timber.*

### RULE I, or COMMON RULE.

Multiply the square of the quarter girt, or of $\frac{1}{4}$ of the mean circumference, by the length, for the content.

### BY THE SLIDING RULE.

As the length upon C : 12 or 10 upon D : : quarter girt, in 12ths or 10ths, on D : content on C.

*Note* 1. When the tree is tapering, take the mean dimensions as in the former problems, either by girting it in the middle for the mean girt, or at the two ends, and take half the sum of the two. But when the tree is very irregular, divide it into several lengths, and find the content of each part separately.

2. This rule, which is commonly used, gives the answer about $\frac{1}{4}$ less than the true quantity in the tree, or

or nearly what the quantity would be after the tree is hewed ſquare in the uſual way: ſo that it ſeems intended to make an allowance for the ſquaring of the tree. When the true quantity is deſired, uſe the 2d rule, given below.

EXAMPLES.

1. A piece of round timber being 9 feet 6 inches long, and its mean quarter girt 42 inches; what is the content?

| *Decimals.* | | *Duodecimals.* |
|---|---|---|
| 3·5 | quarter girt | 3 6 |
| 3·5 | | 3 6 |
| 175 | | 10 6 |
| 105 | | 1 9 |
| 12·25 | | 12 3 |
| 9·5 | length | 9 6 |
| 6125 | | 110 3 |
| 11025 | | 6 1 6 |
| 116·375 | content | 116 4 6 |

BY THE SLIDING RULE.

| | C | D | D | C |
|---|---|---|---|---|
| As | 9·5 | 10 :: 35 | : | $116\frac{1}{4}$ |
| Or | 9·5 | 12 :: 42 | : | $116\frac{1}{4}$ |

Ex. 2. The length of a tree is 24 feet, its girt at the thicker end 6 feet, and at the ſmaller end 2 feet; required the content. Anſ. 96 feet.

Ex. 3. What is the content of a tree, whoſe mean girt is 3·15 feet, and length 14 feet 6 inches?

Anſ. 8·9929 feet.

Ex. 4.

Ex. 4. Required the content of a tree, whose length is 17½ feet, and which girts in five different places as follows, namely, in the firſt place 9·43 feet, in the ſecond 7·92, in the third 6·15, in the fourth 4·74, and in the fifth 3·16. Anſ. 42·6075.

### RULE II.

Multiply the ſquare of ¼ of the mean girt by double the length, and the product will be the content, very near the truth.

### BY THE SLIDING RULE.

As the double length on C : 12 or 10 on D :: ¼ of the girt, in 12ths or 10ths, on D : content on C.

### EXAMPLES.

1. What is the content of a tree, its length being 9 feet 6 inches, and its mean girt 14 feet?

| *Decimals.* | | *Duodecimals.* |
|---|---|---|
| 2·8 | ¼ of girt | 2 9 7 |
| 2·8 | | 2 9 7 |
| 224 | | 5 7 2 |
| 56 | | 2 1 9 |
| 7·84 | | 1 8 |
| 19 | | 7 10 1 |
| 7056 | | 19 |
| 784 | | |
| 148·96 | content | 148 11 7 |

BY

BY THE SLIDING RULE.

C D D C

As 19 : 10 :: 28 : 149.

Or 19 : 12 :: $33\frac{6}{10}$ : 149.

Ex. 2. Required the content of a tree, which is 24 feet long, and mean girt 8 feet. Anſ. 122·88 feet.

Ex. 3. The length of a tree is $14\frac{1}{2}$ feet, and mean girt 3·15 feet, what is the content? Anſw. 11·51 feet.

Ex. 4. The length of a tree is $17\frac{1}{4}$ feet, and its mean girt 6·28; what is the content?

Anſ. 54·4065 feet.

Ex. 5. Sold an oak tree whoſe girt at the low end was $9\frac{1}{4}$ feet, and its length, to the part where it becomes of 2 feet girt, is $27\frac{1}{2}$ feet; it hath alſo two boughs, the girt at the thicker end of the one is 4·3, and at the thicker end of the other 3·94, the length of the timber of the former, that is to the part where it becomes of only 2 feet girt, is 9 feet; and the length of the latter $7\frac{3}{4}$ feet; required the price at 2*l.* 3*s.* 9*d.* a load of 50 feet, allowing $\frac{1}{10}$ of the mean girt for the bark of the trunk, and $\frac{1}{12}$ of the ſame in the boughs, by both the rules. Anſ. true value £2 18 6.

falſe value £2 5 $9\frac{1}{2}$.

*Note* 1. That part of a tree or of the branches, which is leſs than 2 feet in circumference, or 6 inches quarter girt, is cut off; not being accounted timber.

*Note* 2. A cuſtom has of late been creeping into uſe, where the buyers of timber can introduce it, of allowing an inch on every foot of quarter girt, for bark. This practice, however, is unreaſonable, and ought to be diſcouraged. Elm timber is the chief kind in which an allowance ought to be made, and it will be found on examination, that the common allowance of one inch on the tree, is abundantly ſufficient for an average allowance.

*Note* 3. Fifty cubic feet of timber make a load; and therefore to reduce feet to loads, divide them by 50.

# A TABLE

## For readily finding the Content of Trees, according to the common Method of measuring Timber.

### RULE.

Seek the quarter girt in the first column towards the left-hand, and take out the number opposite. Multiply that number by the length of the tree in feet &c, and the product will be the content in solid feet &c.

| Ins. | Feet | In. | " | ''' | '''' |
|---|---|---|---|---|---|
| 6 | 0 | 3 | 0 | 0 | 0 |
| 1/4 | 0 | 3 | 3 | 0 | 9 |
| 1/2 | 0 | 3 | 6 | 3 | 0 |
| 3/4 | 0 | 3 | 9 | 6 | 9 |
| 7 | 0 | 4 | 1 | 0 | 0 |
| 1/4 | 0 | 4 | 4 | 6 | 9 |
| 1/2 | 0 | 4 | 8 | 3 | 0 |
| 3/4 | 0 | 5 | 0 | 0 | 9 |
| 8 | 0 | 5 | 4 | 0 | 0 |
| 1/4 | 0 | 5 | 8 | 0 | 9 |
| 1/2 | 0 | 6 | 0 | 3 | 0 |
| 3/4 | 0 | 6 | 4 | 6 | 9 |
| 9 | 0 | 6 | 9 | 0 | 0 |
| 1/4 | 0 | 7 | 1 | 6 | 9 |
| 1/2 | 0 | 7 | 6 | 3 | 0 |
| 3/4 | 0 | 7 | 11 | 0 | 9 |
| 10 | 0 | 8 | 4 | 0 | 0 |
| 1/4 | 0 | 8 | 9 | 0 | 9 |
| 1/2 | 0 | 9 | 2 | 3 | 0 |
| 3/4 | 0 | 9 | 7 | 6 | 9 |
| 11 | 0 | 10 | 1 | 0 | 0 |
| 1/4 | 0 | 10 | 6 | 6 | 9 |
| 1/2 | 0 | 11 | 0 | 3 | 0 |
| 3/4 | 0 | 11 | 6 | 0 | 9 |
| 12 | 1 | 0 | 0 | 0 | 0 |
| 1/4 | 1 | 0 | 6 | 0 | 9 |
| 1/2 | 1 | 1 | 0 | 3 | 0 |
| 3/4 | 1 | 1 | 6 | 6 | 9 |
| 13 | 1 | 2 | 1 | 0 | 0 |
| 1/4 | 1 | 2 | 7 | 6 | 9 |
| 1/2 | 1 | 3 | 2 | 3 | 0 |
| 3/4 | 1 | 3 | 9 | 0 | 9 |
| 14 | 1 | 4 | 4 | 0 | 0 |
| 1/4 | 1 | 4 | 11 | 0 | 9 |
| 1/2 | 1 | 5 | 6 | 3 | 0 |
| 3/4 | 1 | 6 | 1 | 6 | 9 |
| 15 | 1 | 6 | 9 | 0 | 0 |
| 1/4 | 1 | 7 | 4 | 6 | 9 |
| 1/2 | 1 | 8 | 0 | 3 | 0 |
| 3/4 | 1 | 8 | 8 | 0 | 9 |
| 16 | 1 | 9 | 4 | 0 | 0 |
| 1/4 | 1 | 10 | 0 | 0 | 9 |
| 1/2 | 1 | 10 | 8 | 3 | 0 |
| 3/4 | 1 | 11 | 4 | 6 | 9 |
| 17 | 2 | 0 | 1 | 0 | 0 |
| 1/4 | 2 | 0 | 9 | 6 | 9 |
| 1/2 | 2 | 1 | 6 | 3 | 0 |
| 3/4 | 2 | 2 | 3 | 0 | 9 |
| 18 | 2 | 3 | 0 | 0 | 0 |
| 1/4 | 2 | 3 | 9 | 0 | 9 |
| 1/2 | 2 | 4 | 6 | 3 | 0 |
| 3/4 | 2 | 5 | 3 | 6 | 9 |
| 19 | 2 | 6 | 1 | 0 | 0 |
| 1/4 | 2 | 6 | 10 | 6 | 9 |
| 1/2 | 2 | 7 | 8 | 3 | 0 |
| 3/4 | 2 | 8 | 6 | 0 | 9 |
| 20 | 2 | 9 | 4 | 0 | 0 |
| 1/4 | 2 | 10 | 2 | 0 | 9 |
| 1/2 | 2 | 11 | 0 | 3 | 0 |
| 3/4 | 2 | 11 | 10 | 6 | 9 |
| 21 | 3 | 0 | 9 | 0 | 0 |
| 1/4 | 3 | 1 | 7 | 6 | 9 |
| 1/2 | 3 | 2 | 6 | 3 | 0 |
| 3/4 | 3 | 3 | 5 | 0 | 9 |
| 22 | 3 | 4 | 4 | 0 | 0 |
| 1/4 | 3 | 5 | 3 | 0 | 9 |
| 1/2 | 3 | 6 | 2 | 3 | 0 |
| 3/4 | 3 | 7 | 1 | 6 | 9 |
| 23 | 3 | 8 | 1 | 0 | 0 |
| 1/4 | 3 | 9 | 0 | 6 | 9 |
| 1/2 | 3 | 10 | 0 | 3 | 0 |
| 3/4 | 3 | 11 | 0 | 0 | 9 |
| 24 | 4 | 0 | 0 | 0 | 0 |
| 1/4 | 4 | 1 | 0 | 0 | 9 |
| 1/2 | 4 | 2 | 0 | 3 | 0 |
| 3/4 | 4 | 3 | 0 | 6 | 9 |
| 25 | 4 | 4 | 1 | 0 | 0 |
| 1/4 | 4 | 5 | 1 | 6 | 9 |
| 1/2 | 4 | 6 | 2 | 3 | 0 |
| 3/4 | 4 | 7 | 3 | 0 | 9 |
| 26 | 4 | 8 | 4 | 0 | 0 |
| 1/4 | 4 | 9 | 5 | 0 | 9 |
| 1/2 | 4 | 10 | 6 | 3 | 0 |
| 3/4 | 4 | 11 | 7 | 6 | 9 |
| 27 | 5 | 0 | 9 | 0 | 0 |
| 1/4 | 5 | 1 | 10 | 6 | 9 |
| 1/2 | 5 | 3 | 0 | 3 | 0 |
| 3/4 | 5 | 4 | 2 | 0 | 9 |
| 28 | 5 | 5 | 4 | 0 | 0 |
| 1/4 | 5 | 6 | 6 | 0 | 9 |
| 1/2 | 5 | 7 | 8 | 3 | 0 |
| 3/4 | 5 | 8 | 10 | 6 | 9 |
| 29 | 5 | 10 | 1 | 0 | 0 |
| 1/4 | 5 | 11 | 3 | 6 | 9 |
| 1/2 | 6 | 0 | 6 | 3 | 0 |
| 3/4 | 6 | 1 | 9 | 0 | 9 |
| 30 | 6 | 3 | 0 | 0 | 0 |
| 1/4 | 6 | 4 | 3 | 0 | 9 |
| 1/2 | 6 | 5 | 6 | 3 | 0 |
| 3/4 | 6 | 6 | 9 | 6 | 9 |
| 31 | 6 | 8 | 1 | 0 | 0 |
| 1/4 | 6 | 9 | 4 | 6 | 9 |
| 1/2 | 6 | 10 | 8 | 3 | 0 |
| 3/4 | 7 | 0 | 0 | 0 | 9 |
| 32 | 7 | 1 | 4 | 0 | 0 |
| 1/4 | 7 | 2 | 8 | 0 | 9 |
| 1/2 | 7 | 4 | 0 | 3 | 0 |
| 3/4 | 7 | 5 | 4 | 6 | 9 |

SCHOLIUM.

In measuring squared timber, unskilful measurers usually take $\frac{1}{4}$ of the circumference, or girt, for the side of a mean square; which quarter girt therefore multiplied by itself, and the product multiplied by the length, they account the solidity, or content: but it is really always above the truth. Indeed when the breadth and thickness are nearly equal, this method will give the solidity pretty near the truth; but if the breadth and thickness differ considerably from each other, the error will be so great as that it ought not by any means to be neglected.

Thus, suppose we take, for an example, a balk of 24 feet long, and a foot square throughout; and consequently its solidity 24 cubic feet. Now if this balk be slit exactly in two from end to end, making each piece 6 inches broad, and 12 inches thick; it is evident that the true solidity of each will be 12 feet; but by the quarter girt method, they would amount to much more; for the false quarter girt, being equal to half the sum of the breadth and thickness, in this case will be 9 inches, the square of which is 81, which being divided by 144, and the quotient multiplied by 24 the length, we obtain $13\frac{1}{2}$ feet for the solidity of each part, and consequently the two solidities together make 27 feet, instead of 24.

Again, suppose the balk to be so cut, as that the breadth of the one piece may be only 4 inches, and consequently that of the other 8 inches. Here the true content of the less piece will be 8 feet, and that of the greater 16 feet. But, proceeding by the other method, the quarter girt of the less piece will be 8, whose square 64, multiplied by 24, and the product divided by 144, gives $10\frac{2}{3}$ feet instead of 8. And, by the same method, the content of the greater piece will be $16\frac{2}{3}$ feet, instead of 16. And the sum of both is $27\frac{1}{3}$ feet, instead of 24.

Farther, if the less piece be cut only 2 inches broad, and consequently the greater 10 inches; the

true content of the leſs piece would be 4 feet, and that of the greater 20. But, by the other method, the quarter girt of the leſs piece would be 7 inches, whoſe ſquare 49 being divided by 6, gives $8\frac{1}{6}$ feet, inſtead of 4, for the content. And by the ſame method the content of the greater piece would be $20\frac{1}{6}$, inſtead of 20 feet. So that their ſum would be $28\frac{1}{3}$, inſtead of 24 feet.

Hence it is evident, that the greater the proportion is between the breadth and depth, the greater will the error be, by uſing the falſe method; that the ſum of the two parts, by the ſame method, is greater, as the difference of the ſame two parts is greater, and conſequently the ſum is leaſt when the two parts are equal to each other, or when the balk is cut equally in two; and laſtly, that when the ſides of a balk differ not above an inch or two from each other, the quarter girt method may then be uſed, without inducing an error that will be of any material conſequence.

From the preceding examples it appears that this new method, which is very near the truth, is full as eaſy in practice as the common falſe one. But there are many other reaſons for changing the method; and one in particular, is the preventing of the ſellers from playing any tricks with their timber, by cutting trees into different lengths, ſo as to make them meaſure to more than the whole did; for, by the falſe method, this may be done in many reſpects, as will appear in the three following problems, which contain the chief caſes of this artifice, but which however I do not explain to teach them to uſe theſe means.

PROBLEM

PROBLEM IV.

*To find where a Piece of Round Tapering Timber muſt be cut, ſo that the Two Parts, meaſured ſeparately, according to the Common Method of meaſuring, ſhall produce a Greater Solidity than when cut in any other Part, and Greater than the Whole.*

* Cut it through exactly in the middle, or at $\frac{1}{2}$ of the length, and the two parts will meaſure to the moſt poſſible by the common method.

EXAMPLE.

Suppoſing a tree to gift 14 feet at the greater end, 2 feet at the leſs, and conſequently 8 feet in the middle; and that the length is 32 feet.

Then, by the common method, the whole tree meaſures to only 128 feet; but when cut through at the middle, the greater part meaſures to 121, and the leſs part to 25 feet; whoſe ſum is 146 feet; which exceeds the whole by 18 feet, and is the moſt that it can be made to meaſure to by cutting it into two parts.



* DEMONSTRATION.

Put $G$ = the greateſt girt, $g$ = the leaſt, and $x$ = the girt at the ſection; alſo $L$ = the whole length, and $z$ = the length to be cut off the leſs end.

Then, by ſimilar figures, $L : z :: G-g : x-g$, hence $x = \frac{Gz-gz}{L} + g$. But $(g+x)^2 . z + (G+x)^2 . (L-z) =$ a maximum; whoſe fluxion being put equal to nothing, and the value of $x$ ſubſtituted inſtead of it, there reſults $z = \frac{1}{2} L$. *Q. E. D.*

COROLLARY.

By thus biſecting the length of a tree, and then each of the parts, and ſo on, continually biſecting the lengths of the ſeveral parts, the meaſure of the whole will be continually increaſed.

PROBLEM V.

*To find where a Tree ſhould be Cut, ſo that the Part next the Greater End may meaſure to the Moſt Poſſible.*

* From the greater girt take 3 times the leſs; then, as the difference of the girts is to the remainder, ſo is $\frac{1}{3}$ of the whole length, to the length from the leſs end to be cut off.

Or, cut it where the girt is $\frac{1}{3}$ of the greateſt girt.

*Note.* If the greateſt girt do not exceed 3 times the leaſt, the tree cannot be cut as is required by this problem. For, when the leaſt girt is exactly equal to $\frac{1}{3}$ of the greater, the tree already meaſures to the greateſt poſſible; that is, none can be cut off, nor indeed added to it, continuing the ſame taper, that the remainder or ſum may meaſure to ſo much as the whole: And when the leaſt girt exceeds $\frac{1}{3}$ of the greater, the reſult by the rule ſhews how much in length muſt be *added*, that the reſult may meaſure to the moſt poſſible.

EXAMPLE.

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* DEMONSTRATION.

Uſing the ſame notation as in the laſt demonſtration; we ſhall have here alſo $x = \frac{Gx - gx}{L} + g$, and $(G+x)^2 . (L-x) =$ a maximum; which, treated as before, gives $x = \frac{G-3g}{G-g} \times \frac{1}{3}L$. And $x = \frac{G-g}{L}x + g = \frac{1}{3}G$, by ſubſtituting the above value of $x$.

Q. E. D

EXAMPLE.

Taking here the ſame example as before, we ſhall have as 12 : 8 : : $\frac{32}{3}$ : $7\frac{1}{9}$ = the length to be cut off; and conſequently the length of the remaining part is $24\frac{8}{9}$; alſo $\frac{14}{3} = 4\frac{2}{3}$ is the girt at the ſection. Hence the content of the remaining part is $135\frac{41}{81}$ feet; whereas the whole tree, by the ſame method, meaſures only to 128 feet.

PROBLEM VI.

*To Cut a Tree ſo as that the Part next the Greater End may meaſure, by the Common Method, to exactly the ſame Quantity as the Whole meaſures to.*

* Call the ſum of the girts of the two ends $s$, and their difference $d$. Then multiply $d$ by the ſum of $d$ and $4s$, and from the root of the product take the difference between $d$ and $2s$; then, as $2d$ is to the remainder, ſo is the whole length, to the length to be cut off the ſmall end.

And if $s$ be taken from the ſaid root, half the remainder will be the girt at the ſection.



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* DEMONSTRATION.

Uſing ſtill the ſame notation, we ſhall have $s^2L = (L-z).(G+x)^2$; hence, inſtead of $x$, ſubſtituting its value $\frac{dz}{L} + g$, we obtain $z = \frac{L}{2d} \times (\sqrt{(4s+d).d} - 2s + d)$. And hence $x = \frac{1}{2}\sqrt{(4s+d).d} - \frac{1}{2}s$. Q. E. D.

EXAMPLE.

Using ſtill the ſame example, we have $s = 16$, $d = 12$, and the length $L = 32$; hence $\frac{L}{2d} \times (\sqrt{(4s+d).d} - 2s + d) = \frac{32}{24} \times (\sqrt{76 \times 12} - 20) = \frac{16}{3}\sqrt{57} - 26\frac{2}{3} = 13.599118 =$ the length to be cut off; and conſequently 18.400882 is the length of the remaining part. Alſo $\frac{1}{2}\sqrt{(4s+d).d} - \frac{1}{2}s = 2\sqrt{57} - 8 = 7.099669$ is the girt at the ſection. Hence the girt in the middle of the greater part is $\frac{14+7.099669}{2} = 10.549834$, whoſe $\frac{1}{4}$th part is 2.637458; and conſequently the content of the ſame part is $2.637458^2 \times 18.400882 = 128$, the very ſame as the whole tree meaſures to, notwithſtanding above $\frac{1}{3}$ part is cut off the length.

A NEW

A

NEW AND ACCURATE

# TABLE

OF THE

AREAS OF CIRCULAR SEGMENTS,

FOR

EVERY TEN-THOUSANDTH PART

OF THE

DIAMETER.

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 0001 | 00000133 | 0046 | 00041540 | 0091 | 00115427 |
| 0002 | 00000377 | 0047 | 00042901 | 0092 | 00117332 |
| 0003 | 00000693 | 0048 | 00044277 | 0093 | 00119247 |
| 0004 | 00001067 | 0049 | 00045666 | 0094 | 00121172 |
| 0005 | 00001490 | 0050 | 00047070 | 0095 | 00123107 |
| 0006 | 00001959 | 0051 | 00048487 | 0096 | 00125052 |
| 0007 | 00002469 | 0052 | 00049919 | 0097 | 00127007 |
| 0008 | 00003016 | 0053 | 00051364 | 0098 | 00128972 |
| 0009 | 00003599 | 0054 | 00052823 | 0099 | 00130947 |
| 0010 | 00004215 | 0055 | 00054296 | 0100 | 00132932 |
| 0011 | 00004863 | 0056 | 00055781 | 0101 | 00134927 |
| 0012 | 00005541 | 0057 | 00057281 | 0102 | 00136932 |
| 0013 | 00006247 | 0058 | 00058793 | 0103 | 00138947 |
| 0014 | 00006981 | 0059 | 00060318 | 0104 | 00140971 |
| 0015 | 00007742 | 0060 | 00061856 | 0105 | 00143005 |
| 0016 | 00008529 | 0061 | 00063407 | 0106 | 00145048 |
| 0017 | 00009341 | 0062 | 00064971 | 0107 | 00147101 |
| 0018 | 00010177 | 0063 | 00066547 | 0108 | 00149163 |
| 0019 | 00011036 | 0064 | 00068135 | 0109 | 00151235 |
| 0020 | 00011919 | 0065 | 00069736 | 0110 | 00153317 |
| 0021 | 00012823 | 0066 | 00071350 | 0111 | 00155407 |
| 0022 | 00013749 | 0067 | 00072975 | 0112 | 00157507 |
| 0023 | 00014697 | 0068 | 00074613 | 0113 | 00159617 |
| 0024 | 00015665 | 0069 | 00076263 | 0114 | 00161735 |
| 0025 | 00016654 | 0070 | 00077924 | 0115 | 00163863 |
| 0026 | 00017663 | 0071 | 00079597 | 0116 | 00166000 |
| 0027 | 00018691 | 0072 | 00081282 | 0117 | 00168146 |
| 0028 | 00019738 | 0073 | 00082979 | 0118 | 00170301 |
| 0029 | 00020805 | 0074 | 00084688 | 0119 | 00172466 |
| 0030 | 00021889 | 0075 | 00086407 | 0120 | 00174639 |
| 0031 | 00022992 | 0076 | 00088139 | 0121 | 00176821 |
| 0032 | 00024113 | 0077 | 00089881 | 0122 | 00179012 |
| 0033 | 00025251 | 0078 | 00091635 | 0123 | 00181212 |
| 0034 | 00026407 | 0079 | 00093400 | 0124 | 00183421 |
| 0035 | 00027579 | 0080 | 00095176 | 0125 | 00185639 |
| 0036 | 00028769 | 0081 | 00096963 | 0126 | 00187865 |
| 0037 | 00029975 | 0082 | 00098762 | 0127 | 00190100 |
| 0038 | 00031197 | 0083 | 00100571 | 0128 | 00192344 |
| 0039 | 00032436 | 0084 | 00102391 | 0129 | 00194597 |
| 0040 | 00033690 | 0085 | 00104221 | 0130 | 00196858 |
| 0041 | 00034961 | 0086 | 00106063 | 0131 | 00199128 |
| 0042 | 00036247 | 0087 | 00107915 | 0132 | 00201406 |
| 0043 | 00037547 | 0088 | 00109777 | 0133 | 00203693 |
| 0044 | 00038864 | 0089 | 00111651 | 0134 | 00205988 |
| 0045 | 00040195 | 0090 | 00113534 | 0135 | 00208292 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 0136 | 00210604 | 0181 | 00322912 | 0226 | 00449919 |
| 0137 | 00212925 | 0182 | 00325582 | 0227 | 00452895 |
| 0138 | 00215254 | 0183 | 00328259 | 0228 | 00455877 |
| 0139 | 00217591 | 0184 | 00330943 | 0229 | 00458866 |
| 0140 | 00219937 | 0185 | 00333635 | 0230 | 00461861 |
| 0141 | 00222291 | 0186 | 00336333 | 0231 | 00464862 |
| 0142 | 00224653 | 0187 | 00339039 | 0232 | 00467869 |
| 0143 | 00227024 | 0188 | 00341752 | 0233 | 00470883 |
| 0144 | 00229402 | 0189 | 00344472 | 0234 | 00473904 |
| 0145 | 00231789 | 0190 | 00347199 | 0235 | 00476930 |
| 0146 | 00234184 | 0191 | 00349933 | 0236 | 00479963 |
| 0147 | 00236587 | 0192 | 00352674 | 0237 | 00483002 |
| 0148 | 00238998 | 0193 | 00355422 | 0238 | 00486047 |
| 0149 | 00241417 | 0194 | 00358177 | 0239 | 00489099 |
| 0150 | 00243844 | 0195 | 00360939 | 0240 | 00492157 |
| 0151 | 00246279 | 0196 | 00363708 | 0241 | 00495221 |
| 0152 | 00248722 | 0197 | 00366484 | 0242 | 00498291 |
| 0153 | 00251173 | 0198 | 00369266 | 0243 | 00501368 |
| 0154 | 00253631 | 0199 | 00372056 | 0244 | 00504451 |
| 0155 | 00256098 | 0200 | 00374853 | 0245 | 00507539 |
| 0156 | 00258573 | 0201 | 00377656 | 0246 | 00510634 |
| 0157 | 00261055 | 0202 | 00380466 | 0247 | 00513736 |
| 0158 | 00263545 | 0203 | 00383283 | 0248 | 00516843 |
| 0159 | 00266043 | 0204 | 00386107 | 0249 | 00519956 |
| 0160 | 00268549 | 0205 | 00388938 | 0250 | 00523076 |
| 0161 | 00271062 | 0206 | 00391775 | 0251 | 00526201 |
| 0162 | 00273583 | 0207 | 00394620 | 0252 | 00529333 |
| 0163 | 00276112 | 0208 | 00397471 | 0253 | 00532470 |
| 0164 | 00278648 | 0209 | 00400328 | 0254 | 00535614 |
| 0165 | 00281192 | 0210 | 00403193 | 0255 | 00538764 |
| 0166 | 00283744 | 0211 | 00406064 | 0256 | 00541920 |
| 0167 | 00286303 | 0212 | 00408941 | 0257 | 00545081 |
| 0168 | 00288870 | 0213 | 00411826 | 0258 | 00548249 |
| 0169 | 00291444 | 0214 | 00414717 | 0259 | 00551423 |
| 0170 | 00284025 | 0215 | 00417614 | 0260 | 00554603 |
| 0171 | 00296614 | 0216 | 00420518 | 0261 | 00557788 |
| 0172 | 00299211 | 0217 | 00423429 | 0262 | 00560980 |
| 0173 | 00301815 | 0218 | 00426346 | 0263 | 00564178 |
| 0174 | 00304427 | 0219 | 00429270 | 0264 | 00567381 |
| 0175 | 00307045 | 0220 | 00432201 | 0265 | 00570590 |
| 0176 | 00309672 | 0221 | 00435137 | 0266 | 00573806 |
| 0177 | 00312305 | 0222 | 00438081 | 0267 | 00577027 |
| 0178 | 00314946 | 0223 | 00441031 | 0268 | 00580254 |
| 0179 | 00317594 | 0224 | 00443987 | 0269 | 00583487 |
| 0180 | 00320249 | 0225 | 00446950 | 0270 | 00586726 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area. | Verſ. ſine | Seg. area. |
|---|---|---|---|---|---|
| 0271 | 00589970 | 0316 | 00741838 | 0361 | 00904564 |
| 0272 | 00593221 | 0317 | 00745339 | 0362 | 00908297 |
| 0273 | 00596477 | 0318 | 00748846 | 0363 | 00912036 |
| 0274 | 00599739 | 0319 | 00752358 | 0364 | 00915779 |
| 0275 | 00603007 | 0320 | 00755875 | 0365 | 00919527 |
| 0276 | 00606280 | 0321 | 00759398 | 0366 | 00923280 |
| 0277 | 00609560 | 0322 | 00762926 | 0367 | 00927038 |
| 0278 | 00612845 | 0323 | 00766459 | 0368 | 00930801 |
| 0279 | 00616135 | 0324 | 00769997 | 0369 | 00934569 |
| 0280 | 00619432 | 0325 | 00773541 | 0370 | 00938342 |
| 0281 | 00622734 | 0326 | 00777090 | 0371 | 00942119 |
| 0282 | 00626042 | 0327 | 00780645 | 0372 | 00945902 |
| 0283 | 00629356 | 0328 | 00784204 | 0373 | 00949686 |
| 0284 | 00632676 | 0329 | 00787769 | 0374 | 00953482 |
| 0285 | 00636001 | 0330 | 00791339 | 0375 | 00957279 |
| 0286 | 00639331 | 0331 | 00794915 | 0376 | 00961081 |
| 0287 | 00642668 | 0332 | 00798495 | 0377 | 00964888 |
| 0288 | 00646010 | 0333 | 00802081 | 0378 | 00968700 |
| 0289 | 00649358 | 0334 | 00805672 | 0379 | 00972517 |
| 0290 | 00652711 | 0335 | 00809268 | 0380 | 00976338 |
| 0291 | 00656070 | 0336 | 00812869 | 0381 | 00980164 |
| 0292 | 00659434 | 0337 | 00816476 | 0382 | 00983996 |
| 0293 | 00662804 | 0338 | 00820088 | 0383 | 00987831 |
| 0294 | 00666180 | 0339 | 00823705 | 0384 | 00991672 |
| 0295 | 00669561 | 0340 | 00827327 | 0385 | 00995517 |
| 0296 | 00672948 | 0341 | 00830954 | 0386 | 00999368 |
| 0297 | 00676341 | 0342 | 00834586 | 0387 | 01003224 |
| 0298 | 00679739 | 0343 | 00838224 | 0388 | 01007083 |
| 0299 | 00683142 | 0344 | 00841867 | 0389 | 01010948 |
| 0300 | 00686551 | 0345 | 00845514 | 0390 | 01014818 |
| 0301 | 00689966 | 0346 | 00849166 | 0391 | 01018692 |
| 0302 | 00693386 | 0347 | 00852824 | 0392 | 01022571 |
| 0303 | 00696811 | 0348 | 00856487 | 0393 | 01026455 |
| 0304 | 00700242 | 0349 | 00860155 | 0394 | 01030343 |
| 0305 | 00703679 | 0350 | 00863828 | 0395 | 01034237 |
| 0306 | 00707120 | 0351 | 00867506 | 0396 | 01038135 |
| 0307 | 00710568 | 0352 | 00871190 | 0397 | 01042038 |
| 0308 | 00714021 | 0353 | 00874878 | 0398 | 01045945 |
| 0309 | 00717479 | 0354 | 00878571 | 0399 | 01049857 |
| 0310 | 00720942 | 0355 | 00882269 | 0400 | 01053774 |
| 0311 | 00724411 | 0356 | 00885973 | 0401 | 01057695 |
| 0312 | 00727886 | 0357 | 00889681 | 0402 | 01061622 |
| 0313 | 00731366 | 0358 | 00893394 | 0403 | 01065553 |
| 0314 | 00734851 | 0359 | 00897113 | 0404 | 01069488 |
| 0315 | 00738342 | 0360 | 00900836 | 0405 | 01073428 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 0406 | 01077373 | 0451 | 01259617 | 0496 | 01450745 |
| 0407 | 01081323 | 0452 | 01263770 | 0497 | 01455090 |
| 0408 | 01085277 | 0453 | 01267927 | 0498 | 01459438 |
| 0409 | 01089236 | 0454 | 01272088 | 0499 | 01463791 |
| 0410 | 01093199 | 0455 | 01276254 | 0500 | 01468148 |
| 0411 | 01097167 | 0456 | 01280424 | 0501 | 01472509 |
| 0412 | 01101140 | 0457 | 01284599 | 0502 | 01476874 |
| 0413 | 01105117 | 0458 | 01288778 | 0503 | 01481243 |
| 0414 | 01109099 | 0459 | 01292961 | 0504 | 01485616 |
| 0415 | 01113086 | 0460 | 01297148 | 0505 | 01489994 |
| 0416 | 01117077 | 0461 | 01301340 | 0506 | 01494375 |
| 0417 | 01121073 | 0462 | 01305536 | 0507 | 01498761 |
| 0418 | 01125073 | 0463 | 01309737 | 0508 | 01503151 |
| 0419 | 01129078 | 0464 | 01313942 | 0509 | 01507544 |
| 0420 | 01133088 | 0465 | 01318151 | 0510 | 01511942 |
| 0421 | 01137102 | 0466 | 01322364 | 0511 | 01516344 |
| 0422 | 01141120 | 0467 | 01326582 | 0512 | 01520750 |
| 0423 | 01145144 | 0468 | 01330804 | 0513 | 01525161 |
| 0424 | 01149171 | 0469 | 01335031 | 0514 | 01529575 |
| 0425 | 01153203 | 0470 | 01339261 | 0515 | 01533993 |
| 0426 | 01157240 | 0471 | 01343496 | 0516 | 01538415 |
| 0427 | 01161282 | 0472 | 01347735 | 0517 | 01542842 |
| 0428 | 01165328 | 0473 | 01351979 | 0518 | 01547272 |
| 0429 | 01169378 | 0474 | 01356226 | 0519 | 01551707 |
| 0430 | 01173433 | 0475 | 01360478 | 0520 | 01556145 |
| 0431 | 01177492 | 0476 | 01364735 | 0521 | 01560588 |
| 0432 | 01181556 | 0477 | 01368995 | 0522 | 01565034 |
| 0433 | 01185625 | 0478 | 01373260 | 0523 | 01569485 |
| 0434 | 01189697 | 0479 | 01377529 | 0524 | 01573940 |
| 0435 | 01193775 | 0480 | 01381802 | 0525 | 01578398 |
| 0436 | 01197857 | 0481 | 01386079 | 0526 | 01582861 |
| 0437 | 01201943 | 0482 | 01390361 | 0527 | 01587328 |
| 0438 | 01206034 | 0483 | 01394647 | 0528 | 01591798 |
| 0439 | 01210129 | 0484 | 01398937 | 0529 | 01596273 |
| 0440 | 01214229 | 0485 | 01403231 | 0530 | 01600752 |
| 0441 | 01218333 | 0486 | 01407530 | 0531 | 01605234 |
| 0442 | 01222441 | 0487 | 01411833 | 0532 | 01609720 |
| 0443 | 01226554 | 0488 | 01416140 | 0533 | 01614211 |
| 0444 | 01230672 | 0489 | 01420451 | 0534 | 01618706 |
| 0445 | 01234794 | 0490 | 01424766 | 0535 | 01623205 |
| 0446 | 01238920 | 0491 | 01429085 | 0536 | 01627707 |
| 0447 | 01243050 | 0492 | 01433409 | 0537 | 01632214 |
| 0448 | 01247186 | 0493 | 01437737 | 0538 | 01636724 |
| 0449 | 01251325 | 0494 | 01442069 | 0539 | 01641239 |
| 0450 | 01255469 | 0495 | 01446405 | 0540 | 01645757 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
|---|---|---|---|---|---|
| 0541 | 01650279 | 0586 | 01857800 | 0631 | 02073936 |
| 0542 | 01654806 | 0587 | 01862500 | 0632 | 02078801 |
| 0543 | 01659336 | 0588 | 01867203 | 0633 | 02083669 |
| 0544 | 01663870 | 0589 | 01871910 | 0634 | 02088541 |
| 0545 | 01668408 | 0590 | 01876621 | 0635 | 02093416 |
| 0546 | 01672950 | 0591 | 01881335 | 0636 | 02098295 |
| 0547 | 01677496 | 0592 | 01886053 | 0637 | 02103178 |
| 0548 | 01682046 | 0593 | 01890775 | 0638 | 02108064 |
| 0549 | 01686600 | 0594 | 01895500 | 0639 | 02112953 |
| 0550 | 01691157 | 0595 | 01900230 | 0640 | 02117847 |
| 0551 | 01695719 | 0596 | 01904963 | 0641 | 02122744 |
| 0552 | 01700284 | 0597 | 01909700 | 0642 | 02127644 |
| 0553 | 01704854 | 0598 | 01914440 | 0643 | 02132548 |
| 0554 | 01709427 | 0599 | 01919184 | 0644 | 02137455 |
| 0555 | 01714004 | 0600 | 01923932 | 0645 | 02142366 |
| 0556 | 01718585 | 0601 | 01928684 | 0646 | 02147281 |
| 0557 | 01723170 | 0602 | 01933439 | 0647 | 02152195 |
| 0558 | 01727759 | 0603 | 01938198 | 0648 | 02157121 |
| 0559 | 01732351 | 0604 | 01942961 | 0649 | 02162046 |
| 0560 | 01736948 | 0605 | 01947727 | 0650 | 02166975 |
| 0561 | 01741548 | 0606 | 01952497 | 0651 | 02171906 |
| 0562 | 01746152 | 0607 | 01957271 | 0652 | 02176842 |
| 0563 | 01750760 | 0608 | 01962048 | 0653 | 02181781 |
| 0564 | 01755372 | 0609 | 01966829 | 0654 | 02186724 |
| 0565 | 01759988 | 0610 | 01971614 | 0655 | 02191671 |
| 0566 | 01764608 | 0611 | 01976403 | 0656 | 02196620 |
| 0567 | 01769231 | 0612 | 01981195 | 0657 | 02201574 |
| 0568 | 01773859 | 0613 | 01986990 | 0658 | 02206531 |
| 0569 | 01778490 | 0614 | 01991790 | 0659 | 02211491 |
| 0570 | 01783125 | 0615 | 01996593 | 0660 | 02216455 |
| 0571 | 01787763 | 0616 | 02001400 | 0661 | 02220422 |
| 0572 | 01792406 | 0617 | 02006210 | 0662 | 02225393 |
| 0573 | 01797052 | 0618 | 02011024 | 0663 | 02230368 |
| 0574 | 01801702 | 0619 | 02015842 | 0664 | 02235346 |
| 0575 | 01806356 | 0620 | 02020663 | 0665 | 02240327 |
| 0576 | 01811014 | 0621 | 02025488 | 0666 | 02245312 |
| 0577 | 01815676 | 0622 | 02030317 | 0667 | 02250300 |
| 0578 | 01820341 | 0623 | 02035149 | 0668 | 02255292 |
| 0579 | 01825010 | 0624 | 02039985 | 0669 | 02260287 |
| 0580 | 01829683 | 0625 | 02044824 | 0670 | 02265286 |
| 0581 | 01834360 | 0626 | 02049667 | 0671 | 02270288 |
| 0582 | 01839041 | 0627 | 02054514 | 0672 | 02275294 |
| 0583 | 01843725 | 0628 | 02059364 | 0673 | 02280303 |
| 0584 | 01848413 | 0629 | 02064218 | 0674 | 02285316 |
| 0585 | 01853105 | 0630 | 02069075 | 0675 | 02290332 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
|---|---|---|---|---|---|
| 0676 | 02295351 | 0721 | 02524745 | 0766 | 02760840 |
| 0677 | 02300374 | 0722 | 02529920 | 0767 | 02766161 |
| 0678 | 02305401 | 0723 | 02535098 | 0768 | 02771485 |
| 0679 | 02310431 | 0724 | 02540279 | 0769 | 02776812 |
| 0680 | 02315484 | 0725 | 02545464 | 0770 | 02782142 |
| 0681 | 02320500 | 0726 | 02550652 | 0771 | 02787475 |
| 0682 | 02325540 | 0727 | 02555843 | 0772 | 02792812 |
| 0683 | 02330584 | 0728 | 02561037 | 0773 | 02798152 |
| 0684 | 02335631 | 0729 | 02566235 | 0774 | 02803495 |
| 0685 | 02340681 | 0730 | 02571437 | 0775 | 02808841 |
| 0686 | 02345735 | 0731 | 02576641 | 0776 | 02814190 |
| 0687 | 02350792 | 0732 | 02581848 | 0777 | 02819543 |
| 0688 | 02355852 | 0733 | 02587059 | 0778 | 02824898 |
| 0689 | 02360916 | 0734 | 02592273 | 0779 | 02830257 |
| 0690 | 02365984 | 0735 | 02597491 | 0780 | 02835619 |
| 0691 | 02371055 | 0736 | 02602712 | 0781 | 02840984 |
| 0692 | 02376129 | 0737 | 02607936 | 0782 | 02846352 |
| 0693 | 02381206 | 0738 | 02613163 | 0783 | 02851723 |
| 0694 | 02386287 | 0739 | 02618393 | 0784 | 02857097 |
| 0695 | 02391372 | 0740 | 02623627 | 0785 | 02862475 |
| 0696 | 02396459 | 0741 | 02628864 | 0786 | 02867856 |
| 0697 | 02401550 | 0742 | 02634105 | 0787 | 02873240 |
| 0698 | 02406646 | 0743 | 02639348 | 0788 | 02878627 |
| 0699 | 02411743 | 0744 | 02644598 | 0789 | 02884017 |
| 0700 | 02416845 | 0745 | 02649845 | 0790 | 02889410 |
| 0701 | 02421949 | 0746 | 02655098 | 0791 | 02894806 |
| 0702 | 02427057 | 0747 | 02660355 | 0792 | 02900206 |
| 0703 | 02432169 | 0748 | 02665614 | 0793 | 02905608 |
| 0704 | 02437283 | 0749 | 02670877 | 0794 | 02911014 |
| 0705 | 02442401 | 0750 | 02676144 | 0795 | 02916423 |
| 0706 | 02447523 | 0751 | 02681413 | 0796 | 02921835 |
| 0707 | 02452648 | 0752 | 02686686 | 0797 | 02927250 |
| 0708 | 02457776 | 0753 | 02691962 | 0798 | 02932668 |
| 0709 | 02462908 | 0754 | 02697241 | 0799 | 02938089 |
| 0710 | 02468042 | 0755 | 02702526 | 0800 | 02943513 |
| 0711 | 02473180 | 0756 | 02707809 | 0801 | 02948941 |
| 0712 | 02478322 | 0757 | 02713097 | 0802 | 02954371 |
| 0713 | 02483467 | 0758 | 02718389 | 0803 | 02959805 |
| 0714 | 02488615 | 0759 | 02723684 | 0804 | 02965241 |
| 0715 | 02493766 | 0760 | 02728982 | 0805 | 02970681 |
| 0716 | 02498921 | 0761 | 02734284 | 0806 | 02976124 |
| 0717 | 02504079 | 0762 | 02739589 | 0807 | 02981570 |
| 0718 | 02509241 | 0763 | 02744897 | 0808 | 02987019 |
| 0719 | 02514405 | 0764 | 02750208 | 0809 | 02992471 |
| 0720 | 02519574 | 0765 | 02755523 | 0810 | 02997926 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
|---|---|---|---|---|---|
| 0811 | 03003385 | 0856 | 03252145 | 0901 | 03506905 |
| 0812 | 03008846 | 0857 | 03257742 | 0902 | 03512633 |
| 0813 | 03014310 | 0858 | 03263342 | 0903 | 03518364 |
| 0814 | 03019778 | 0859 | 03268945 | 0904 | 03524098 |
| 0815 | 03025248 | 0860 | 03274550 | 0905 | 03529834 |
| 0816 | 03030722 | 0861 | 03280159 | 0906 | 03535574 |
| 0817 | 03036198 | 0862 | 03285771 | 0907 | 03541316 |
| 0818 | 03041678 | 0863 | 03291386 | 0908 | 03547061 |
| 0819 | 03047161 | 0864 | 03297003 | 0909 | 03552809 |
| 0820 | 03052646 | 0865 | 03302624 | 0910 | 03558560 |
| 0821 | 03058135 | 0866 | 03308247 | 0911 | 03564313 |
| 0822 | 03063627 | 0867 | 03313874 | 0912 | 03570070 |
| 0823 | 03069122 | 0868 | 03319503 | 0913 | 03575829 |
| 0824 | 03074620 | 0869 | 03325135 | 0914 | 03581591 |
| 0825 | 03080121 | 0870 | 03330771 | 0915 | 03587356 |
| 0826 | 03085625 | 0871 | 03336409 | 0916 | 03593124 |
| 0827 | 03091132 | 0872 | 03342050 | 0917 | 03598895 |
| 0828 | 03096642 | 0873 | 03347694 | 0918 | 03604668 |
| 0829 | 03102155 | 0874 | 03353341 | 0919 | 03610444 |
| 0830 | 03107671 | 0875 | 03358991 | 0920 | 03616223 |
| 0831 | 03113190 | 0876 | 03364643 | 0921 | 03622005 |
| 0832 | 03118713 | 0877 | 03370299 | 0922 | 03627790 |
| 0833 | 03124238 | 0878 | 03375958 | 0923 | 03633578 |
| 0834 | 03129766 | 0879 | 03381619 | 0924 | 03639368 |
| 0835 | 03135297 | 0880 | 03387284 | 0925 | 03645161 |
| 0836 | 03140832 | 0881 | 03392951 | 0926 | 03650957 |
| 0837 | 03146369 | 0882 | 03398621 | 0927 | 03656756 |
| 0838 | 03151909 | 0883 | 03404294 | 0928 | 03662558 |
| 0839 | 03157452 | 0884 | 03409970 | 0929 | 03668362 |
| 0840 | 03162999 | 0885 | 03415649 | 0930 | 03674169 |
| 0841 | 03168548 | 0886 | 03421331 | 0931 | 03679980 |
| 0842 | 03174100 | 0887 | 03427016 | 0932 | 03685792 |
| 0843 | 03179655 | 0888 | 03432704 | 0933 | 03691608 |
| 0844 | 03185214 | 0889 | 03438394 | 0934 | 03697426 |
| 0845 | 03190775 | 0890 | 03444088 | 0935 | 03703248 |
| 0846 | 03196339 | 0891 | 03449784 | 0936 | 03709072 |
| 0847 | 03201906 | 0892 | 03455483 | 0937 | 03714899 |
| 0848 | 03207476 | 0893 | 03461185 | 0938 | 03720728 |
| 0849 | 03213050 | 0894 | 03466890 | 0939 | 03726561 |
| 0850 | 03218626 | 0895 | 03472598 | 0940 | 03732396 |
| 0851 | 03224205 | 0896 | 03478309 | 0941 | 03738234 |
| 0852 | 03229787 | 0897 | 03484022 | 0942 | 03744074 |
| 0853 | 03235372 | 0898 | 03489739 | 0943 | 03749918 |
| 0854 | 03240960 | 0899 | 03495458 | 0944 | 03755764 |
| 0855 | 03246551 | 0900 | 03501180 | 0945 | 03761613 |

| Verf. fine | Seg. area | Verf. fine | Seg. area | Verf. fine | Seg. area |
|---|---|---|---|---|---|
| 0946 | 03767465 | 0991 | 04033636 | 1036 | 04305241 |
| 0947 | 03773320 | 0992 | 04039613 | 1037 | 04311338 |
| 0948 | 03779177 | 0993 | 04045593 | 1038 | 04317436 |
| 0949 | 03785037 | 0994 | 04051576 | 1039 | 04323538 |
| 0950 | 03790900 | 0995 | 04057561 | 1040 | 04329642 |
| 0951 | 03796766 | 0996 | 04063549 | 1041 | 04335748 |
| 0952 | 03802634 | 0997 | 04069540 | 1042 | 04341857 |
| 0953 | 03808506 | 0998 | 04075533 | 1043 | 04347969 |
| 0954 | 03814380 | 0999 | 04081529 | 1044 | 04354083 |
| 0955 | 03820256 | 1000 | 04087528 | 1045 | 04360200 |
| 0956 | 03826136 | 1001 | 04093529 | 1046 | 04366319 |
| 0957 | 03832018 | 1002 | 04099533 | 1047 | 04372442 |
| 0958 | 03837903 | 1003 | 04105540 | 1048 | 04378566 |
| 0959 | 03843791 | 1004 | 04111549 | 1049 | 04384693 |
| 0960 | 03849681 | 1005 | 04117561 | 1050 | 04390823 |
| 0961 | 03855574 | 1006 | 04123576 | 1051 | 04396955 |
| 0962 | 03861470 | 1007 | 04129593 | 1052 | 04403090 |
| 0963 | 03867369 | 1008 | 04135613 | 1053 | 04409228 |
| 0964 | 03873270 | 1009 | 04141635 | 1054 | 04415368 |
| 0965 | 03879174 | 1010 | 04147661 | 1055 | 04421511 |
| 0966 | 03885081 | 1011 | 04153689 | 1056 | 04427656 |
| 0967 | 03890991 | 1012 | 04159719 | 1057 | 04433804 |
| 0968 | 03896903 | 1013 | 04165752 | 1058 | 04439954 |
| 0969 | 03902818 | 1014 | 04171788 | 1059 | 04446107 |
| 0970 | 03908736 | 1015 | 04177827 | 1060 | 04452262 |
| 0971 | 03914657 | 1016 | 04183868 | 1061 | 04458420 |
| 0972 | 03920580 | 1017 | 04189912 | 1062 | 04464581 |
| 0973 | 03926506 | 1018 | 04195958 | 1063 | 04470744 |
| 0974 | 03932435 | 1019 | 04202007 | 1064 | 04476910 |
| 0975 | 03938366 | 1020 | 04208059 | 1065 | 04483078 |
| 0976 | 03944300 | 1021 | 04214113 | 1066 | 04489249 |
| 0977 | 03950237 | 1022 | 04220170 | 1067 | 04495422 |
| 0978 | 03956176 | 1023 | 04226229 | 1068 | 04501598 |
| 0979 | 03962119 | 1024 | 04232291 | 1069 | 04507777 |
| 0980 | 03968064 | 1025 | 04238356 | 1070 | 04513958 |
| 0981 | 03974011 | 1026 | 04244424 | 1071 | 04520141 |
| 0982 | 03979962 | 1027 | 04250494 | 1072 | 04526327 |
| 0983 | 03985915 | 1028 | 04256566 | 1073 | 04532516 |
| 0984 | 03991870 | 1029 | 04262642 | 1074 | 04538707 |
| 0985 | 03997829 | 1030 | 04268719 | 1075 | 04544901 |
| 0986 | 04003790 | 1031 | 04274800 | 1076 | 04551097 |
| 0987 | 04009754 | 1032 | 04280883 | 1077 | 04557296 |
| 0988 | 04015720 | 1033 | 04286969 | 1078 | 04563497 |
| 0989 | 04021689 | 1034 | 04293057 | 1079 | 04569701 |
| 0990 | 04027661 | 1035 | 04299148 | 1080 | 04575907 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 1081 | 04582116 | 1126 | 04864103 | 1171 | 05151054 |
| 1082 | 04588327 | 1127 | 04870426 | 1172 | 05157486 |
| 1083 | 04594541 | 1128 | 04876752 | 1173 | 05163920 |
| 1084 | 04600758 | 1129 | 04883080 | 1174 | 05170357 |
| 1085 | 04606977 | 1130 | 04889411 | 1175 | 05176796 |
| 1086 | 04613198 | 1131 | 04895744 | 1176 | 05183237 |
| 1087 | 04619422 | 1132 | 04902079 | 1177 | 05189681 |
| 1088 | 04625649 | 1133 | 04908417 | 1178 | 05196127 |
| 1089 | 04631878 | 1134 | 04914758 | 1179 | 05202576 |
| 1090 | 04638109 | 1135 | 04921100 | 1180 | 05209027 |
| 1091 | 04644343 | 1136 | 04927446 | 1181 | 05215480 |
| 1092 | 04650580 | 1137 | 04933793 | 1182 | 05221936 |
| 1093 | 04656819 | 1138 | 04940144 | 1183 | 05228394 |
| 1094 | 04663060 | 1139 | 04946496 | 1184 | 05234855 |
| 1095 | 04669304 | 1140 | 04952851 | 1185 | 05241317 |
| 1096 | 04675551 | 1141 | 04959209 | 1186 | 05247783 |
| 1097 | 04681800 | 1142 | 04965568 | 1187 | 05254250 |
| 1098 | 04688052 | 1143 | 04971931 | 1188 | 05260720 |
| 1099 | 04694306 | 1144 | 04978295 | 1189 | 05267192 |
| 1100 | 04700562 | 1145 | 04984663 | 1190 | 05273667 |
| 1101 | 04706821 | 1146 | 04991032 | 1191 | 05280144 |
| 1102 | 04713083 | 1147 | 04997404 | 1192 | 05286623 |
| 1103 | 04719347 | 1148 | 05003779 | 1193 | 05293105 |
| 1104 | 04725613 | 1149 | 05010155 | 1194 | 05299589 |
| 1105 | 04731882 | 1150 | 05016535 | 1195 | 05306075 |
| 1106 | 04738154 | 1151 | 05022916 | 1196 | 05312564 |
| 1107 | 04744428 | 1152 | 05029300 | 1197 | 05319055 |
| 1108 | 04750704 | 1153 | 05035687 | 1198 | 05325548 |
| 1109 | 04756983 | 1154 | 05042076 | 1199 | 05332044 |
| 1110 | 04763265 | 1155 | 05048467 | 1200 | 05338542 |
| 1111 | 04769548 | 1156 | 05054861 | 1201 | 05345042 |
| 1112 | 04775835 | 1157 | 05061257 | 1202 | 05351545 |
| 1113 | 04782124 | 1158 | 05067655 | 1203 | 05358050 |
| 1114 | 04788415 | 1159 | 05074056 | 1204 | 05364558 |
| 1115 | 04794709 | 1160 | 05080459 | 1205 | 05371067 |
| 1116 | 04801005 | 1161 | 05086865 | 1206 | 05377579 |
| 1117 | 04807304 | 1162 | 05093273 | 1207 | 05384094 |
| 1118 | 04813605 | 1163 | 05099684 | 1208 | 05390610 |
| 1119 | 04819908 | 1164 | 05106097 | 1209 | 05397130 |
| 1120 | 04826215 | 1165 | 05112512 | 1210 | 05403651 |
| 1121 | 04832523 | 1166 | 05118930 | 1211 | 05410175 |
| 1122 | 04838834 | 1167 | 05125350 | 1212 | 05416701 |
| 1123 | 04845148 | 1168 | 05131772 | 1213 | 05423229 |
| 1124 | 04851464 | 1169 | 05138197 | 1214 | 05429760 |
| 1125 | 04857782 | 1170 | 05144624 | 1215 | 05436293 |

| Verf. fine | Seg. area | Verf. fine | Seg. area | Verf. fine | Seg. area |
|---|---|---|---|---|---|
| 1216 | 05442828 | 1261 | 05739292 | 1306 | 06040318 |
| 1217 | 05449366 | 1262 | 05745932 | 1307 | 06047058 |
| 1218 | 05455906 | 1263 | 05752575 | 1308 | 06053801 |
| 1219 | 05462448 | 1264 | 05759220 | 1309 | 06060546 |
| 1220 | 05468992 | 1265 | 05765867 | 1310 | 06067293 |
| 1221 | 05475539 | 1266 | 05772516 | 1311 | 06074042 |
| 1222 | 05482088 | 1267 | 05779168 | 1312 | 06080793 |
| 1223 | 05488640 | 1268 | 05785822 | 1313 | 06087546 |
| 1224 | 05495194 | 1269 | 05792478 | 1314 | 06094302 |
| 1225 | 05501750 | 1270 | 05799136 | 1315 | 06101060 |
| 1226 | 05508308 | 1271 | 05805797 | 1316 | 06107820 |
| 1227 | 05514869 | 1272 | 05812460 | 1317 | 06114582 |
| 1228 | 05521432 | 1273 | 05819125 | 1318 | 06121347 |
| 1229 | 05527997 | 1274 | 05825792 | 1319 | 06128113 |
| 1230 | 05534565 | 1275 | 05832461 | 1320 | 06134882 |
| 1231 | 05541135 | 1276 | 05839133 | 1321 | 06141653 |
| 1232 | 05547707 | 1277 | 05845807 | 1322 | 06148426 |
| 1233 | 05554281 | 1278 | 05852483 | 1323 | 06155201 |
| 1234 | 05560858 | 1279 | 05859162 | 1324 | 06161978 |
| 1235 | 05567437 | 1280 | 05865843 | 1325 | 06168758 |
| 1236 | 05574019 | 1281 | 05872525 | 1326 | 06175540 |
| 1237 | 05580602 | 1282 | 05879211 | 1327 | 06182324 |
| 1238 | 05587188 | 1283 | 05885898 | 1328 | 06189110 |
| 1239 | 05593776 | 1284 | 05892588 | 1329 | 06195898 |
| 1240 | 05600367 | 1285 | 05899279 | 1330 | 06202688 |
| 1241 | 05606960 | 1286 | 05905973 | 1331 | 06209481 |
| 1242 | 05613555 | 1287 | 05912670 | 1332 | 06216276 |
| 1243 | 05620152 | 1288 | 05919368 | 1333 | 06223073 |
| 1244 | 05626752 | 1289 | 05926069 | 1334 | 06229872 |
| 1245 | 05633353 | 1290 | 05932772 | 1335 | 06236673 |
| 1246 | 05639958 | 1291 | 05939477 | 1336 | 06243476 |
| 1247 | 05646564 | 1292 | 05946184 | 1337 | 06250282 |
| 1248 | 05653173 | 1293 | 05952894 | 1338 | 06257090 |
| 1249 | 05659784 | 1294 | 05959605 | 1339 | 06263899 |
| 1250 | 05666397 | 1295 | 05966319 | 1340 | 06270711 |
| 1251 | 05673012 | 1296 | 05973035 | 1341 | 06277525 |
| 1252 | 05679730 | 1297 | 05979754 | 1342 | 06284342 |
| 1253 | 05686250 | 1298 | 05986474 | 1343 | 06291160 |
| 1254 | 05692873 | 1299 | 05993197 | 1344 | 06297981 |
| 1255 | 05699497 | 1300 | 05999922 | 1345 | 06304803 |
| 1256 | 05706124 | 1301 | 06006649 | 1346 | 06311628 |
| 1257 | 05712753 | 1302 | 06013379 | 1347 | 06318455 |
| 1258 | 05719384 | 1303 | 06020110 | 1348 | 06325284 |
| 1259 | 05726018 | 1304 | 06026844 | 1349 | 06332116 |
| 1260 | 05732654 | 1305 | 06033580 | 1350 | 06338949 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
|---|---|---|---|---|---|
| 1351 | 06345785 | 1396 | 06655575 | 1441 | 06969577 |
| 1352 | 06352622 | 1397 | 06662507 | 1442 | 06976602 |
| 1353 | 06359462 | 1398 | 06669442 | 1443 | 06983629 |
| 1354 | 06366304 | 1399 | 06676378 | 1444 | 06990657 |
| 1355 | 06373148 | 1400 | 06683317 | 1445 | 06997688 |
| 1356 | 06379994 | 1401 | 06690258 | 1446 | 07004721 |
| 1357 | 06386843 | 1402 | 06697201 | 1447 | 07011756 |
| 1358 | 06393693 | 1403 | 06704146 | 1448 | 07018793 |
| 1359 | 06400546 | 1404 | 06711093 | 1449 | 07025832 |
| 1360 | 06407400 | 1405 | 06718042 | 1450 | 07032873 |
| 1361 | 06414257 | 1406 | 06724993 | 1451 | 07039916 |
| 1362 | 06421116 | 1407 | 06731946 | 1452 | 07046961 |
| 1363 | 06427977 | 1408 | 06738901 | 1453 | 07054008 |
| 1364 | 06434840 | 1409 | 06745859 | 1454 | 07061057 |
| 1365 | 06441706 | 1410 | 06752818 | 1455 | 07068108 |
| 1366 | 06448573 | 1411 | 06759780 | 1456 | 07075162 |
| 1367 | 06455443 | 1412 | 06766743 | 1457 | 07082217 |
| 1368 | 06462314 | 1413 | 06773709 | 1458 | 07089274 |
| 1369 | 06469188 | 1414 | 06780676 | 1459 | 07096333 |
| 1370 | 06476064 | 1415 | 06787646 | 1460 | 07103394 |
| 1371 | 06482942 | 1416 | 06794618 | 1461 | 07110457 |
| 1372 | 06489822 | 1417 | 06801592 | 1462 | 07117522 |
| 1373 | 06496704 | 1418 | 06808567 | 1463 | 07124589 |
| 1374 | 06503589 | 1419 | 06815545 | 1464 | 07131659 |
| 1375 | 06510475 | 1420 | 06822525 | 1465 | 07138730 |
| 1376 | 06517364 | 1421 | 06829507 | 1466 | 07145803 |
| 1377 | 06524254 | 1422 | 06836491 | 1467 | 07152878 |
| 1378 | 06531147 | 1423 | 06843478 | 1468 | 07159955 |
| 1379 | 06538042 | 1424 | 06850466 | 1469 | 07167034 |
| 1380 | 06544939 | 1425 | 06857456 | 1470 | 07174115 |
| 1381 | 06551838 | 1426 | 06864448 | 1471 | 07181198 |
| 1382 | 06558739 | 1427 | 06871443 | 1472 | 07188284 |
| 1383 | 06565642 | 1428 | 06878439 | 1473 | 07195371 |
| 1384 | 06572548 | 1429 | 06885437 | 1474 | 07202460 |
| 1385 | 06579455 | 1430 | 06892438 | 1475 | 07209551 |
| 1386 | 06586365 | 1431 | 06899440 | 1476 | 07216644 |
| 1387 | 06593276 | 1432 | 06906445 | 1477 | 07223739 |
| 1388 | 06600190 | 1433 | 06913451 | 1478 | 07230836 |
| 1389 | 06607106 | 1434 | 06920460 | 1479 | 07237935 |
| 1390 | 06614024 | 1435 | 06927470 | 1480 | 07245036 |
| 1391 | 06620944 | 1436 | 06934483 | 1481 | 07252139 |
| 1392 | 06627866 | 1437 | 06941498 | 1482 | 07259244 |
| 1393 | 06634790 | 1438 | 06948515 | 1483 | 07266351 |
| 1394 | 06641716 | 1439 | 06955533 | 1484 | 07273460 |
| 1395 | 06648644 | 1440 | 06962554 | 1485 | 07280571 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 1486 | 07287084 | 1531 | 07609792 | 1576 | 07936802 |
| 1487 | 07294798 | 1532 | 07616994 | 1577 | 07943090 |
| 1488 | 07301915 | 1533 | 07624199 | 1578 | 07950380 |
| 1489 | 07309034 | 1534 | 07631406 | 1579 | 07957672 |
| 1490 | 07316155 | 1535 | 07638614 | 1580 | 07964966 |
| 1491 | 07323278 | 1536 | 07645824 | 1581 | 07972262 |
| 1492 | 07330402 | 1537 | 07653037 | 1582 | 07979560 |
| 1493 | 07337529 | 1538 | 07660251 | 1583 | 07986859 |
| 1494 | 07344658 | 1539 | 07667467 | 1584 | 07994161 |
| 1495 | 07351788 | 1540 | 07674685 | 1585 | 08001464 |
| 1496 | 07358921 | 1541 | 07681905 | 1586 | 08008769 |
| 1497 | 07366056 | 1542 | 07689127 | 1587 | 08016076 |
| 1498 | 07373192 | 1543 | 07696350 | 1588 | 08023385 |
| 1499 | 07380331 | 1544 | 07703576 | 1589 | 08030696 |
| 1500 | 07387471 | 1545 | 07710804 | 1590 | 08038008 |
| 1501 | 07394613 | 1546 | 07718033 | 1591 | 08045323 |
| 1502 | 07401758 | 1547 | 07725265 | 1592 | 08052639 |
| 1503 | 07408904 | 1548 | 07732498 | 1593 | 08059957 |
| 1504 | 07416052 | 1549 | 07739733 | 1594 | 08067277 |
| 1505 | 07423203 | 1550 | 07746970 | 1595 | 08074599 |
| 1506 | 07430355 | 1551 | 07754209 | 1596 | 08081923 |
| 1507 | 07437509 | 1552 | 07761450 | 1597 | 08089248 |
| 1508 | 07444665 | 1553 | 07768693 | 1598 | 08096576 |
| 1509 | 07451823 | 1554 | 07775938 | 1599 | 08103905 |
| 1510 | 07458983 | 1555 | 07783185 | 1600 | 08111236 |
| 1511 | 07466145 | 1556 | 07790433 | 1601 | 08118569 |
| 1512 | 07473309 | 1557 | 07797684 | 1602 | 08125904 |
| 1513 | 07480475 | 1558 | 07804936 | 1603 | 08133241 |
| 1514 | 07487643 | 1559 | 07812190 | 1604 | 08140580 |
| 1515 | 07494812 | 1560 | 07819446 | 1605 | 08147920 |
| 1516 | 07501984 | 1561 | 07826704 | 1606 | 08155262 |
| 1517 | 07509158 | 1562 | 07833964 | 1607 | 08162607 |
| 1518 | 07516333 | 1563 | 07841226 | 1608 | 08169953 |
| 1519 | 07523511 | 1564 | 07848490 | 1609 | 08177300 |
| 1520 | 07530690 | 1565 | 07855756 | 1610 | 08184650 |
| 1521 | 07537872 | 1566 | 07863023 | 1611 | 08192002 |
| 1522 | 07545055 | 1567 | 07870292 | 1612 | 08199355 |
| 1523 | 07552240 | 1568 | 07877564 | 1613 | 08206710 |
| 1524 | 07559427 | 1569 | 07884837 | 1614 | 08214067 |
| 1525 | 07566616 | 1570 | 07892112 | 1615 | 08221426 |
| 1526 | 07573808 | 1571 | 07899389 | 1616 | 08228789 |
| 1527 | 07581001 | 1572 | 07906668 | 1617 | 08236150 |
| 1528 | 07588195 | 1573 | 07913949 | 1618 | 08243514 |
| 1529 | 07595392 | 1574 | 07921231 | 1619 | 08250880 |
| 1530 | 07602591 | 1575 | 07928516 | 1620 | 08258248 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 1621 | 08265618 | 1666 | 08599148 | 1711 | 08936302 |
| 1622 | 08272990 | 1667 | 08606601 | 1712 | 08943835 |
| 1623 | 08280364 | 1668 | 08614056 | 1713 | 08951369 |
| 1624 | 08287739 | 1669 | 08621513 | 1714 | 08958906 |
| 1625 | 08295116 | 1670 | 08628972 | 1715 | 08966444 |
| 1626 | 08302495 | 1671 | 08636432 | 1716 | 08973983 |
| 1627 | 08309876 | 1672 | 08643894 | 1717 | 08981525 |
| 1628 | 08317259 | 1673 | 08651358 | 1718 | 08989068 |
| 1629 | 08324644 | 1674 | 08658824 | 1719 | 08996613 |
| 1630 | 08332030 | 1675 | 08666292 | 1720 | 09004160 |
| 1631 | 08339418 | 1676 | 08673761 | 1721 | 09011709 |
| 1632 | 08346808 | 1677 | 08681232 | 1722 | 09019259 |
| 1633 | 08354200 | 1678 | 08688705 | 1723 | 09026811 |
| 1634 | 08361594 | 1679 | 08696180 | 1724 | 09034364 |
| 1635 | 08368989 | 1680 | 08703656 | 1725 | 09041920 |
| 1636 | 08376387 | 1681 | 08711134 | 1726 | 09049477 |
| 1637 | 08383786 | 1682 | 08718614 | 1727 | 09057036 |
| 1638 | 08391187 | 1683 | 08726096 | 1728 | 09064596 |
| 1639 | 08398590 | 1684 | 08733580 | 1729 | 09072159 |
| 1640 | 08405994 | 1685 | 08741065 | 1730 | 09079723 |
| 1641 | 08413401 | 1686 | 08748552 | 1731 | 09087289 |
| 1642 | 08420809 | 1687 | 08756041 | 1732 | 09094856 |
| 1643 | 08428219 | 1688 | 08763532 | 1733 | 09102425 |
| 1644 | 08435631 | 1689 | 08771024 | 1734 | 09109996 |
| 1645 | 08443044 | 1690 | 08778518 | 1735 | 09117569 |
| 1646 | 08450460 | 1691 | 08786014 | 1736 | 09125143 |
| 1647 | 08457877 | 1692 | 08793512 | 1737 | 09132720 |
| 1648 | 08465296 | 1693 | 08801011 | 1738 | 09140298 |
| 1649 | 08472717 | 1694 | 08808512 | 1739 | 09147877 |
| 1650 | 08480140 | 1695 | 08816015 | 1740 | 09155458 |
| 1651 | 08487564 | 1696 | 08823520 | 1741 | 09163042 |
| 1652 | 08494991 | 1697 | 08831027 | 1742 | 09170626 |
| 1653 | 08502419 | 1698 | 08838535 | 1743 | 09178213 |
| 1654 | 08509849 | 1699 | 08846045 | 1744 | 09185801 |
| 1655 | 08517281 | 1700 | 08853557 | 1745 | 09193391 |
| 1656 | 08524714 | 1701 | 08861070 | 1746 | 09200983 |
| 1657 | 08532149 | 1702 | 08868585 | 1747 | 09208576 |
| 1658 | 08539587 | 1703 | 08876103 | 1748 | 09216171 |
| 1659 | 08547025 | 1704 | 08883621 | 1749 | 09223768 |
| 1660 | 08554466 | 1705 | 08891142 | 1750 | 09231366 |
| 1661 | 08561909 | 1706 | 08898664 | 1751 | 09238966 |
| 1662 | 08569353 | 1707 | 08906188 | 1752 | 09246568 |
| 1663 | 08576799 | 1708 | 08913714 | 1753 | 09254172 |
| 1664 | 08584247 | 1709 | 08921242 | 1754 | 09261777 |
| 1665 | 08591697 | 1710 | 08928771 | 1755 | 09269384 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
|---|---|---|---|---|---|
| 1756 | 09276993 | 1801 | 09621138 | 1846 | 09968654 |
| 1757 | 09284603 | 1802 | 09628824 | 1847 | 09976414 |
| 1758 | 09292216 | 1803 | 09636512 | 1848 | 09984176 |
| 1759 | 09299829 | 1804 | 09644201 | 1849 | 09991939 |
| 1760 | 09307445 | 1805 | 09651893 | 1850 | 09999704 |
| 1761 | 09315062 | 1806 | 09659585 | 1851 | 10007471 |
| 1762 | 09322681 | 1807 | 09667280 | 1852 | 10015240 |
| 1763 | 09330302 | 1808 | 09674976 | 1853 | 10023010 |
| 1764 | 09337924 | 1809 | 09682674 | 1854 | 10030781 |
| 1765 | 09345548 | 1810 | 09690374 | 1855 | 10038555 |
| 1766 | 09353174 | 1811 | 09698075 | 1856 | 10046330 |
| 1767 | 09360802 | 1812 | 09705778 | 1857 | 10054106 |
| 1768 | 09368431 | 1813 | 09713482 | 1858 | 10061884 |
| 1769 | 09376062 | 1814 | 09721188 | 1859 | 10069664 |
| 1770 | 09383694 | 1815 | 09728896 | 1860 | 10077445 |
| 1771 | 09391328 | 1816 | 09736606 | 1861 | 10085228 |
| 1772 | 09398964 | 1817 | 09744317 | 1862 | 10093012 |
| 1773 | 09406602 | 1818 | 09752029 | 1863 | 10100799 |
| 1774 | 09414241 | 1819 | 09759744 | 1864 | 10108586 |
| 1775 | 09421882 | 1820 | 09767460 | 1865 | 10116376 |
| 1776 | 09429525 | 1821 | 09775178 | 1866 | 10124167 |
| 1777 | 09437169 | 1822 | 09782897 | 1867 | 10131959 |
| 1778 | 09444815 | 1823 | 09790618 | 1868 | 10139754 |
| 1779 | 09452463 | 1824 | 09798341 | 1869 | 10147549 |
| 1780 | 09460112 | 1825 | 09806065 | 1870 | 10155347 |
| 1781 | 09467764 | 1826 | 09813791 | 1871 | 10163146 |
| 1782 | 09475416 | 1827 | 09821519 | 1872 | 10170947 |
| 1783 | 09483071 | 1828 | 09829248 | 1873 | 10178749 |
| 1784 | 09490727 | 1829 | 09836979 | 1874 | 10186553 |
| 1785 | 09498385 | 1830 | 09844711 | 1875 | 10194358 |
| 1786 | 09506044 | 1831 | 09852445 | 1876 | 10202165 |
| 1787 | 09513705 | 1832 | 09860181 | 1877 | 10209974 |
| 1788 | 09521368 | 1833 | 09867919 | 1878 | 10217784 |
| 1789 | 09529033 | 1834 | 09875658 | 1879 | 10225596 |
| 1790 | 09536699 | 1835 | 09883398 | 1880 | 10233409 |
| 1791 | 09544367 | 1836 | 09891141 | 1881 | 10241224 |
| 1792 | 09552036 | 1837 | 09898885 | 1882 | 10249041 |
| 1793 | 09559708 | 1838 | 09906630 | 1883 | 10256859 |
| 1794 | 09567381 | 1839 | 09914377 | 1884 | 10264679 |
| 1795 | 09575055 | 1840 | 09922126 | 1885 | 10272500 |
| 1796 | 09582731 | 1841 | 09929877 | 1886 | 10280323 |
| 1797 | 09590409 | 1842 | 09937629 | 1887 | 10288148 |
| 1798 | 09598089 | 1843 | 09945383 | 1888 | 10295974 |
| 1799 | 09605770 | 1844 | 09953138 | 1889 | 10303802 |
| 1800 | 09613453 | 1845 | 09960895 | 1890 | 10311631 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
|---|---|---|---|---|---|
| 1891 | 10319462 | 1936 | 10673487 | 1981 | 11030652 |
| 1892 | 10327295 | 1937 | 10681390 | 1982 | 11038624 |
| 1893 | 10335129 | 1938 | 10689295 | 1983 | 11046598 |
| 1894 | 10342965 | 1939 | 10697201 | 1984 | 11054573 |
| 1895 | 10350802 | 1940 | 10705109 | 1985 | 11062550 |
| 1896 | 10358641 | 1941 | 10713018 | 1986 | 11070528 |
| 1897 | 10366481 | 1942 | 10720929 | 1987 | 11078507 |
| 1898 | 10374324 | 1943 | 10728842 | 1988 | 11086489 |
| 1899 | 10382167 | 1944 | 10736756 | 1989 | 11094471 |
| 1900 | 10390013 | 1945 | 10744671 | 1990 | 11102456 |
| 1901 | 10397859 | 1946 | 10752588 | 1991 | 11110441 |
| 1902 | 10405708 | 1947 | 10760507 | 1992 | 11118429 |
| 1903 | 10413558 | 1948 | 10768427 | 1993 | 11126417 |
| 1904 | 10421409 | 1949 | 10776349 | 1994 | 11134407 |
| 1905 | 10429262 | 1950 | 10784272 | 1995 | 11142399 |
| 1906 | 10437117 | 1951 | 10792197 | 1996 | 11150392 |
| 1907 | 10444973 | 1952 | 10800123 | 1997 | 11158387 |
| 1908 | 10452831 | 1953 | 10808051 | 1998 | 11166383 |
| 1909 | 10460691 | 1954 | 10815980 | 1999 | 11174381 |
| 1910 | 10468552 | 1955 | 10823911 | 2000 | 11182380 |
| 1911 | 10476414 | 1956 | 10831844 | 2001 | 11190381 |
| 1912 | 10484278 | 1957 | 10839778 | 2002 | 11198383 |
| 1913 | 10492144 | 1958 | 10847713 | 2003 | 11206387 |
| 1914 | 10500011 | 1959 | 10855650 | 2004 | 11214392 |
| 1915 | 10507880 | 1960 | 10863589 | 2005 | 11222399 |
| 1916 | 10515751 | 1961 | 10871529 | 2006 | 11230407 |
| 1917 | 10523623 | 1962 | 10879471 | 2007 | 11238417 |
| 1918 | 10531496 | 1963 | 10887414 | 2008 | 11246428 |
| 1919 | 10539371 | 1964 | 10895359 | 2009 | 11254441 |
| 1920 | 10547248 | 1965 | 10903305 | 2010 | 11262455 |
| 1921 | 10555126 | 1966 | 10911253 | 2011 | 11270471 |
| 1922 | 10563006 | 1967 | 10919202 | 2012 | 11278488 |
| 1923 | 10570887 | 1968 | 10927153 | 2013 | 11286507 |
| 1924 | 10578770 | 1969 | 10935105 | 2014 | 11294527 |
| 1925 | 10586655 | 1970 | 10943059 | 2015 | 11302549 |
| 1926 | 10594541 | 1971 | 10951014 | 2016 | 11310572 |
| 1927 | 10602428 | 1972 | 10958971 | 2017 | 11318597 |
| 1928 | 10610318 | 1973 | 10966930 | 2018 | 11326623 |
| 1929 | 10618208 | 1974 | 10974890 | 2019 | 11334650 |
| 1930 | 10626101 | 1975 | 10982851 | 2020 | 11342679 |
| 1931 | 10633994 | 1976 | 10990814 | 2021 | 11350710 |
| 1932 | 10641890 | 1977 | 10998779 | 2022 | 11358742 |
| 1933 | 10649787 | 1978 | 11006745 | 2023 | 11366776 |
| 1934 | 10657685 | 1979 | 11014713 | 2024 | 11374811 |
| 1935 | 10665585 | 1980 | 11022682 | 2025 | 11382847 |

| Verf. fine | Seg. area | Verf. fine | Seg. area | Verf. fine | Seg. area |
|---|---|---|---|---|---|
| 2026 | 11390885 | 2071 | 11754115 | 2116 | 12120272 |
| 2027 | 11398925 | 2072 | 11762220 | 2117 | 12128442 |
| 2028 | 11406966 | 2073 | 11770327 | 2118 | 12136613 |
| 2029 | 11415008 | 2074 | 11778435 | 2119 | 12144785 |
| 2030 | 11423052 | 2075 | 11786545 | 2120 | 12152959 |
| 2031 | 11431097 | 2076 | 11794656 | 2121 | 12161134 |
| 2032 | 11439144 | 2077 | 11802768 | 2122 | 12169311 |
| 2033 | 11447193 | 2078 | 11810882 | 2123 | 12177489 |
| 2034 | 11455242 | 2079 | 11818998 | 2124 | 12185668 |
| 2035 | 11463294 | 2080 | 11827114 | 2125 | 12193849 |
| 2036 | 11471346 | 2081 | 11835233 | 2126 | 12202031 |
| 2037 | 11479401 | 2082 | 11843352 | 2127 | 12210215 |
| 2238 | 11487456 | 2083 | 11851474 | 2128 | 12218400 |
| 2039 | 11495514 | 2084 | 11859596 | 2129 | 12226587 |
| 2040 | 11503572 | 2085 | 11867720 | 2130 | 12234774 |
| 2041 | 11511632 | 2086 | 11875846 | 2131 | 12242964 |
| 2042 | 11519694 | 2087 | 11883972 | 2132 | 12251154 |
| 2043 | 11527757 | 2088 | 11892101 | 2133 | 12259346 |
| 2044 | 11535821 | 2089 | 11900230 | 2134 | 12267540 |
| 2045 | 11543887 | 2090 | 11908362 | 2135 | 12275735 |
| 2046 | 11551955 | 2091 | 11916494 | 2136 | 12283931 |
| 2047 | 11560024 | 2092 | 11924628 | 2137 | 12292129 |
| 2048 | 11568094 | 2093 | 11932764 | 2138 | 12300328 |
| 2049 | 11576166 | 2094 | 11940901 | 2139 | 12308528 |
| 2050 | 11584239 | 2095 | 11949039 | 2140 | 12316730 |
| 2051 | 11592314 | 2096 | 11957179 | 2141 | 12324933 |
| 2052 | 11600390 | 2097 | 11965320 | 2142 | 12333138 |
| 2053 | 11608468 | 2098 | 11973462 | 2143 | 12341344 |
| 2054 | 11616547 | 2099 | 11981606 | 2144 | 12349551 |
| 2055 | 11624628 | 2100 | 11989752 | 2145 | 12357760 |
| 2056 | 11632710 | 2101 | 11997899 | 2146 | 12365970 |
| 2057 | 11640793 | 2102 | 12006047 | 2147 | 12374182 |
| 2058 | 11648878 | 2103 | 12014197 | 2148 | 12382395 |
| 2059 | 11656965 | 2104 | 12022348 | 2149 | 12390609 |
| 2060 | 11665053 | 2105 | 12030501 | 2150 | 12398825 |
| 2061 | 11673142 | 2106 | 12038655 | 2151 | 12407042 |
| 2062 | 11681233 | 2107 | 12046810 | 2152 | 12415261 |
| 2063 | 11689325 | 2108 | 12054967 | 2153 | 12423481 |
| 2064 | 11697419 | 2109 | 12063125 | 2154 | 12431702 |
| 2065 | 11705514 | 2110 | 12071285 | 2155 | 12439924 |
| 2066 | 11713610 | 2111 | 12079446 | 2156 | 12448149 |
| 2067 | 11721708 | 2112 | 12087608 | 2157 | 12456374 |
| 2068 | 11729808 | 2113 | 12095772 | 2158 | 12464601 |
| 2069 | 11737909 | 2114 | 12103938 | 2159 | 12472829 |
| 2070 | 11746011 | 2115 | 12112104 | 2160 | 12481059 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 2161 | 12489290 | 2206 | 12861101 | 2251 | 13235641 |
| 2162 | 12497522 | 2207 | 12869394 | 2252 | 13243994 |
| 2163 | 12505756 | 2208 | 12877689 | 2253 | 13252349 |
| 2164 | 12513991 | 2209 | 12885986 | 2254 | 13260705 |
| 2165 | 12522227 | 2210 | 12894283 | 2255 | 13269063 |
| 2166 | 12530465 | 2211 | 12902583 | 2256 | 13277422 |
| 2167 | 12538704 | 2212 | 12910883 | 2257 | 13285782 |
| 2168 | 12546945 | 2213 | 12919185 | 2258 | 13294143 |
| 2169 | 12555187 | 2214 | 12927488 | 2259 | 13302506 |
| 2170 | 12563430 | 2215 | 12935792 | 2260 | 13310870 |
| 2171 | 12571675 | 2216 | 12944098 | 2261 | 13319236 |
| 2172 | 12579921 | 2217 | 12952405 | 2262 | 13327603 |
| 2173 | 12588169 | 2218 | 12960714 | 2263 | 13335971 |
| 2174 | 12596418 | 2219 | 12969024 | 2264 | 13344340 |
| 2175 | 12604668 | 2220 | 12977335 | 2265 | 13352711 |
| 2176 | 12612919 | 2221 | 12985647 | 2266 | 13361083 |
| 2177 | 12621172 | 2222 | 12993961 | 2267 | 13369456 |
| 2178 | 12629427 | 2223 | 13002276 | 2268 | 13377831 |
| 2179 | 12637682 | 2224 | 13010593 | 2269 | 13386206 |
| 2180 | 12645939 | 2225 | 13018911 | 2270 | 13394584 |
| 2181 | 12654198 | 2226 | 13027230 | 2271 | 13402962 |
| 2182 | 12662458 | 2227 | 13035550 | 2272 | 13411342 |
| 2183 | 12670719 | 2228 | 13043872 | 2273 | 13419723 |
| 2184 | 12678981 | 2229 | 13052195 | 2274 | 13428105 |
| 2185 | 12687245 | 2230 | 13060520 | 2275 | 13436489 |
| 2186 | 12695511 | 2231 | 13068846 | 2276 | 13444874 |
| 2187 | 12703777 | 2232 | 13077173 | 2277 | 13453260 |
| 2188 | 12712045 | 2233 | 13085501 | 2278 | 13461648 |
| 2189 | 12720314 | 2234 | 13093831 | 2279 | 13470037 |
| 2190 | 12728585 | 2235 | 13102162 | 2280 | 13478427 |
| 2191 | 12736857 | 2236 | 13110495 | 2281 | 13486819 |
| 2192 | 12745131 | 2237 | 13118828 | 2282 | 13495212 |
| 2193 | 12753405 | 2238 | 13127164 | 2283 | 13503606 |
| 2194 | 12761682 | 2239 | 13135500 | 2284 | 13512001 |
| 2195 | 12769959 | 2240 | 13143838 | 2285 | 13520398 |
| 2196 | 12778238 | 2241 | 13152177 | 2286 | 13528796 |
| 2197 | 12786518 | 2242 | 13160517 | 2287 | 13537195 |
| 2198 | 12794800 | 2243 | 13168859 | 2288 | 13545595 |
| 2199 | 12803082 | 2244 | 13177203 | 2289 | 13553997 |
| 2200 | 12811367 | 2245 | 13185546 | 2290 | 13562400 |
| 2201 | 12819652 | 2246 | 13193892 | 2291 | 13570805 |
| 2202 | 12827939 | 2247 | 13202239 | 2292 | 13579211 |
| 2203 | 12836228 | 2248 | 13210588 | 2293 | 13587618 |
| 2204 | 12844517 | 2249 | 13218937 | 2294 | 13596026 |
| 2205 | 12852808 | 2250 | 13227288 | 2295 | 13604435 |

| Verf. fine | Seg. area | Verf. fine | Seg. area | Verf. fine | Seg. area |
|---|---|---|---|---|---|
| 2296 | 13612846 | 2341 | 13992656 | 2386 | 14375009 |
| 2297 | 13621259 | 2342 | 14001126 | 2387 | 14383535 |
| 2298 | 13629672 | 2343 | 14009596 | 2388 | 14392061 |
| 2299 | 13638087 | 2344 | 14018068 | 2389 | 14400589 |
| 2300 | 13646503 | 2345 | 14026541 | 2390 | 14409117 |
| 2301 | 13654920 | 2346 | 14035015 | 2391 | 14417647 |
| 2302 | 13663339 | 2347 | 14043491 | 2392 | 14426179 |
| 2303 | 13671758 | 2348 | 14051968 | 2393 | 14434711 |
| 2304 | 13680180 | 2349 | 14060446 | 2394 | 14443245 |
| 2305 | 13688602 | 2350 | 14068925 | 2395 | 14451780 |
| 2306 | 13697026 | 2351 | 14077406 | 2396 | 14460316 |
| 2307 | 13705451 | 2352 | 14085888 | 2397 | 14468854 |
| 2308 | 13713877 | 2353 | 14094371 | 2398 | 14477392 |
| 2309 | 13722304 | 2354 | 14102855 | 2399 | 14485932 |
| 2310 | 13730733 | 2355 | 14111341 | 2400 | 14494473 |
| 2311 | 13739163 | 2356 | 14119828 | 2401 | 14503015 |
| 2312 | 13747595 | 2357 | 14128316 | 2402 | 14511559 |
| 2313 | 13756027 | 2358 | 14136805 | 2403 | 14520103 |
| 2314 | 13764461 | 2359 | 14145296 | 2404 | 14528649 |
| 2315 | 13772896 | 2360 | 14153787 | 2405 | 14537197 |
| 2316 | 13781333 | 2361 | 14162281 | 2406 | 14545745 |
| 2317 | 13789771 | 2362 | 14170775 | 2407 | 14554294 |
| 2318 | 13798210 | 2363 | 14179270 | 2408 | 14562845 |
| 2319 | 13806650 | 2364 | 14187767 | 2409 | 14571397 |
| 2320 | 13815091 | 2365 | 14196265 | 2410 | 14579950 |
| 2321 | 13823534 | 2366 | 14204764 | 2411 | 14588505 |
| 2322 | 13831978 | 2367 | 14213265 | 2412 | 14597060 |
| 2323 | 13840424 | 2368 | 14221767 | 2413 | 14605617 |
| 2324 | 13848870 | 2369 | 14230270 | 2414 | 14614175 |
| 2325 | 13857318 | 2370 | 14238774 | 2415 | 14622735 |
| 2326 | 13865767 | 2371 | 14247279 | 2416 | 14631295 |
| 2327 | 13874218 | 2372 | 14255786 | 2417 | 14639857 |
| 2328 | 13882669 | 2373 | 14264294 | 2418 | 14648419 |
| 2329 | 13891123 | 2374 | 14272803 | 2419 | 14656984 |
| 2330 | 13899577 | 2375 | 14281314 | 2420 | 14665549 |
| 2331 | 13908032 | 2376 | 14289825 | 2421 | 14674115 |
| 2332 | 13916489 | 2377 | 14298338 | 2422 | 14682683 |
| 2333 | 13924947 | 2378 | 14306852 | 2423 | 14691252 |
| 2334 | 13933406 | 2379 | 14315368 | 2424 | 14699822 |
| 2335 | 13941867 | 2380 | 14323884 | 2425 | 14708393 |
| 2336 | 13950328 | 2381 | 14332402 | 2426 | 14716966 |
| 2337 | 13958791 | 2382 | 14340921 | 2427 | 14725540 |
| 2338 | 13967256 | 2383 | 14349441 | 2428 | 14734114 |
| 2339 | 13975721 | 2384 | 14357963 | 2429 | 14742691 |
| 2340 | 13984188 | 2385 | 14366485 | 2430 | 14751268 |

| Verf. fine | Seg. area | Verf. fine | Seg. area | Verf. fine | Seg. area |
|---|---|---|---|---|---|
| 2431 | 14759846 | 2476 | 15147109 | 2521 | 15536740 |
| 2432 | 14768426 | 2477 | 15155742 | 2522 | 15545425 |
| 2433 | 14777007 | 2478 | 15164376 | 2523 | 15554111 |
| 2434 | 14785589 | 2479 | 15173011 | 2524 | 15562798 |
| 2435 | 14794172 | 2480 | 15181648 | 2525 | 15571487 |
| 2436 | 14802757 | 2481 | 15190285 | 2526 | 15580176 |
| 2437 | 14811342 | 2482 | 15198924 | 2527 | 15588867 |
| 2438 | 14819929 | 2483 | 15207564 | 2528 | 15597559 |
| 2439 | 14828517 | 2484 | 15216205 | 2529 | 15606252 |
| 2440 | 14837107 | 2485 | 15224848 | 2530 | 15614946 |
| 2441 | 14845697 | 2486 | 15233491 | 2531 | 15623641 |
| 2442 | 14854289 | 2487 | 15242136 | 2532 | 15632337 |
| 2443 | 14862882 | 2488 | 15250781 | 2533 | 15641035 |
| 2444 | 14871476 | 2489 | 15259428 | 2534 | 15649733 |
| 2445 | 14880071 | 2490 | 15268077 | 2535 | 15658433 |
| 2446 | 14888667 | 2491 | 15276726 | 2536 | 15667134 |
| 2447 | 14897265 | 2492 | 15285376 | 2537 | 15675836 |
| 2448 | 14905864 | 2493 | 15294028 | 2538 | 15684539 |
| 2449 | 14914464 | 2494 | 15302681 | 2539 | 15693243 |
| 2450 | 14923065 | 2495 | 15311334 | 2540 | 15701949 |
| 2451 | 14931667 | 2496 | 15319989 | 2541 | 15710655 |
| 2452 | 14940271 | 2497 | 15328646 | 2542 | 15719363 |
| 2453 | 14948875 | 2498 | 15337303 | 2543 | 15728071 |
| 2454 | 14957481 | 2499 | 15345961 | 2544 | 15736781 |
| 2455 | 14966088 | 2500 | 15354621 | 2545 | 15745492 |
| 2456 | 14974696 | 2501 | 15363282 | 2546 | 15754205 |
| 2457 | 14983306 | 2502 | 15371944 | 2547 | 15762918 |
| 2458 | 14991917 | 2503 | 15380607 | 2548 | 15771632 |
| 2459 | 15000528 | 2504 | 15389271 | 2549 | 15780348 |
| 2460 | 15009141 | 2505 | 15397937 | 2550 | 15789064 |
| 2461 | 15017755 | 2506 | 15406603 | 2551 | 15797782 |
| 2462 | 15026371 | 2507 | 15415271 | 2552 | 15806501 |
| 2463 | 15034987 | 2508 | 15423940 | 2553 | 15815221 |
| 2464 | 15043605 | 2509 | 15432610 | 2554 | 15823942 |
| 2465 | 15052224 | 2510 | 15441281 | 2555 | 15832665 |
| 2466 | 15060844 | 2511 | 15449954 | 2556 | 15841388 |
| 2467 | 15069465 | 2512 | 15458627 | 2557 | 15850113 |
| 2468 | 15078088 | 2513 | 15467302 | 2558 | 15858838 |
| 2469 | 15086711 | 2514 | 15475978 | 2559 | 15867565 |
| 2470 | 15095336 | 2515 | 15484655 | 2560 | 15876293 |
| 2471 | 15103962 | 2516 | 15493333 | 2561 | 15885022 |
| 2472 | 15112589 | 2517 | 15502012 | 2562 | 15893752 |
| 2473 | 15121217 | 2518 | 15510692 | 2563 | 15902483 |
| 2474 | 15129847 | 2519 | 15519374 | 2564 | 15911215 |
| 2475 | 15138477 | 2520 | 15528056 | 2565 | 15919949 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 2566 | 15928684 | 2611 | 16322884 | 2656 | 16719287 |
| 2567 | 15937419 | 2612 | 16331669 | 2657 | 16728121 |
| 2568 | 15946157 | 2613 | 16340456 | 2658 | 16736956 |
| 2569 | 15954894 | 2614 | 16349243 | 2659 | 16745791 |
| 2570 | 15963633 | 2615 | 16358032 | 2660 | 16754628 |
| 2571 | 15972373 | 2616 | 16366821 | 2661 | 16763466 |
| 2572 | 15981114 | 2617 | 16375612 | 2662 | 16772305 |
| 2573 | 15989856 | 2618 | 16384404 | 2663 | 16781145 |
| 2574 | 15998600 | 2619 | 16393196 | 2664 | 16789986 |
| 2575 | 16007345 | 2620 | 16401990 | 2665 | 16798828 |
| 2576 | 16016091 | 2621 | 16410785 | 2666 | 16807671 |
| 2577 | 16024837 | 2622 | 16419581 | 2667 | 16816515 |
| 2578 | 16033585 | 2623 | 16428379 | 2668 | 16825360 |
| 2579 | 16042334 | 2624 | 16437177 | 2669 | 16834207 |
| 2580 | 16051084 | 2625 | 16445976 | 2670 | 16843054 |
| 2581 | 16059836 | 2626 | 16454777 | 2671 | 16851902 |
| 2582 | 16068588 | 2627 | 16463578 | 2672 | 16860752 |
| 2583 | 16077341 | 2628 | 16472381 | 2673 | 16869602 |
| 2584 | 16086096 | 2629 | 16481184 | 2674 | 16878454 |
| 2585 | 16094852 | 2630 | 16489989 | 2675 | 16881306 |
| 2586 | 16103608 | 2631 | 16498795 | 2676 | 16896160 |
| 2587 | 16112366 | 2632 | 16507602 | 2677 | 16905015 |
| 2588 | 16121125 | 2633 | 16516410 | 2678 | 16913870 |
| 2589 | 16129885 | 2634 | 16525219 | 2679 | 16922727 |
| 2590 | 16138646 | 2635 | 16534029 | 2680 | 16931585 |
| 2591 | 16147409 | 2636 | 16542840 | 2681 | 16940444 |
| 2592 | 16156172 | 2637 | 16551652 | 2682 | 16949304 |
| 2593 | 16164937 | 2638 | 16560465 | 2683 | 16958165 |
| 2594 | 16173702 | 2639 | 16569280 | 2684 | 16967027 |
| 2595 | 16182469 | 2640 | 16578095 | 2685 | 16975890 |
| 2596 | 16191237 | 2641 | 16586912 | 2686 | 16984754 |
| 2597 | 16200005 | 2642 | 16595729 | 2687 | 16993619 |
| 2598 | 16208775 | 2643 | 16604548 | 2688 | 17002485 |
| 2599 | 16217546 | 2644 | 16613368 | 2689 | 17011352 |
| 2600 | 16226319 | 2645 | 16622188 | 2690 | 17020221 |
| 2601 | 16235092 | 2646 | 16631010 | 2691 | 17029090 |
| 2602 | 16243866 | 2647 | 16639833 | 2692 | 17037960 |
| 2603 | 16252641 | 2648 | 16648657 | 2693 | 17046832 |
| 2604 | 16261418 | 2649 | 16657482 | 2694 | 17055704 |
| 2605 | 16270196 | 2650 | 16666308 | 2695 | 17064578 |
| 2606 | 16278974 | 2651 | 16675136 | 2696 | 17073452 |
| 2607 | 16287754 | 2652 | 16683964 | 2697 | 17082328 |
| 2608 | 16296535 | 2653 | 16692793 | 2698 | 17091204 |
| 2609 | 16305317 | 2654 | 16701624 | 2699 | 17100082 |
| 2610 | 16314100 | 2655 | 16710455 | 2700 | 17108961 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 2701 | 17117840 | 2746 | 17518490 | 2791 | 17921186 |
| 2702 | 17126721 | 2747 | 17527417 | 2792 | 17930157 |
| 2703 | 17135603 | 2748 | 17536345 | 2793 | 17939130 |
| 2704 | 17144486 | 2749 | 17545274 | 2794 | 17948104 |
| 2705 | 17153370 | 2750 | 17554203 | 2795 | 17957079 |
| 2706 | 17162254 | 2751 | 17563134 | 2796 | 17966054 |
| 2707 | 17171140 | 2752 | 17572066 | 2797 | 17975030 |
| 2708 | 17180027 | 2753 | 17580999 | 2798 | 17984008 |
| 2709 | 17188915 | 2754 | 17589933 | 2799 | 17992986 |
| 2710 | 17197804 | 2755 | 17598867 | 2800 | 18001966 |
| 2711 | 17206694 | 2756 | 17607803 | 2801 | 18010946 |
| 2712 | 17215585 | 2757 | 17616740 | 2802 | 18019928 |
| 2713 | 17224477 | 2758 | 17625678 | 2803 | 18028910 |
| 2714 | 17233371 | 2759 | 17634617 | 2804 | 18037894 |
| 2715 | 17242265 | 2760 | 17643556 | 2805 | 18046878 |
| 2716 | 17251160 | 2761 | 17652497 | 2806 | 18055863 |
| 2717 | 17260056 | 2762 | 17661439 | 2807 | 18064850 |
| 2718 | 17268953 | 2763 | 17670382 | 2808 | 18073837 |
| 2719 | 17277852 | 2764 | 17679326 | 2809 | 18082825 |
| 2720 | 17286751 | 2765 | 17688271 | 2810 | 18091815 |
| 2721 | 17295651 | 2766 | 17697217 | 2811 | 18100805 |
| 2722 | 17304553 | 2767 | 17706163 | 2812 | 18109796 |
| 2723 | 17313455 | 2768 | 17715111 | 2813 | 18118788 |
| 2724 | 17322358 | 2769 | 17724060 | 2814 | 18127781 |
| 2725 | 17331263 | 2770 | 17733010 | 2815 | 18136775 |
| 2726 | 17340168 | 2771 | 17741961 | 2816 | 18145771 |
| 2727 | 17349075 | 2772 | 17750913 | 2817 | 18154767 |
| 2728 | 17357982 | 2773 | 17759865 | 2818 | 18163764 |
| 2729 | 17366891 | 2774 | 17768819 | 2819 | 18172762 |
| 2730 | 17375800 | 2775 | 17777774 | 2820 | 18181761 |
| 2731 | 17384711 | 2776 | 17786730 | 2821 | 18190761 |
| 2732 | 17393622 | 2777 | 17795687 | 2822 | 18199762 |
| 2733 | 17402535 | 2778 | 17804644 | 2823 | 18208763 |
| 2734 | 17411448 | 2779 | 17813603 | 2824 | 18217766 |
| 2735 | 17420363 | 2780 | 17822563 | 2825 | 18226770 |
| 2736 | 17429278 | 2781 | 17831524 | 2826 | 18235775 |
| 2737 | 17438195 | 2782 | 17840486 | 2827 | 18244781 |
| 2738 | 17447113 | 2783 | 17849448 | 2828 | 18253787 |
| 2739 | 17456031 | 2784 | 17858412 | 2829 | 18262795 |
| 2740 | 17464951 | 2785 | 17867377 | 2830 | 18271804 |
| 2741 | 17473872 | 2786 | 17876342 | 2831 | 18280813 |
| 2742 | 17482793 | 2787 | 17885309 | 2832 | 18289824 |
| 2743 | 17491716 | 2788 | 17894277 | 2833 | 18298835 |
| 2744 | 17500640 | 2789 | 17903245 | 2834 | 18307848 |
| 2745 | 17509565 | 2790 | 17912215 | 2835 | 18316861 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area. | Verſ. fine | Seg. area. |
|---|---|---|---|---|---|
| 2836 | 18325876 | 2881 | 18732510 | 2926 | 19141040 |
| 2837 | 18334891 | 2882 | 18741568 | 2927 | 19150139 |
| 2838 | 18343907 | 2883 | 18750627 | 2928 | 19159240 |
| 2839 | 18352925 | 2884 | 18759687 | 2929 | 19168341 |
| 2840 | 18361943 | 2885 | 18768748 | 2930 | 19177443 |
| 2841 | 18370962 | 2886 | 18777810 | 2931 | 19186547 |
| 2842 | 18379982 | 2887 | 18786872 | 2932 | 19195651 |
| 2843 | 18389004 | 2888 | 18795936 | 2933 | 19204756 |
| 2844 | 18398026 | 2889 | 18805001 | 2934 | 19213862 |
| 2845 | 18407049 | 2890 | 18814066 | 2935 | 19222969 |
| 2846 | 18416073 | 2891 | 18823132 | 2936 | 19232076 |
| 2847 | 18425098 | 2892 | 18832200 | 2937 | 19241185 |
| 2848 | 18434124 | 2893 | 18841268 | 2938 | 19250295 |
| 2849 | 18443150 | 2894 | 18850337 | 2939 | 19259405 |
| 2850 | 18452178 | 2895 | 18859407 | 2940 | 19268517 |
| 2851 | 18461207 | 2896 | 18868479 | 2941 | 19277629 |
| 2852 | 18470237 | 2897 | 18877551 | 2942 | 19286742 |
| 2853 | 18479267 | 2898 | 18886623 | 2943 | 19295856 |
| 2854 | 18488299 | 2899 | 18895697 | 2944 | 19304971 |
| 2855 | 18497332 | 2900 | 18904772 | 2945 | 19314087 |
| 2856 | 18506365 | 2901 | 18913848 | 2946 | 19323204 |
| 2857 | 18515400 | 2902 | 18922924 | 2947 | 19332321 |
| 2858 | 18524435 | 2903 | 18932002 | 2948 | 19341440 |
| 2859 | 18533471 | 2904 | 18941080 | 2949 | 19350560 |
| 2860 | 18542509 | 2905 | 18950160 | 2950 | 19359680 |
| 2861 | 18551547 | 2906 | 18959240 | 2951 | 19368801 |
| 2862 | 18560586 | 2907 | 18968321 | 2952 | 19377924 |
| 2863 | 18569626 | 2908 | 18977404 | 2953 | 19387047 |
| 2864 | 18578667 | 2909 | 18986487 | 2954 | 19396171 |
| 2865 | 18587709 | 2910 | 18995571 | 2955 | 19405296 |
| 2866 | 18596752 | 2911 | 19004656 | 2956 | 19414421 |
| 2867 | 18605796 | 2912 | 19013741 | 2957 | 19423548 |
| 2868 | 18614841 | 2913 | 19022828 | 2958 | 19432676 |
| 2869 | 18623887 | 2914 | 19031916 | 2959 | 19441804 |
| 2870 | 18632934 | 2915 | 19041005 | 2960 | 19450933 |
| 2871 | 18641981 | 2916 | 19050094 | 2961 | 19460064 |
| 2872 | 18651030 | 2917 | 19059184 | 2962 | 19469195 |
| 2873 | 18660080 | 2918 | 19068276 | 2963 | 19478327 |
| 2874 | 18669130 | 2919 | 19077368 | 2964 | 19487460 |
| 2875 | 18678182 | 2920 | 19086461 | 2965 | 19496594 |
| 2876 | 18687234 | 2921 | 19095555 | 2966 | 19505728 |
| 2877 | 18696287 | 2922 | 19104650 | 2967 | 19514864 |
| 2878 | 18705342 | 2923 | 19113746 | 2968 | 19524001 |
| 2879 | 18714397 | 2924 | 19122843 | 2969 | 19533138 |
| 2880 | 18723453 | 2925 | 19131941 | 2970 | 19542276 |

| Verf. fine | Seg. area | Verf. fine | Seg. area | Verf. fine | Seg. area |
|---|---|---|---|---|---|
| 2971 | 19551415 | 3016 | 19963589 | 3061 | 20377514 |
| 2972 | 19560556 | 3017 | 19972769 | 3062 | 20386732 |
| 2973 | 19569696 | 3018 | 19981949 | 3063 | 20395951 |
| 2974 | 19578838 | 3019 | 19991130 | 3064 | 20405170 |
| 2975 | 19587981 | 3020 | 20000313 | 3065 | 20414391 |
| 2976 | 19597125 | 3021 | 20009495 | 3066 | 20423612 |
| 2977 | 19606269 | 3022 | 20018679 | 3067 | 20432834 |
| 2978 | 19615415 | 3023 | 20027864 | 3068 | 20442057 |
| 2979 | 19624561 | 3024 | 20037049 | 3069 | 20451281 |
| 2980 | 19633708 | 3025 | 20046236 | 3070 | 20460505 |
| 2981 | 19642856 | 3026 | 20055423 | 3071 | 20469731 |
| 2982 | 19652005 | 3027 | 20064611 | 3072 | 20478957 |
| 2983 | 19661155 | 3028 | 20073800 | 3073 | 20488184 |
| 2984 | 19679305 | 3029 | 20082990 | 3074 | 20497412 |
| 2985 | 19679457 | 3030 | 20092181 | 3075 | 20506641 |
| 2986 | 19688609 | 3031 | 20101372 | 3076 | 20515870 |
| 2987 | 19697763 | 3032 | 20110565 | 3077 | 20525101 |
| 2988 | 19706917 | 3033 | 20119758 | 3078 | 20534332 |
| 2989 | 19716072 | 3034 | 20128952 | 3079 | 20543564 |
| 2990 | 19725228 | 3035 | 20138147 | 3080 | 20552797 |
| 2991 | 19734385 | 3036 | 20147343 | 3081 | 20562031 |
| 2992 | 19743542 | 3037 | 20156539 | 3082 | 20571265 |
| 2993 | 19752701 | 3038 | 20165737 | 3083 | 20580501 |
| 2994 | 19761861 | 3039 | 20174935 | 3084 | 20589737 |
| 2995 | 19771021 | 3040 | 20184134 | 3085 | 20598974 |
| 2996 | 19780182 | 3041 | 20193335 | 3086 | 20608212 |
| 2997 | 19789344 | 3042 | 20202535 | 3087 | 20617451 |
| 2998 | 19798507 | 3043 | 20211737 | 3088 | 20626690 |
| 2999 | 19807671 | 3044 | 20220940 | 3089 | 20635930 |
| 3000 | 19816836 | 3045 | 20230143 | 3090 | 20645172 |
| 3001 | 19826001 | 3046 | 20239348 | 3091 | 20654414 |
| 3002 | 19835168 | 3047 | 20248553 | 3092 | 20663657 |
| 3003 | 19844335 | 3048 | 20257759 | 3093 | 20672900 |
| 3004 | 19853503 | 3049 | 20266966 | 3094 | 20682145 |
| 3005 | 19862672 | 3050 | 20276173 | 3095 | 20691390 |
| 3006 | 19871842 | 3051 | 20285382 | 3096 | 20700636 |
| 3007 | 19881013 | 3052 | 20294591 | 3097 | 20709883 |
| 3008 | 19890185 | 3053 | 20303802 | 3098 | 20719131 |
| 3009 | 19899357 | 3054 | 20313013 | 3099 | 20728380 |
| 3010 | 19908531 | 3055 | 20322225 | 3100 | 20737629 |
| 3011 | 19917705 | 3056 | 20331438 | 3101 | 20746879 |
| 3012 | 19926880 | 3057 | 20340651 | 3102 | 20756131 |
| 3013 | 19936056 | 3058 | 20349866 | 3103 | 20765382 |
| 3014 | 19945233 | 3059 | 20359081 | 3104 | 20774635 |
| 3015 | 19954411 | 3060 | 20368297 | 3105 | 20783889 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
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| 3106 | 20793143 | 3151 | 21210430 | 3196 | 21629328 |
| 3107 | 20802398 | 3152 | 21219721 | 3197 | 21638655 |
| 3108 | 20811654 | 3153 | 21229013 | 3198 | 21647983 |
| 3109 | 20820911 | 3154 | 21238307 | 3199 | 21657311 |
| 3110 | 20830169 | 3155 | 21247600 | 3200 | 21666640 |
| 3111 | 20839427 | 3156 | 21256895 | 3201 | 21675970 |
| 3112 | 20848687 | 3157 | 21266191 | 3202 | 21685301 |
| 3113 | 20857947 | 3158 | 21275487 | 3203 | 21694632 |
| 3114 | 20867208 | 3159 | 21284784 | 3204 | 21703964 |
| 3115 | 20876469 | 3160 | 21294082 | 3205 | 21713297 |
| 3116 | 20885732 | 3161 | 21303980 | 3206 | 21722631 |
| 3117 | 20894995 | 3162 | 21312680 | 3207 | 21731966 |
| 3118 | 20904259 | 3163 | 21321380 | 3208 | 21741301 |
| 3119 | 20913524 | 3164 | 21331281 | 3209 | 21750637 |
| 3120 | 20922790 | 3165 | 21340583 | 3210 | 21759974 |
| 3121 | 20932057 | 3166 | 21349886 | 3211 | 21769311 |
| 3122 | 20941324 | 3167 | 21359189 | 3212 | 21778650 |
| 3123 | 20950592 | 3168 | 21368493 | 3213 | 21787989 |
| 3124 | 20959861 | 3169 | 21377798 | 3214 | 21797329 |
| 3125 | 20969131 | 3170 | 21387104 | 3215 | 21806669 |
| 3126 | 20978402 | 3171 | 21396410 | 3216 | 21816011 |
| 3127 | 20987673 | 3172 | 21405718 | 3217 | 21825353 |
| 3128 | 20996946 | 3173 | 21415026 | 3218 | 21834696 |
| 3129 | 21006219 | 3174 | 21424335 | 3219 | 21844040 |
| 3130 | 21015493 | 3175 | 21433644 | 3220 | 21853384 |
| 3131 | 21024767 | 3176 | 21442955 | 3221 | 21862729 |
| 3132 | 21034043 | 3177 | 21452266 | 3222 | 21872076 |
| 3133 | 21043319 | 3178 | 21461578 | 3223 | 21881422 |
| 3134 | 21052596 | 3179 | 21470891 | 3224 | 21890770 |
| 3135 | 21061874 | 3180 | 21480205 | 3225 | 21900118 |
| 3136 | 21071153 | 3181 | 21489519 | 3226 | 21909467 |
| 3137 | 21080432 | 3182 | 21498834 | 3227 | 21918817 |
| 3138 | 21089713 | 3183 | 21508150 | 3228 | 21928168 |
| 3139 | 21098994 | 3184 | 21517467 | 3229 | 21937519 |
| 3140 | 21108276 | 3185 | 21526784 | 3230 | 21946871 |
| 3141 | 21117558 | 3186 | 21536103 | 3231 | 21956224 |
| 3142 | 21126842 | 3187 | 21545422 | 3232 | 21965577 |
| 3143 | 21136126 | 3188 | 21554742 | 3233 | 21974932 |
| 3144 | 21145411 | 3189 | 21564062 | 3234 | 21984287 |
| 3145 | 21154697 | 3190 | 21573384 | 3235 | 21993643 |
| 3146 | 21163984 | 3191 | 21582706 | 3236 | 22002999 |
| 3147 | 21173272 | 3192 | 21592029 | 3237 | 22012357 |
| 3148 | 21182560 | 3193 | 21601352 | 3238 | 22021715 |
| 3149 | 21191849 | 3194 | 21610677 | 3239 | 22031074 |
| 3150 | 21201139 | 3195 | 21620002 | 3240 | 22040433 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
|---|---|---|---|---|---|
| 3241 | 22049794 | 3286 | 22471781 | 3331 | 22895246 |
| 3242 | 22059155 | 3287 | 22481175 | 3332 | 22904673 |
| 3243 | 22068517 | 3288 | 22490571 | 3333 | 22914101 |
| 3244 | 22077879 | 3289 | 22499966 | 3334 | 22923529 |
| 3245 | 22087243 | 3290 | 22509363 | 3335 | 22932958 |
| 3246 | 22096607 | 3291 | 22518761 | 3336 | 22942388 |
| 3247 | 22105972 | 3292 | 22528159 | 3337 | 22951818 |
| 3248 | 22115337 | 3293 | 22537557 | 3338 | 22961249 |
| 3249 | 22124704 | 3294 | 22546957 | 3339 | 22970681 |
| 3250 | 22134071 | 3295 | 22556357 | 3340 | 22980113 |
| 3251 | 22143439 | 3296 | 22565758 | 3341 | 22989546 |
| 3252 | 22152807 | 3297 | 22575160 | 3342 | 22998980 |
| 3253 | 22162177 | 3298 | 22584562 | 3343 | 23008415 |
| 3254 | 22171547 | 3299 | 22593966 | 3344 | 23017850 |
| 3255 | 22180918 | 3300 | 22603370 | 3345 | 23027286 |
| 3256 | 22190289 | 3301 | 22612774 | 3346 | 23036722 |
| 3257 | 22199661 | 3302 | 22622179 | 3347 | 23046160 |
| 3258 | 22209035 | 3303 | 22631586 | 3348 | 23055598 |
| 3259 | 22218408 | 3304 | 22640992 | 3349 | 23065037 |
| 3260 | 22227783 | 3305 | 22650400 | 3350 | 23074476 |
| 3261 | 22237158 | 3306 | 22659808 | 3351 | 23083916 |
| 3262 | 22246534 | 3307 | 22669217 | 3352 | 23093357 |
| 3263 | 22255911 | 3308 | 22678627 | 3353 | 23102799 |
| 3264 | 22265289 | 3309 | 22688037 | 3354 | 23112241 |
| 3265 | 22274667 | 3310 | 22697448 | 3355 | 23121684 |
| 3266 | 22284046 | 3311 | 22706860 | 3356 | 23131128 |
| 3267 | 22293426 | 3312 | 22716273 | 3357 | 23140572 |
| 3268 | 22302806 | 3313 | 22725686 | 3358 | 23150017 |
| 3269 | 22312187 | 3314 | 22735100 | 3359 | 23159463 |
| 3270 | 22321569 | 3315 | 22744514 | 3360 | 23168909 |
| 3271 | 22330952 | 3316 | 22753930 | 3361 | 23178356 |
| 3272 | 22340336 | 3317 | 22763346 | 3362 | 23187804 |
| 3273 | 22349720 | 3318 | 22772763 | 3363 | 23197253 |
| 3274 | 22359105 | 3319 | 22782180 | 3364 | 23206702 |
| 3275 | 22368490 | 3320 | 22791599 | 3365 | 23216152 |
| 3276 | 22377877 | 3321 | 22801018 | 3366 | 23225602 |
| 3277 | 22387264 | 3322 | 22810437 | 3367 | 23235054 |
| 3278 | 22396652 | 3323 | 22819858 | 3368 | 23244505 |
| 3279 | 22406040 | 3324 | 22829279 | 3369 | 23253958 |
| 3280 | 22415430 | 3325 | 22838701 | 3370 | 23263411 |
| 3281 | 22424820 | 3326 | 22848123 | 3371 | 23272866 |
| 3282 | 22434211 | 3327 | 22857546 | 3372 | 23282320 |
| 3283 | 22443602 | 3328 | 22866970 | 3373 | 23291776 |
| 3284 | 22452994 | 3329 | 22876395 | 3374 | 23301232 |
| 3285 | 22462387 | 3330 | 22885820 | 3375 | 23310689 |

| Verf. fine | Seg. area | Verf. fine | Seg. area | Verf. fine | Seg. area |
|---|---|---|---|---|---|
| 3376 | 23320146 | 3421 | 23746437 | 3466 | 24174075 |
| 3377 | 23329624 | 3422 | 23755925 | 3467 | 24183593 |
| 3378 | 23339063 | 3423 | 23765414 | 3468 | 24193112 |
| 3379 | 23348523 | 3424 | 23774904 | 3469 | 24202631 |
| 3380 | 23357983 | 3425 | 23784395 | 3470 | 24212151 |
| 3381 | 23367444 | 3426 | 23793886 | 3471 | 24221672 |
| 3382 | 23376905 | 3427 | 23803378 | 3472 | 24231193 |
| 3383 | 23386368 | 3428 | 23812871 | 3473 | 24240715 |
| 3384 | 23395831 | 3429 | 23822364 | 3474 | 24250238 |
| 3385 | 23405294 | 3430 | 23831858 | 3475 | 24259761 |
| 3386 | 23414759 | 3431 | 23841352 | 3476 | 24269285 |
| 3387 | 23424224 | 3432 | 23850848 | 3477 | 24278809 |
| 3388 | 23433689 | 3433 | 23860343 | 3478 | 24288334 |
| 3389 | 23443156 | 3434 | 23869840 | 3479 | 24297860 |
| 3390 | 23452623 | 3435 | 23879337 | 3480 | 24307386 |
| 3391 | 23462090 | 3436 | 23888835 | 3481 | 24316913 |
| 3392 | 23471559 | 3437 | 23898334 | 3482 | 24326441 |
| 3393 | 23481028 | 3438 | 23907833 | 3483 | 24335969 |
| 3394 | 23490498 | 3439 | 23917333 | 3484 | 24345498 |
| 3395 | 23499968 | 3440 | 23926833 | 3485 | 24355028 |
| 3396 | 23509439 | 3441 | 23936334 | 3486 | 24364558 |
| 3397 | 23518911 | 3442 | 23945836 | 3487 | 24374089 |
| 3398 | 23528384 | 3443 | 23955339 | 3488 | 24383621 |
| 3399 | 23537857 | 3444 | 23964842 | 3489 | 24393153 |
| 3400 | 23547331 | 3445 | 23974345 | 3490 | 24402685 |
| 3401 | 23556805 | 3446 | 23983850 | 3491 | 24412219 |
| 3402 | 23566280 | 3447 | 23993355 | 3492 | 24421753 |
| 3403 | 23575756 | 3448 | 24002861 | 3493 | 24431288 |
| 3404 | 23585233 | 3449 | 24012367 | 3494 | 24440823 |
| 3405 | 23594710 | 3450 | 24021874 | 3495 | 24450359 |
| 3406 | 23604188 | 3451 | 24031382 | 3496 | 24459895 |
| 3407 | 23613666 | 3452 | 24040890 | 3497 | 24469432 |
| 3408 | 23623146 | 3453 | 24050399 | 3498 | 24478970 |
| 3409 | 23632626 | 3454 | 24059909 | 3499 | 24488509 |
| 3410 | 23642106 | 3455 | 24069419 | 3500 | 24498048 |
| 3411 | 23651587 | 3456 | 24078930 | 3501 | 24507588 |
| 3412 | 23661069 | 3457 | 24088442 | 3502 | 24517128 |
| 3413 | 23670552 | 3458 | 24097954 | 3503 | 24526669 |
| 3414 | 23680035 | 3459 | 24107467 | 3504 | 24536210 |
| 3415 | 23689519 | 3460 | 24116980 | 3505 | 24545753 |
| 3416 | 23699004 | 3461 | 24126494 | 3506 | 24555295 |
| 3417 | 23708489 | 3462 | 24136009 | 3507 | 24564839 |
| 3418 | 23717975 | 3463 | 24145525 | 3508 | 24574383 |
| 3419 | 23727461 | 3464 | 24155041 | 3509 | 24583928 |
| 3420 | 23736949 | 3465 | 24164558 | 3510 | 24593473 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
|---|---|---|---|---|---|
| 3511 | 24603019 | 3556 | 25033227 | 3601 | 25464656 |
| 3512 | 24612566 | 3557 | 25042801 | 3602 | 25474257 |
| 3513 | 24622113 | 3558 | 25052376 | 3603 | 25483858 |
| 3514 | 24631661 | 3559 | 25061951 | 3604 | 25493460 |
| 3515 | 24641209 | 3560 | 25071527 | 3605 | 25503063 |
| 3516 | 24650758 | 3561 | 25081104 | 3606 | 25512666 |
| 3517 | 24660308 | 3562 | 25090681 | 3607 | 25522270 |
| 3518 | 24669858 | 3563 | 25100259 | 3608 | 25531874 |
| 3519 | 24679409 | 3564 | 25109837 | 3609 | 25541479 |
| 3520 | 24688961 | 3565 | 25119416 | 3610 | 25551085 |
| 3521 | 24698513 | 3566 | 25128996 | 3611 | 25560691 |
| 3522 | 24708066 | 3567 | 25138576 | 3612 | 25570297 |
| 3523 | 24717619 | 3568 | 25148157 | 3613 | 25579905 |
| 3524 | 24727173 | 3569 | 25157738 | 3614 | 25589512 |
| 3525 | 24736728 | 3570 | 25167320 | 3615 | 25599121 |
| 3526 | 24746283 | 3571 | 25176903 | 3616 | 25608730 |
| 3527 | 24755839 | 3572 | 25186486 | 3617 | 25618339 |
| 3528 | 24765396 | 3573 | 25196070 | 3618 | 25627950 |
| 3529 | 24774953 | 3574 | 25205654 | 3619 | 25637560 |
| 3530 | 24784511 | 3575 | 25215239 | 3620 | 25647172 |
| 3531 | 24794069 | 3576 | 25224824 | 3621 | 25656783 |
| 3532 | 24803628 | 3577 | 25234411 | 3622 | 25666396 |
| 3533 | 24813187 | 3578 | 25243997 | 3623 | 25676009 |
| 3534 | 24822748 | 3579 | 25253585 | 3624 | 25685622 |
| 3535 | 24832308 | 3580 | 25263173 | 3625 | 25695237 |
| 3536 | 24841870 | 3581 | 25272761 | 3626 | 25704851 |
| 3537 | 24851432 | 3582 | 25282350 | 3627 | 25714467 |
| 3538 | 24860995 | 3583 | 25291940 | 3628 | 25724082 |
| 3539 | 24870558 | 3584 | 25301530 | 3629 | 25733699 |
| 3540 | 24880122 | 3585 | 25311121 | 3630 | 25743316 |
| 3541 | 24889686 | 3586 | 25320713 | 3631 | 25752933 |
| 3542 | 24899251 | 3587 | 25330305 | 3632 | 25762552 |
| 3543 | 24908817 | 3588 | 25339898 | 3633 | 25772170 |
| 3544 | 24918383 | 3589 | 25349491 | 3634 | 25781790 |
| 3545 | 24927950 | 3590 | 25359085 | 3635 | 25791410 |
| 3546 | 24937518 | 3591 | 25368679 | 3636 | 25801030 |
| 3547 | 24947086 | 3592 | 25378274 | 3637 | 25810651 |
| 3548 | 24956655 | 3593 | 25387870 | 3638 | 25820272 |
| 3549 | 24966224 | 3594 | 25397466 | 3639 | 25829895 |
| 3550 | 24975794 | 3595 | 25407063 | 3640 | 25839517 |
| 3551 | 24985365 | 3596 | 25416660 | 3641 | 25849141 |
| 3552 | 24994936 | 3597 | 25426258 | 3642 | 25858764 |
| 3553 | 25004508 | 3598 | 25435857 | 3643 | 25868389 |
| 3554 | 25014080 | 3599 | 25445456 | 3644 | 25878014 |
| 3555 | 25023653 | 3600 | 25455055 | 3645 | 25887639 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 3646 | 25897265 | 3691 | 26331014 | 3736 | 26765861 |
| 3647 | 25906892 | 3692 | 26340666 | 3737 | 26775536 |
| 3648 | 25916519 | 3693 | 26350318 | 3738 | 26785212 |
| 3649 | 25926147 | 3694 | 26359970 | 3739 | 26794889 |
| 3650 | 25935775 | 3695 | 26369623 | 3740 | 26804566 |
| 3651 | 25945404 | 3696 | 26379277 | 3741 | 26814243 |
| 3652 | 25955034 | 3697 | 26388931 | 3742 | 26823921 |
| 3653 | 25964664 | 3698 | 26398586 | 3743 | 26833601 |
| 3654 | 25974294 | 3699 | 26408241 | 3744 | 26843280 |
| 3655 | 25983925 | 3700 | 26417897 | 3745 | 26852960 |
| 3656 | 25993557 | 3701 | 26427553 | 3746 | 26862640 |
| 3657 | 26003189 | 3702 | 26437210 | 3747 | 26872320 |
| 3658 | 26012822 | 3703 | 26446868 | 3748 | 26882001 |
| 3659 | 26022455 | 3704 | 26456526 | 3749 | 26891683 |
| 3660 | 26032089 | 3705 | 26466184 | 3750 | 26901365 |
| 3661 | 26041724 | 3706 | 26475843 | 3751 | 26911048 |
| 3662 | 26051359 | 3707 | 26485503 | 3752 | 26920731 |
| 3663 | 26060994 | 3708 | 26495163 | 3753 | 26930415 |
| 3664 | 26070630 | 3709 | 26504824 | 3754 | 26940099 |
| 3665 | 26080267 | 3710 | 26514485 | 3755 | 26949784 |
| 3666 | 26089904 | 3711 | 26524147 | 3756 | 26959469 |
| 3667 | 26099542 | 3712 | 26533809 | 3757 | 26969155 |
| 3668 | 26109180 | 3713 | 26543472 | 3758 | 26978841 |
| 3669 | 26118819 | 3714 | 26553135 | 3759 | 26988528 |
| 3670 | 26128459 | 3715 | 26562799 | 3760 | 26998216 |
| 3671 | 26138099 | 3716 | 26572463 | 3761 | 27007903 |
| 3672 | 26147739 | 3717 | 26582128 | 3762 | 27017592 |
| 3673 | 26157381 | 3718 | 26591793 | 3763 | 27027281 |
| 3674 | 26167022 | 3719 | 26601459 | 3764 | 27036970 |
| 3675 | 26176664 | 3720 | 26611125 | 3765 | 27046660 |
| 3676 | 26186307 | 3721 | 26620792 | 3766 | 27056350 |
| 3677 | 26195950 | 3722 | 26630460 | 3767 | 27066041 |
| 3678 | 26205594 | 3723 | 26640128 | 3768 | 27075733 |
| 3679 | 26215239 | 3724 | 26649796 | 3769 | 27085425 |
| 3680 | 26224884 | 3725 | 26659465 | 3770 | 27095117 |
| 3681 | 26234529 | 3726 | 26669135 | 3771 | 27104810 |
| 3682 | 26244175 | 3727 | 26678805 | 3772 | 27114503 |
| 3683 | 26253822 | 3728 | 26688476 | 3773 | 27124198 |
| 3684 | 26263469 | 3729 | 26698147 | 3774 | 27133892 |
| 3685 | 26273117 | 3730 | 26707818 | 3775 | 27143587 |
| 3686 | 26282765 | 3731 | 26717491 | 3776 | 27153282 |
| 3687 | 26292414 | 3732 | 26727164 | 3777 | 27162978 |
| 3688 | 26302063 | 3733 | 26736838 | 3778 | 27172675 |
| 3689 | 26311713 | 3734 | 26746511 | 3779 | 27182372 |
| 3690 | 26321363 | 3735 | 26756186 | 3780 | 27192069 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
|---|---|---|---|---|---|
| 3781 | 27201767 | 3826 | 27638691 | 3871 | 28076594 |
| 3782 | 27211466 | 3827 | 27648412 | 3872 | 28086336 |
| 3783 | 27221165 | 3828 | 27658133 | 3873 | 28096078 |
| 3784 | 27230864 | 3829 | 27667855 | 3874 | 28105821 |
| 3785 | 27240565 | 3830 | 27677577 | 3875 | 28115564 |
| 3786 | 27250265 | 3831 | 27687300 | 3876 | 28125308 |
| 3787 | 27259966 | 3832 | 27697023 | 3877 | 28135052 |
| 3788 | 27269668 | 3833 | 27706746 | 3878 | 28144797 |
| 3789 | 27279370 | 3834 | 27716470 | 3879 | 28154542 |
| 3790 | 27289072 | 3835 | 27726195 | 3880 | 28164288 |
| 3791 | 27298775 | 3836 | 27735920 | 3881 | 28174034 |
| 3792 | 27308479 | 3837 | 27745645 | 3882 | 28183781 |
| 3793 | 27318183 | 3838 | 27755371 | 3883 | 28193528 |
| 3794 | 27327887 | 3839 | 27765098 | 3884 | 28203275 |
| 3795 | 27337592 | 3840 | 27774825 | 3885 | 28213023 |
| 3796 | 27347298 | 3841 | 27784552 | 3886 | 28222772 |
| 3797 | 27357004 | 3842 | 27794280 | 3887 | 28232520 |
| 3798 | 27366710 | 3843 | 27804008 | 3888 | 28242270 |
| 3799 | 27376417 | 3844 | 27813737 | 3889 | 28252019 |
| 3800 | 27386125 | 3845 | 27823466 | 3890 | 28261770 |
| 3801 | 27395833 | 3846 | 27833196 | 3891 | 28271520 |
| 3802 | 27405541 | 3847 | 27842926 | 3892 | 28281272 |
| 3803 | 27415259 | 3848 | 27852657 | 3893 | 28291023 |
| 3804 | 27424959 | 3849 | 27862388 | 3894 | 28300775 |
| 3805 | 27434669 | 3850 | 27872120 | 3895 | 28310528 |
| 3806 | 27444380 | 3851 | 27881852 | 3896 | 28320281 |
| 3807 | 27454091 | 3852 | 27891585 | 3897 | 28330034 |
| 3808 | 27463802 | 3853 | 27901318 | 3898 | 28339788 |
| 3809 | 27473514 | 3854 | 27911051 | 3899 | 28349342 |
| 3810 | 27483226 | 3855 | 27920785 | 3900 | 28359297 |
| 3811 | 27492939 | 3856 | 27930520 | 3901 | 28369052 |
| 3812 | 27502653 | 3857 | 27940255 | 3902 | 28378808 |
| 3813 | 27512367 | 3858 | 27949990 | 3903 | 28388564 |
| 3814 | 27522081 | 3859 | 27959726 | 3904 | 28398321 |
| 3815 | 27531796 | 3860 | 27969463 | 3905 | 28408078 |
| 3816 | 27541511 | 3861 | 27979199 | 3906 | 28417835 |
| 3817 | 27551227 | 3862 | 27988937 | 3907 | 28427593 |
| 3818 | 27560943 | 3863 | 27998675 | 3908 | 28437351 |
| 3819 | 27570660 | 3864 | 28008413 | 3909 | 28447110 |
| 3820 | 27580377 | 3865 | 28018152 | 3910 | 28456870 |
| 3821 | 27590095 | 3866 | 28027891 | 3911 | 28466629 |
| 3822 | 27599813 | 3867 | 28037630 | 3912 | 28476389 |
| 3823 | 27609532 | 3868 | 28047370 | 3913 | 28486150 |
| 3824 | 27619251 | 3869 | 28057111 | 3914 | 28495911 |
| 3825 | 27628971 | 3870 | 28066852 | 3915 | 28505673 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 3916 | 28515435 | 3961 | 28955175 | 4006 | 29395776 |
| 3917 | 28525197 | 3962 | 28964957 | 4007 | 29405576 |
| 3918 | 28534960 | 3963 | 28974739 | 4008 | 29415378 |
| 3919 | 28544723 | 3964 | 28984522 | 4009 | 29425179 |
| 3920 | 28554487 | 3965 | 28994305 | 4010 | 29434981 |
| 3921 | 28564251 | 3966 | 29004089 | 4011 | 29444783 |
| 3922 | 28574015 | 3967 | 29013873 | 4012 | 29454585 |
| 3923 | 28583781 | 3968 | 29023657 | 4013 | 29464388 |
| 3924 | 28593546 | 3969 | 29033442 | 4014 | 29474192 |
| 3925 | 28603312 | 3970 | 29043228 | 4015 | 29483996 |
| 3926 | 28613078 | 3971 | 29053013 | 4016 | 29493800 |
| 3927 | 28622845 | 3972 | 29062799 | 4017 | 29503605 |
| 3928 | 28632612 | 3973 | 29072586 | 4018 | 29513410 |
| 3929 | 28642380 | 3974 | 29082373 | 4019 | 29523215 |
| 3930 | 28652148 | 3975 | 29092160 | 4020 | 29533021 |
| 3931 | 28661917 | 3976 | 29101948 | 4021 | 29542827 |
| 3932 | 28671686 | 3977 | 29111736 | 4022 | 29552634 |
| 3933 | 28681455 | 3978 | 29121525 | 4023 | 29562441 |
| 3934 | 28691225 | 3979 | 29131314 | 4024 | 29572248 |
| 3935 | 28700995 | 3980 | 29141104 | 4025 | 29582056 |
| 3936 | 28710766 | 3981 | 29150894 | 4026 | 29591864 |
| 3937 | 28720537 | 3982 | 29160684 | 4027 | 29601673 |
| 3938 | 28730309 | 3983 | 29170475 | 4028 | 29611482 |
| 3939 | 28740081 | 3984 | 29180266 | 4029 | 29621291 |
| 3940 | 28749853 | 3985 | 29190057 | 4030 | 29631101 |
| 3941 | 28759626 | 3986 | 29199849 | 4031 | 29640911 |
| 3942 | 28769400 | 3987 | 29209642 | 4032 | 29650722 |
| 3943 | 28779173 | 3988 | 29219435 | 4033 | 29660533 |
| 3944 | 28788948 | 3989 | 29229228 | 4034 | 29670344 |
| 3945 | 28798722 | 3990 | 29239022 | 4035 | 29680156 |
| 3946 | 28808497 | 3991 | 29248816 | 4036 | 29689968 |
| 3947 | 28818273 | 3992 | 29258610 | 4037 | 29699781 |
| 3948 | 28828049 | 3993 | 29268405 | 4038 | 29709594 |
| 3949 | 28837825 | 3994 | 29278200 | 4039 | 29719407 |
| 3950 | 28847602 | 3995 | 29287996 | 4040 | 29729221 |
| 3951 | 28857379 | 3996 | 29297792 | 4041 | 29739035 |
| 3952 | 28867157 | 3997 | 29307589 | 4042 | 29748850 |
| 3953 | 28876935 | 3998 | 29317386 | 4043 | 29758665 |
| 3954 | 28886713 | 3999 | 29327183 | 4044 | 29768480 |
| 3955 | 28896492 | 4000 | 29336981 | 4045 | 29778296 |
| 3956 | 28906272 | 4001 | 29346779 | 4046 | 29788112 |
| 3957 | 28916051 | 4002 | 29356577 | 4047 | 29797928 |
| 3958 | 28925832 | 4003 | 29366376 | 4048 | 29807745 |
| 3959 | 28935612 | 4004 | 29376176 | 4049 | 29817562 |
| 3960 | 28945393 | 4005 | 29385976 | 4050 | 29827380 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 4051 | 29837198 | 4096 | 30279403 | 4141 | 30722353 |
| 4052 | 29847017 | 4097 | 30289239 | 4142 | 30732205 |
| 4053 | 29856835 | 4098 | 30299075 | 4143 | 30742057 |
| 4054 | 29866655 | 4099 | 30308911 | 4144 | 30751909 |
| 4055 | 29876474 | 4100 | 30318747 | 4145 | 30761761 |
| 4056 | 29886294 | 4101 | 30328584 | 4146 | 30771614 |
| 4057 | 29896114 | 4102 | 30338421 | 4147 | 30781467 |
| 4058 | 29905935 | 4103 | 30348259 | 4148 | 30791321 |
| 4059 | 29915756 | 4104 | 30358097 | 4149 | 30801175 |
| 4060 | 29925578 | 4105 | 30367935 | 4150 | 30811029 |
| 4061 | 29935400 | 4106 | 30377774 | 4151 | 30820884 |
| 4062 | 29945222 | 4107 | 30387613 | 4152 | 30830739 |
| 4063 | 29955045 | 4108 | 30397452 | 4153 | 30840594 |
| 4064 | 29964868 | 4109 | 30407292 | 4154 | 30850450 |
| 4065 | 29974691 | 4110 | 30417132 | 4155 | 30860306 |
| 4066 | 29984515 | 4111 | 30426973 | 4156 | 30870162 |
| 4067 | 29994339 | 4112 | 30436813 | 4157 | 30880019 |
| 4068 | 30004164 | 4113 | 30446655 | 4158 | 30889876 |
| 4069 | 30013988 | 4114 | 30456496 | 4159 | 30899733 |
| 4070 | 30023814 | 4115 | 30466338 | 4160 | 30909591 |
| 4071 | 30033639 | 4116 | 30476180 | 4161 | 30919449 |
| 4072 | 30043466 | 4117 | 30486023 | 4162 | 30929307 |
| 4073 | 30053292 | 4118 | 30495866 | 4163 | 30939166 |
| 4074 | 30063119 | 4119 | 30505709 | 4164 | 30949025 |
| 4075 | 30072946 | 4120 | 30515553 | 4165 | 30958884 |
| 4076 | 30082774 | 4121 | 30525397 | 4166 | 30968744 |
| 4077 | 30092602 | 4122 | 30535242 | 4167 | 30978604 |
| 4078 | 30102430 | 4123 | 30545086 | 4168 | 30988465 |
| 4079 | 30112259 | 4124 | 30554932 | 4169 | 30998325 |
| 4080 | 30122088 | 4125 | 30564777 | 4170 | 31008186 |
| 4081 | 30131917 | 4126 | 30574623 | 4171 | 31018048 |
| 4082 | 30141747 | 4127 | 30584469 | 4172 | 31027910 |
| 4083 | 30151577 | 4128 | 30594316 | 4173 | 31037772 |
| 4084 | 30161408 | 4129 | 30604163 | 4174 | 31047634 |
| 4085 | 30171239 | 4130 | 30614010 | 4175 | 31057497 |
| 4086 | 30181070 | 4131 | 30623858 | 4176 | 31067360 |
| 4087 | 30190902 | 4132 | 30633706 | 4177 | 31077223 |
| 4088 | 30200734 | 4133 | 30643554 | 4178 | 31087087 |
| 4089 | 30210566 | 4134 | 30653403 | 4179 | 31096951 |
| 4090 | 30220399 | 4135 | 30663252 | 4180 | 31106816 |
| 4091 | 30230232 | 4136 | 30673101 | 4181 | 31116681 |
| 4092 | 30240066 | 4137 | 30682951 | 4182 | 31126546 |
| 4093 | 30249899 | 4138 | 30692801 | 4183 | 31136411 |
| 4094 | 30259734 | 4139 | 30702651 | 4184 | 31146277 |
| 4095 | 30269568 | 4140 | 30712502 | 4185 | 31156143 |

| Verf. fine | Seg. area | Verf. fine | Seg. area | Verf. fine | Seg. area |
|---|---|---|---|---|---|
| 4186 | 31166009 | 4231 | 31610334 | 4276 | 32055289 |
| 4187 | 31175876 | 4232 | 31620215 | 4277 | 32065183 |
| 4188 | 31185743 | 4233 | 31630096 | 4278 | 32075078 |
| 4189 | 31195611 | 4234 | 31639978 | 4279 | 32084974 |
| 4190 | 31205478 | 4235 | 31649860 | 4280 | 32094869 |
| 4191 | 31215346 | 4236 | 31659743 | 4281 | 32104765 |
| 4192 | 31225215 | 4237 | 31669625 | 4282 | 32114661 |
| 4193 | 31235084 | 4238 | 31679509 | 4283 | 32124558 |
| 4194 | 31244953 | 4239 | 31689392 | 4284 | 32134455 |
| 4195 | 31254822 | 4240 | 31699276 | 4285 | 32144352 |
| 4196 | 31264692 | 4241 | 31709159 | 4286 | 32154249 |
| 4197 | 31274562 | 4242 | 31719044 | 4287 | 32164147 |
| 4198 | 31284482 | 4243 | 31728928 | 4288 | 32174045 |
| 4199 | 31294303 | 4244 | 31738813 | 4289 | 32183943 |
| 4200 | 31304174 | 4245 | 31748698 | 4290 | 32193842 |
| 4201 | 31314045 | 4246 | 31758584 | 4291 | 32203740 |
| 4202 | 31323917 | 4247 | 31768470 | 4292 | 32213640 |
| 4203 | 31333789 | 4248 | 31778356 | 4293 | 32223539 |
| 4204 | 31343661 | 4249 | 31788242 | 4294 | 32233439 |
| 4205 | 31353534 | 4250 | 31798129 | 4295 | 32243339 |
| 4206 | 31363407 | 4251 | 31808016 | 4296 | 32253239 |
| 4207 | 31373280 | 4252 | 31817903 | 4297 | 32263139 |
| 4208 | 31383153 | 4253 | 31827791 | 4298 | 32273040 |
| 4209 | 31393027 | 4254 | 31837679 | 4299 | 32282941 |
| 4210 | 31402902 | 4255 | 31847567 | 4300 | 32292843 |
| 4211 | 31412776 | 4256 | 31857455 | 4301 | 32302744 |
| 4212 | 31422651 | 4257 | 31867344 | 4302 | 32312646 |
| 4213 | 31432526 | 4258 | 31877233 | 4303 | 32322548 |
| 4214 | 31442402 | 4259 | 31887123 | 4304 | 32332451 |
| 4215 | 31452278 | 4260 | 31897013 | 4305 | 32342354 |
| 4216 | 31462154 | 4261 | 31906903 | 4306 | 32352257 |
| 4217 | 31472030 | 4262 | 31916793 | 4307 | 32362160 |
| 4218 | 31481907 | 4263 | 31926684 | 4308 | 32372064 |
| 4219 | 31491784 | 4264 | 31936574 | 4309 | 32381968 |
| 4220 | 31501661 | 4265 | 31946466 | 4310 | 32391872 |
| 4221 | 31511539 | 4266 | 31956357 | 4311 | 32401776 |
| 4222 | 31521417 | 4267 | 31966249 | 4312 | 32411681 |
| 4223 | 31531296 | 4268 | 31976141 | 4313 | 32421586 |
| 4224 | 31541174 | 4269 | 31986033 | 4314 | 32431491 |
| 4225 | 31551053 | 4270 | 31995926 | 4315 | 32441397 |
| 4226 | 31560933 | 4271 | 32005819 | 4316 | 32451303 |
| 4227 | 31570812 | 4272 | 32015712 | 4317 | 32461209 |
| 4228 | 31580692 | 4273 | 32025606 | 4318 | 32471115 |
| 4229 | 31590572 | 4274 | 32035500 | 4319 | 32481022 |
| 4230 | 31600453 | 4275 | 32045394 | 4320 | 32490929 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
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| 4322 | 32510744 | 4367 | 32956858 | 4412 | 33403490 |
| 4323 | 32520651 | 4368 | 32966778 | 4413 | 33413420 |
| 4324 | 32530559 | 4369 | 32976698 | 4414 | 33423351 |
| 4325 | 32540468 | 4370 | 32986618 | 4415 | 33433282 |
| 4326 | 32550376 | 4371 | 32996538 | 4416 | 33443214 |
| 4327 | 32560285 | 4372 | 33006459 | 4417 | 33453146 |
| 4328 | 32570194 | 4373 | 33016380 | 4418 | 33463078 |
| 4329 | 32580104 | 4374 | 33026301 | 4419 | 33473010 |
| 4330 | 32590013 | 4375 | 33036223 | 4420 | 33482942 |
| 4331 | 32599923 | 4376 | 33046144 | 4421 | 33492875 |
| 4332 | 32609834 | 4377 | 33056066 | 4422 | 33502807 |
| 4333 | 32619744 | 4378 | 33065988 | 4423 | 33512741 |
| 4334 | 32629655 | 4379 | 33075911 | 4424 | 33522674 |
| 4335 | 32639566 | 4380 | 33085834 | 4425 | 33532607 |
| 4336 | 32649477 | 4381 | 33095757 | 4426 | 33542541 |
| 4337 | 32659389 | 4382 | 33105680 | 4427 | 33552475 |
| 4338 | 32669301 | 4383 | 33115603 | 4428 | 33562409 |
| 4339 | 32679213 | 4384 | 33125527 | 4429 | 33572344 |
| 4340 | 32689125 | 4385 | 33135451 | 4430 | 33582279 |
| 4341 | 32699038 | 4386 | 33145375 | 4431 | 33592213 |
| 4342 | 32708951 | 4387 | 33155299 | 4432 | 33602149 |
| 4343 | 32718864 | 4388 | 33165224 | 4433 | 33612084 |
| 4344 | 32728777 | 4389 | 33175149 | 4434 | 33622020 |
| 4345 | 32738691 | 4390 | 33185074 | 4435 | 33631955 |
| 4346 | 32748605 | 4391 | 33195000 | 4436 | 33641892 |
| 4347 | 32758529 | 4392 | 33204925 | 4437 | 33651828 |
| 4348 | 32768434 | 4393 | 33214851 | 4438 | 33661764 |
| 4349 | 32778348 | 4394 | 33224777 | 4439 | 33671701 |
| 4350 | 32788263 | 4395 | 33234704 | 4440 | 33681638 |
| 4351 | 32798179 | 4396 | 33244630 | 4441 | 33691575 |
| 4352 | 32808094 | 4397 | 33254557 | 4442 | 33701513 |
| 4353 | 32818010 | 4398 | 33264484 | 4443 | 33711450 |
| 4354 | 32827926 | 4399 | 33274412 | 4444 | 33721388 |
| 4355 | 32837842 | 4400 | 33284339 | 4445 | 33731326 |
| 4356 | 32847759 | 4401 | 33294267 | 4446 | 33741265 |
| 4357 | 32857676 | 4402 | 33304195 | 4447 | 33751203 |
| 4358 | 32867593 | 4403 | 33314124 | 4448 | 33761142 |
| 4359 | 32877510 | 4404 | 33324052 | 4449 | 33771081 |
| 4360 | 32887428 | 4405 | 33333981 | 4450 | 33781020 |
| 4361 | 32897346 | 4406 | 33343910 | 4451 | 33790959 |
| 4362 | 32907264 | 4407 | 33353840 | 4452 | 33800899 |
| 4363 | 32917182 | 4408 | 33363769 | 4453 | 33810839 |
| 4364 | 32927101 | 4409 | 33373699 | 4454 | 33820779 |
| 4365 | 32937020 | 4410 | 33383629 | 4455 | 33830719 |

| Verſ. ſine | Seg. area | Verſ. ſine | Seg. area | Verſ. ſine | Seg. area |
|---|---|---|---|---|---|
| 4456 | 33840660 | 4501 | 34288204 | 4546 | 34736154 |
| 4457 | 33850601 | 4502 | 34298154 | 4547 | 34746113 |
| 4458 | 33860542 | 4503 | 34308105 | 4548 | 34756072 |
| 4459 | 33870483 | 4504 | 34318055 | 4549 | 34766031 |
| 4460 | 33880424 | 4505 | 34328006 | 4550 | 34775991 |
| 4461 | 33890366 | 4506 | 34337957 | 4551 | 34785950 |
| 4462 | 33900308 | 4507 | 34347908 | 4552 | 34795910 |
| 4463 | 33910250 | 4508 | 34357859 | 4553 | 34805870 |
| 4464 | 33920192 | 4509 | 34367811 | 4554 | 34815830 |
| 4465 | 33930134 | 4510 | 34377763 | 4555 | 34825790 |
| 4466 | 33940077 | 4511 | 34387715 | 4556 | 34835750 |
| 4467 | 33950020 | 4512 | 34397667 | 4557 | 34845711 |
| 4468 | 33959963 | 4513 | 34407619 | 4558 | 34855672 |
| 4469 | 33969907 | 4514 | 34417572 | 4559 | 34865633 |
| 4470 | 33979850 | 4515 | 34427525 | 4560 | 34875594 |
| 4471 | 33989794 | 4516 | 34437478 | 4561 | 34885555 |
| 4472 | 33999738 | 4517 | 34447431 | 4562 | 34895516 |
| 4473 | 34009682 | 4518 | 34457384 | 4563 | 34905478 |
| 4474 | 34019626 | 4519 | 34467338 | 4564 | 34915440 |
| 4475 | 34029571 | 4520 | 34477291 | 4565 | 34925402 |
| 4476 | 34039516 | 4521 | 34487245 | 4566 | 34935364 |
| 4477 | 34049461 | 4522 | 34497199 | 4567 | 34945326 |
| 4478 | 34059406 | 4523 | 34507154 | 4568 | 34955289 |
| 4479 | 34069352 | 4524 | 34517108 | 4569 | 34965252 |
| 4480 | 34079297 | 4525 | 34527063 | 4570 | 34975215 |
| 4481 | 34089243 | 4526 | 34537018 | 4571 | 34985178 |
| 4482 | 34099189 | 4527 | 34546973 | 4572 | 34995141 |
| 4483 | 34109136 | 4528 | 34556928 | 4573 | 35005104 |
| 4484 | 34119082 | 4529 | 34566883 | 4574 | 35015068 |
| 4485 | 34129029 | 4530 | 34576839 | 4575 | 35025031 |
| 4486 | 34138976 | 4531 | 34586795 | 4576 | 35034995 |
| 4487 | 34148923 | 4532 | 34596751 | 4577 | 35044959 |
| 4488 | 34158870 | 4533 | 34606707 | 4578 | 35054924 |
| 4489 | 34168818 | 4534 | 34616663 | 4579 | 35064888 |
| 4490 | 34178765 | 4535 | 34626620 | 4580 | 35074853 |
| 4491 | 34188713 | 4536 | 34636577 | 4581 | 35084817 |
| 4492 | 34198662 | 4537 | 34646534 | 4582 | 35094782 |
| 4493 | 34208610 | 4538 | 34656491 | 4583 | 35104747 |
| 4494 | 34218558 | 4539 | 34666448 | 4584 | 35114713 |
| 4495 | 34228507 | 4540 | 34676406 | 4585 | 35124678 |
| 4496 | 34238456 | 4541 | 34686363 | 4586 | 35134644 |
| 4497 | 34248405 | 4542 | 34696321 | 4587 | 35144609 |
| 4498 | 34258355 | 4543 | 34706279 | 4588 | 35154575 |
| 4499 | 34268304 | 4544 | 34716237 | 4589 | 35164541 |
| 4500 | 34278254 | 4545 | 34726196 | 4590 | 35174508 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 4591 | 35184474 | 4636 | 35633126 | 4681 | 36082074 |
| 4592 | 35194441 | 4637 | 35643099 | 4682 | 36092053 |
| 4593 | 35204407 | 4638 | 35653073 | 4683 | 36102033 |
| 4594 | 35214374 | 4639 | 35663047 | 4684 | 36112013 |
| 4595 | 35224341 | 4640 | 35673021 | 4685 | 36121993 |
| 4596 | 35234308 | 4641 | 35682995 | 4686 | 36131973 |
| 4597 | 35244276 | 4642 | 35692969 | 4687 | 36141954 |
| 4598 | 35254243 | 4643 | 35702944 | 4688 | 36151934 |
| 4599 | 35264211 | 4644 | 35712918 | 4689 | 36161915 |
| 4600 | 35274179 | 4645 | 35722893 | 4690 | 36171895 |
| 4601 | 35284147 | 4646 | 35732868 | 4691 | 36181876 |
| 4602 | 35294115 | 4647 | 35742843 | 4692 | 36191857 |
| 4603 | 35304084 | 4648 | 35752818 | 4693 | 36201838 |
| 4604 | 35314052 | 4649 | 35762793 | 4694 | 36211820 |
| 4605 | 35324021 | 4650 | 35772769 | 4695 | 36221801 |
| 4606 | 35333990 | 4651 | 35782744 | 4696 | 36231783 |
| 4607 | 35343958 | 4652 | 35792720 | 4697 | 36241764 |
| 4608 | 35353928 | 4653 | 35802696 | 4698 | 36251746 |
| 4609 | 35363897 | 4654 | 35812672 | 4699 | 36261728 |
| 4610 | 35373866 | 4655 | 35822648 | 4700 | 36271710 |
| 4611 | 35383836 | 4656 | 35832624 | 4701 | 36281691 |
| 4612 | 35393806 | 4657 | 35842600 | 4702 | 36291673 |
| 4613 | 35403776 | 4658 | 35852577 | 4703 | 36301656 |
| 4614 | 35413746 | 4659 | 35862553 | 4704 | 36311638 |
| 4615 | 35423716 | 4660 | 35872530 | 4705 | 36321621 |
| 4616 | 35433686 | 4661 | 35882507 | 4706 | 36331603 |
| 4617 | 35443657 | 4662 | 35892484 | 4707 | 36341586 |
| 4618 | 35453628 | 4663 | 35902461 | 4708 | 36351569 |
| 4619 | 35463598 | 4664 | 35912439 | 4709 | 36361552 |
| 4620 | 35473569 | 4665 | 35922416 | 4710 | 36371535 |
| 4621 | 35483541 | 4666 | 35932394 | 4711 | 36381518 |
| 4622 | 35493512 | 4667 | 35942372 | 4712 | 36391501 |
| 4623 | 35503483 | 4668 | 35952349 | 4713 | 36401485 |
| 4624 | 35513455 | 4669 | 35962327 | 4714 | 36411469 |
| 4625 | 35523427 | 4670 | 35972306 | 4715 | 36421452 |
| 4626 | 35533399 | 4671 | 35982284 | 4716 | 36431436 |
| 4627 | 35543371 | 4672 | 35992262 | 4717 | 36441420 |
| 4628 | 35553343 | 4673 | 36002241 | 4718 | 36451404 |
| 4629 | 35563315 | 4674 | 36012219 | 4719 | 36461388 |
| 4630 | 35573288 | 4675 | 36022198 | 4720 | 36471372 |
| 4631 | 35583260 | 4676 | 36032177 | 4721 | 36481357 |
| 4632 | 35593233 | 4677 | 36042156 | 4722 | 36491341 |
| 4633 | 35603206 | 4678 | 36052135 | 4723 | 36501326 |
| 4634 | 35613179 | 4679 | 36062115 | 4724 | 36511310 |
| 4635 | 35623153 | 4680 | 36072094 | 4725 | 36521295 |

| Verſ. fine | Seg. area | Verſ. fine | Seg. area | Verſ. fine | Seg. area |
|---|---|---|---|---|---|
| 4726 | 36531280 | 4772 | 36990699 | 4818 | 37450310 |
| 4727 | 36541265 | 4773 | 37000688 | 4819 | 37460304 |
| 4728 | 36551250 | 4774 | 37010678 | 4820 | 3747029 |
| 4729 | 36561236 | 4775 | 37020668 | 4821 | 37480291 |
| 4730 | 36571221 | 4776 | 37030658 | 4822 | 37490284 |
| 4731 | 36581206 | 4777 | 37040648 | 4823 | 37500278 |
| 4732 | 36591192 | 4778 | 37050638 | 4824 | 37510272 |
| 4733 | 36601178 | 4779 | 37060628 | 4825 | 37520266 |
| 4734 | 36611163 | 4780 | 37070618 | 4826 | 37530259 |
| 4735 | 36621149 | 4781 | 37080609 | 4827 | 37540253 |
| 4736 | 36631135 | 4782 | 37090599 | 4828 | 37550247 |
| 4737 | 36641121 | 4783 | 37100590 | 4829 | 37560242 |
| 4738 | 36651108 | 4784 | 37110580 | 4830 | 37570236 |
| 4739 | 36661094 | 4785 | 37120571 | 4831 | 37580230 |
| 4740 | 36671080 | 4786 | 37130562 | 4832 | 37590224 |
| 4741 | 36681067 | 4787 | 37140553 | 4833 | 37600219 |
| 4742 | 36691053 | 4788 | 37150544 | 4834 | 37610213 |
| 4743 | 36701040 | 4789 | 37160535 | 4835 | 37620208 |
| 4744 | 36711027 | 4790 | 37170526 | 4836 | 37630202 |
| 4745 | 36721014 | 4791 | 37180517 | 4837 | 37640197 |
| 4746 | 36731001 | 4792 | 37190508 | 4838 | 37650192 |
| 4747 | 36740988 | 4793 | 37200500 | 4839 | 37660186 |
| 4748 | 36750976 | 4794 | 37210491 | 4840 | 37670181 |
| 4749 | 36760963 | 4795 | 37220483 | 4841 | 37680176 |
| 4750 | 36770950 | 4796 | 37230474 | 4842 | 37690171 |
| 4751 | 36780938 | 4797 | 37240466 | 4843 | 37700166 |
| 4752 | 36790925 | 4798 | 37250458 | 4844 | 37710161 |
| 4753 | 36800913 | 4799 | 37260450 | 4845 | 37720156 |
| 4754 | 36810901 | 4800 | 37270442 | 4846 | 37730152 |
| 4755 | 36820889 | 4801 | 37280434 | 4847 | 37740147 |
| 4756 | 36830877 | 4802 | 37290426 | 4848 | 37750142 |
| 4757 | 36840865 | 4803 | 37300418 | 4849 | 37760138 |
| 4758 | 36850853 | 4804 | 37310410 | 4850 | 37770133 |
| 4759 | 36860842 | 4805 | 37320403 | 4851 | 37780129 |
| 4760 | 36870830 | 4806 | 37330395 | 4852 | 37790124 |
| 4761 | 36880819 | 4807 | 37340387 | 4853 | 37800120 |
| 4762 | 36890807 | 4808 | 37350380 | 4854 | 37810116 |
| 4763 | 36900796 | 4809 | 37360373 | 4855 | 37820111 |
| 4764 | 36910785 | 4810 | 37370365 | 4856 | 37830107 |
| 4765 | 36920774 | 4811 | 37380358 | 4857 | 37840103 |
| 4766 | 36930763 | 4812 | 37390351 | 4858 | 37850099 |
| 4767 | 36940752 | 4813 | 37400344 | 4859 | 37860095 |
| 4768 | 36950741 | 4814 | 37410337 | 4860 | 37870091 |
| 4769 | 36960730 | 4815 | 37420330 | 4861 | 37880087 |
| 4770 | 36970720 | 4816 | 37430324 | 4862 | 37890083 |
| 4771 | 36980709 | 4817 | 37440317 | 4863 | 37900080 |

| Vers. fine | Seg. area | Vers. fine | Seg. area | Vers. fine | Seg. area |
|---|---|---|---|---|---|
| 4864 | 37910076 | 4910 | 38369956 | 4956 | 38829914 |
| 4865 | 37920072 | 4911 | 38379955 | 4957 | 38839913 |
| 4866 | 37930069 | 4912 | 38389953 | 4958 | 38849913 |
| 4867 | 37940065 | 4913 | 38399952 | 4959 | 38859913 |
| 4868 | 37950062 | 4914 | 38409950 | 4960 | 38869912 |
| 4869 | 37960058 | 4915 | 38419949 | 4961 | 38879912 |
| 4870 | 37970055 | 4916 | 38429947 | 4962 | 38889912 |
| 4871 | 37980051 | 4917 | 38439946 | 4963 | 38899912 |
| 4672 | 37990048 | 4918 | 38449945 | 4964 | 38909911 |
| 4873 | 38000045 | 4919 | 38459943 | 4965 | 38919911 |
| 4874 | 38010042 | 4920 | 38469942 | 4966 | 38929911 |
| 4875 | 38020038 | 4921 | 38479941 | 4967 | 38939911 |
| 4876 | 38030035 | 4922 | 38489939 | 4968 | 38949910 |
| 4877 | 38040032 | 4923 | 38499938 | 4969 | 38959910 |
| 4878 | 38050029 | 4924 | 38509937 | 4970 | 38969910 |
| 4879 | 38060026 | 4925 | 38519936 | 4971 | 38979910 |
| 4880 | 38070023 | 4926 | 38529935 | 4972 | 38989910 |
| 4881 | 38080020 | 4927 | 38539934 | 4973 | 38999909 |
| 4882 | 38090018 | 4928 | 38549933 | 4974 | 39009909 |
| 4883 | 38100015 | 4929 | 38559932 | 4975 | 39019909 |
| 4884 | 38110012 | 4930 | 38569931 | 4976 | 39029909 |
| 4885 | 38120010 | 4931 | 38579930 | 4977 | 39039909 |
| 4886 | 38130007 | 4932 | 38589929 | 4978 | 39049909 |
| 4887 | 38140004 | 4933 | 38599928 | 4979 | 39059909 |
| 4888 | 38150002 | 4934 | 38609927 | 4980 | 39069909 |
| 4889 | 38159999 | 4935 | 38619926 | 4981 | 39079909 |
| 4890 | 38169997 | 4936 | 38629926 | 4982 | 39089908 |
| 4891 | 38179994 | 4937 | 38639925 | 4983 | 39099908 |
| 4892 | 38189992 | 4938 | 38649924 | 4984 | 39109908 |
| 4893 | 38199990 | 4939 | 38659923 | 4985 | 39119908 |
| 4894 | 38209988 | 4940 | 38669923 | 4986 | 39129908 |
| 4895 | 38219985 | 4941 | 38679922 | 4987 | 39139908 |
| 4896 | 38229983 | 4942 | 38689921 | 4988 | 39149908 |
| 4897 | 38239981 | 4943 | 38699920 | 4989 | 39159908 |
| 4898 | 38249979 | 4944 | 38709920 | 4990 | 39169908 |
| 4899 | 38259977 | 4945 | 38719919 | 4991 | 39179908 |
| 4900 | 38269975 | 4946 | 38729919 | 4992 | 39189908 |
| 4901 | 38279973 | 4947 | 38739918 | 4993 | 39199908 |
| 4902 | 38289971 | 4948 | 38749918 | 4994 | 39209908 |
| 4903 | 38299969 | 4949 | 38759917 | 4995 | 39219908 |
| 4904 | 38309967 | 4950 | 38769917 | 4996 | 39229908 |
| 4905 | 38319965 | 4951 | 38779916 | 4997 | 39239908 |
| 4906 | 38329963 | 4952 | 38789916 | 4998 | 39249908 |
| 4907 | 38339961 | 4953 | 38799915 | 4999 | 39259908 |
| 4908 | 38349960 | 4954 | 38809915 | 5000 | 39269908 |
| 4909 | 38359958 | 4955 | 38819914 | | |

In the preceding table, each number in the column of ſegments is the area of the circular ſegment, whoſe height, or verſed ſine of its half arc, is the number immediately on the left of it, the diameter of the circle being 1, and the whole area 7853981 6; and therefore all the numbers in the table are to be accounted decimals.

This table is here extended to ten times the length which it had uſually been, the diameter being divided into ten thouſand equal parts, inſtead of one thouſand: And this ſize of it alſo allowed me to carry each number out to eight places of figures, inſtead of ſix; for when the diameter is divided only into one thouſand parts, the proportional part for the figures in the verſed ſine, beyond the third, will not give the area true beyond the 6th figure; but in this table it is always found true in the 8th place.

Rule the 9th to the circular ſegment in page 143 explains the uſe of it; to which it may be only neceſſary to add there, when a ſegment greater than a ſemicircle is to be found, ſubtract the quotient of its verſed ſine, divided by its diameter, from 1; then ſubtract the tabular ſegment which correſponds to the remainder, from 7853981 6, the whole tabular circle, and the remainder will be the tabular circle correſponding to the ſegment required; and which muſt then be multiplied by the ſquare of the diameter.

FINIS.